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EXTENSION OF A GENERALIZED PEXIDER EQUATION

JÁNOS ACZÉL

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ABSTRACT. The equations $k(s+t) = \ell(s)+n(t)$ and $k(s+t) = m(s)n(t)$, called Pexider equations, have been completely solved on $\mathbb{R}^2$. If they are assumed to hold only on an open region, they can be extended to $\mathbb{R}^2$ (the second when k is nowhere 0) and solved that way. In this paper their common generalization $k(s+t) = \ell(s) + m(s)n(t)$ is extended from an open region to $\mathbb{R}^2$ and then completely solved if k is not constant on any proper interval. This equation has further interesting particular cases, such as $k(s+t) = \ell(s) + m(s)k(t)$ and $k(s+t) = k(s)+m(s)n(t)$, that arose in characterization of geometric and power means and in a problem of equivalence of certain utility representations, respectively, where the equations may hold only on an open region in $\mathbb{R}^2$. Thus these problems are solved too.

1. INTRODUCTION

It is known (see e.g. [2, pp. 76–80]) that, if the Pexider equation

$$k(s+t) = \ell(s) + n(t) \tag{1.1}$$

holds on an open region (connected open set) R of $\mathbb{R}^2$, then it can be extended to the whole real plane $\mathbb{R}^2$ in the following sense (true also on more general topological spaces, see e.g. [8]): There exist unique functions $K, L, N : \mathbb{R} \to \mathbb{R}$, equal to k, ℓ, n, respectively, on domains of the latter, and satisfying

$$K(x + y) = L(x) + N(y) \quad \text{for all} \quad (x, y) \in \mathbb{R}^2.$$

A similar statement is true for another Pexider equation,

$$k(s+t) = m(s)n(t) \tag{1.2}$$

if k is nowhere 0 on $\{s + t \,|\, (s, t) \in R\}$. No other regularity assumption was made about the functions k, ℓ, m, n.

While looking for certain equivalent utility representations, we found in [4] the functional equation

$$k(s+t) = \ell(s) + m(s)n(t) \tag{1.3}$$

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(actually with $k = \ell$), valid on

$$\{(s,t) \mid s \in]a,b[,\ t \in]0, b-s[\}. \tag{1.4}$$

There are results in [3] and [6] about differentiability of measurable solutions of equation (1.3) and of more general ones, on cartesian rectangles, leading to their general measurable solution. However, (1.4) is not a cartesian rectangle. To remedy that situation, we extend equation (1.3) from any open region in the real plane to all of $\mathbb{R}^2$ when k is not constant on any interval of positive length and determine all solutions, in particular all measurable ones. Actually, in the final form of [4] the measurable solutions of the special equation ($k = \ell$) have been determined by an *ad hoc* method of C.T. Ng, without making use of extensions.

The general equation (1.3) is still of interest, since it generalizes both (1.1) and (1.2), and also the equation $k(s+t) = \ell(s) + m(s)k(t)$, an important tool in characterizing geometric and power means (see e.g. [1, pp. 150–153], cf. [5, pp. 68–69]). Our extension result makes it possible to calculate the general solution of equation (1.3) and of its particular cases on any open region. Moreover, we will assume no measurability (except in the Corollary) but determine the general solution without any regularity assumption. We assume only that k is not constant on any interval of positive length. We call such functions *locally nonconstant* (A. Lundberg [7] and others call them "philandering").

2. Extension from a hexagonal neighborhood to $\mathbb{R}^2$

We first assume that equation (1.3) is valid on

$$H(c,d;r) := \{(s,t) \mid s \in]c-r, c+r[,\ t \in]d-r, d+r[,\ s+t \in]c+d-r, c+d+r[\}.$$

This is a hexagonal neighborhood of (c,d). We prove that equation (1.3) has a unique extension from $H(c,d;r)$ to $\mathbb{R}^2$. This is the hard part. By properties of open regions, what will remain to prove is that the extensions from two intersecting open hexagons are the same.

Proposition. *If k is locally nonconstant and*

$$k(s+t) = \ell(s) + m(s)n(t) \quad \textit{for all} \quad (s,t) \in H(c,d;r), \tag{2.1}$$

then there exists a unique quadruple of functions $K, L, M, N : \mathbb{R} \to \mathbb{R}$ such that

$$L(s) = \ell(s), \quad M(s) = m(s) \qquad (s \in]c-r, c+r[), \tag{2.2}$$

$$N(t) = n(t)\ (t \in]d-r, d+r[), \qquad K(q) = k(q)\ (q \in]c+d-r, c+d+r[), \tag{2.3}$$

and

$$K(x+y) = L(x) + M(x)N(y) \qquad \textit{for all } (x,y) \in \mathbb{R}^2. \tag{2.4}$$

Proof. We prove the Proposition in three steps.

1. *Moving center to origin:*

Put $s = c$, $t = d$ into (2.1) to get

$$k(c+d) = \ell(c) + m(c)n(d). \tag{2.5}$$

With

$$H^0 := \{(u,v) \mid u, v, u+v \in]-r, r[\}$$

and $s = u + c$, $t = v + d$ in (2.1) we obtain

$$k(u+v+c+d) = \ell(u+c) + m(u+c)n(v+d) \qquad ((u,v) \in H^0). \tag{2.6}$$

We subtract (2.5) from (2.6) and get

$$\begin{aligned} &k(u+v+c+d) - k(c+d) \\ &\qquad = \ell(u+c) - \ell(c) + m(u+c)n(v+d) - m(c)n(d). \end{aligned} \tag{2.7}$$

Substitute $v = 0$:

$$k(u+c+d) - k(c+d) = \ell(u+c) - \ell(c) + [m(u+c) - m(c)]n(d). \tag{2.8}$$

Define

$$\lambda(u) = \ell(u+c) - \ell(c), \quad \mu(u) = m(u+c) - m(c), \tag{2.9}$$

$$\nu(v) = n(v+d) - n(d), \qquad \kappa(w) = k(w+c+d) - k(c+d) \tag{2.10}$$

$(u, v, w \in\,]-r, r[$; notice that $\lambda(0) = \mu(0) = \nu(0) = \kappa(0) = 0$). From (2.8), (2.9), (2.10),

$$\kappa(u) = \lambda(u) + \mu(u)n(d) \tag{2.11}$$

and, from (2.7), (2.9), (2.10),

$$\kappa(u+v) = \lambda(u) + [\mu(u) + m(c)][\nu(v) + n(d)] - m(c)n(d);$$

that is, by (2.9), (2.10) and (2.11),

$$\kappa(u+v) = \kappa(u) + m(u+c)\nu(v). \tag{2.12}$$

Putting $u = 0$ gives $\kappa(v) = m(c)\nu(v)$. If $m(c)$ were 0, then $\kappa = 0$ and k would be constant, which was excluded. Thus

$$\nu(v) = \frac{\kappa(v)}{m(c)}. \tag{2.13}$$

Defining

$$e(u) = \frac{m(u+c)}{m(c)}, \tag{2.14}$$

equation (2.12) becomes

$$\kappa(u+v) = \kappa(u) + e(u)\kappa(v) \qquad ((u,v) \in H^0). \tag{2.15}$$

The left-hand side being symmetric, so is the right: $\kappa(u) + e(u)\kappa(v) = \kappa(v) + e(v)\kappa(u)$, that is,

$$\kappa(u)[e(v) - 1] = \kappa(v)[e(u) - 1]. \tag{2.16}$$

There are two cases:

(i) The function e is identically 1. Then, by (2.15),

$$\kappa(u+v) = \kappa(u) + \kappa(v) \quad ((u,v) \in H^0).$$

(ii) The function e is not identically 1. Then, by (2.16),

$$\kappa(v) = \gamma[e(v) - 1] \qquad (\gamma \neq 0). \tag{2.17}$$

Putting this into (2.15), we get

$$e(u+v) = e(u)e(v) \qquad ((u,v) \in H^0). \tag{2.18}$$

2. *Case* (i):

There, by (2.14), $m(u+c)/m(c) = e(u) = 1$, so m = constant = $m(c)$ on $]c-r, c+r[$. Thus, from (2.9), μ is identically 0 and, by (2.11),

$$\lambda(u) = \kappa(u) \quad (u \in]-r, r[). \tag{2.19}$$

The equation $\kappa(u+v) = \kappa(u) + \kappa(v)$ has a *unique extension* from H^0 to $\mathbb{R}^2$ (see e.g. [2, p. 17]; $x + y \in I$ was omitted from the end of eq. (31) there); i.e., there exists a unique $A : \mathbb{R} \to \mathbb{R}$ such that

$$A(w) = \kappa(w) \quad \text{for} \quad w \in]-r, r[\tag{2.20}$$

and

$$A(x+y) = A(x) + A(y) \quad \text{for all} \quad (x,y) \in \mathbb{R}^2. \tag{2.21}$$

We have, from (2.10) and (2.13),

$$n(v) = \nu(v-d) + n(d) = \frac{1}{m(c)}\kappa(v-d) + n(d) \text{ for } v \in]d-r, d+r[.$$

Since κ has the unique extension A to $\mathbb{R}$, the function n has the *unique extension*

$$N(y) = \frac{1}{m(c)}A(y-d) + n(d) \quad (y \in \mathbb{R}). \tag{2.22}$$

Similarly, L, M, K, defined by

$$\begin{cases} L(x) = A(x-c) + \ell(c),\ M(x) = m(c), \\ K(z) = A(z-c-d) + k(c+d)\ (x, z \in \mathbb{R}), \end{cases} \tag{2.23}$$

are the *unique extensions* of ℓ, m and k, respectively. We check this, for instance, for ℓ : by (2.9), (2.19), and (2.20),

$$\ell(s) = \lambda(s-c) + \ell(c) = \kappa(s-c) + \ell(c) = A(s-c) + \ell(c) = L(s) \quad \text{for} \quad s \in]c-r, c+r[.$$

By (2.23), (2.22), (2.5) and (2.21), the extended functions L, M, K, N also *satisfy equation* (2.4):

$$\begin{aligned} L(x) + M(x)N(y) &= A(x-c) + \ell(c) + m(c)[\frac{1}{m(c)}A(y-d) + n(d)] \\ &= A(x+y-c-d) + k(c+d) = K(x+y). \end{aligned}$$

3. *Case* (ii):

If there existed a $u_0 \in]-r, r[$ with $e(u_0) = 0$, then, by (2.18), $e(u_0 + v) = e(u_0)e(v) = 0$; thus e and, by (2.17), κ would be constant on an interval, which we excluded. If e has no zero in $]-r, r[$, then, by (2.18), $\log e$ is additive, so it and with it e has a *unique extension* from H^0 to $\mathbb{R}^2$; i.e., there exists a unique $E : \mathbb{R} \to \mathbb{R}_+ =]0, \infty[$ such that

$$E(w) = e(w) \qquad \text{for} \quad w \in]-r, r[\tag{2.24}$$

and

$$E(x+y) = E(x)E(y) \quad \text{for all} \quad (x,y) \in \mathbb{R}^2. \tag{2.25}$$

[The functions e, E are positive valued because $e(x) = e(x/2)^2 \geq 0$, $E(x) = E(x/2)^2 \geq 0$ and if e or E were 0 at a point, we would get, as above, $e(x) \equiv 0$ or $E(x) \equiv 0$, respectively; these have been excluded; also $e(x) \equiv 1, E(x) \equiv 1$ were excluded.]

From (2.9), (2.11), (2.17) and from (2.14),

$$\begin{aligned}\ell(u) &= \lambda(u-c)+\ell(c)\\ &= \kappa(u-c)-\mu(u-c)n(d)+\ell(c)\\ &= \gamma[e(u-c)-1]-[m(u)-m(c)]n(d)+\ell(c)\\ &= [\gamma-m(c)n(d)][e(u-c)-1]+\ell(c).\end{aligned}$$

Since γ is a constant determined by e and since e has the unique extension E to $\mathbb{R}_+$, the function ℓ has the *unique extension*

$$L(x) = [\gamma - m(c)n(d)]E(x-c) - \gamma + m(c)n(d) + \ell(c) \quad (x \in \mathbb{R}). \tag{2.26}$$

Similarly, N, M, K, defined by

$$N(y) = \frac{\gamma[E(y-d)-1]}{m(c)} + n(d) \qquad (y \in \mathbb{R}) \tag{2.27}$$

and

$$\begin{cases} M(x) = m(c)E(x-c) \ (x \in \mathbb{R}),\\ K(z) = \gamma[E(z-c-d)-1] + k(c+d) \ (z \in \mathbb{R}), \end{cases} \tag{2.28}$$

are the *unique extensions* of n and m, k, respectively. For example, check for n: By (2.10), (2.13), and (2.17),

$$\begin{aligned} n(t) &= \nu(t-d)+n(d)\\ &= \frac{\kappa(t-d)}{m(c)}+n(d)\\ &= \frac{\gamma[e(t-d)-1]}{m(c)}+n(d)\\ &= N(t) \text{ for } t \in\,]d-r, d+r[. \end{aligned}$$

By (2.26), (2.27), (2.28), (2.5) and (2.25) the extended functions L, N, M, K *satisfy equation* (2.4):

$$\begin{aligned} L(x)+M(x)N(y) &= \gamma E(x-c)E(y-d) - \gamma + m(c)n(d) + \ell(c)\\ &= \gamma[E(x+y-c-d)-1] + k(c+d) = K(x+y). \end{aligned}$$

□

3. Extension from open region to $\mathbb{R}^2$

Theorem 1. *Let R be an open region in $\mathbb{R}^2$ and let*

$$\begin{cases} R_s := \{s \mid \exists t : (s,t) \in R\},\\ R_t := \{t \mid \exists s : (s,t) \in R\},\\ R_{s+t} := \{s+t \mid (s,t) \in R\}. \end{cases} \tag{3.1}$$

If k is locally nonconstant and

$$k(s+t) = \ell(s) + m(s)n(t) \quad \textit{for all} \quad (s,t) \in R, \tag{3.2}$$

then there exists a unique quadruple of functions $K, L, M, N : \mathbb{R} \to \mathbb{R}$ such that

$$L(s) = \ell(s),\ M(s) = m(s)\ (s \in R_s),\ N(t) = n(t)\ (t \in R_t),\ K(q) = k(q)\ (q \in R_{s+t})$$

and

$$K(x+y) = L(x) + M(x)N(y) \quad \textit{for all} \quad (x,y) \in \mathbb{R}^2. \tag{3.3}$$

Proof. Take two open sets, S_1 and S_2, with nonempty intersection. Suppose that equation (3.2) has a unique extension from S_1 and a unique extension from S_2. We show that in this case there exists a unique extension from their union and the three extended quadruples L, M, N, K, starting from the two open sets and their union, are the same.

Indeed, for the intersection D of the two open sets, $D_x := \{x \mid \exists y : (x, y) \in D\}$ contains an interval I of positive length. Take an arbitrary $(c, d) \in D$ such that $c \in I$. By (2.10) and (2.20), the two additive functions A_1 and A_2, obtained from the first and second set, respectively, are equal on a neighborhood U of c. So $A_1 - A_2$, that is also additive on $\mathbb{R}^2$, is zero on U. But an additive function that is zero on an interval of positive length is identically zero everywhere, so $A_1(x) = A_2(x)$ for all $x \in \mathbb{R}$. The L, N, K built from them in case (i) and the constant M are also the same for the two sets. The argument is similar in case (ii) (there M_1, M_2, E_1, E_2 and E_1/E_2 are to be considered).

Finally, if R is an open region (connected open set), then it is path-connected, so a standard compactness argument (we move on strings of intersecting hexagonal neighborhoods) gives the result. □

NOTE 1. The general solutions (with locally nonconstant K) of equation (3.3) are given by (2.22), (2.23) and by (2.26), (2.27), (2.28) with arbitrary constants $\gamma \neq 0$, c, d, arbitrary constants in place of $\ell(c), m(c), n(d)$ with $k(c+d) = \ell(c) + m(c)n(d)$, and with arbitrary locally nonconstant exponential E and additive A. *Exponential* functions are the solutions of

$$E(x+y) = E(x)E(y) \qquad ((x, y) \in \mathbb{R}^2). \tag{3.4}$$

Thus we have the following.

Theorem 2. *Let $R \subseteq \mathbb{R}^2$ be an open region and R_s, R_t, R_{s+t} defined as in* (3.1). *Then the general solutions, with locally nonconstant k, of equation* (3.2) *are given by*

$$m(s) = \omega, \quad \ell(s) = A(s) + B \qquad (s \in R_s), \tag{3.5}$$

$$n(t) = \frac{1}{\omega}A(t) + P \quad (t \in R_t), \qquad k(q) = A(q) + B + P\omega \quad (q \in R_{s+t}) \tag{3.6}$$

and by

$$m(s) = \omega E(s), \quad \ell(s) = \alpha E(s) + B \qquad (s \in R_s), \tag{3.7}$$

$$n(t) = \delta E(t) - \frac{\alpha}{\omega} \quad (t \in R_t), \qquad k(q) = \omega\delta E(q) + B \quad (q \in R_{s+t}), \tag{3.8}$$

where $A : \mathbb{R} \to \mathbb{R}$ and $E : \mathbb{R} \to \mathbb{R}_+$ are arbitrary locally nonconstant additive or exponential functions (*solutions of* (3.4)), *respectively, and $\omega \neq 0, \delta \neq 0, \alpha, B, P$ are arbitrary constants.*

Proof. From equations (2.2), (2.3), (2.23), (2.22) (with $\omega := m(C)$) we get, since A is additive,

$$m(s) = \omega, \ell(s) = A(s) + B \ (s \in R_s),$$
$$n(t) = \frac{1}{\omega}A(t) + P \ (t \in R_t),$$
$$k(q) = A(q) + Q \ (q \in R_{s+t}).$$

Substitution into (3.2) gives $Q = B + P\omega$; thus (3.5) and (3.6) hold.

On the other hand, since E satisfies (3.4), we get from (2.2), (2.3), (2.26), (2.27), (2.28) that

$$m(s) = \omega E(s), \quad \ell(s) = \alpha E(s) + B \qquad (s \in R_s),$$
$$n(t) = \delta E(t) + P \quad (t \in R_t), \qquad k(q) = \varepsilon E(q) + Q \quad (q \in R_{s+t})$$

($\varepsilon \neq 0$ because k is locally nonconstant). Putting these into (3.2), we get $\varepsilon = \omega\delta \neq 0$, $P = -\alpha/\omega, Q = B$, thus (3.7) and (3.8). □

Since the nonconstant measurable solutions of (2.21) and (3.4) are given by $A(x) = Cx$ and $E(x) = e^{Cx}$ ($C \neq 0$), respectively, we have also the following:

Corollary. *Let $R \subseteq \mathbb{R}^2$ be an open region and R_s, R_t, R_{s+t} defined as in* (3.1). *Then the general solutions of equation* (3.2), *with locally nonconstant k, measurable on an interval, are given by*

$$m(s) = \omega,\ \ell(s) = Cs + B\ (s \in R_s),$$
$$n(t) = \frac{C}{\omega}t + P\ (t \in R_t),\ k(q) = Cq + B + P\omega\ (q \in R_{s+t})$$

and by

$$m(s) = \omega e^{Cs}, \quad \ell(s) = \alpha e^{Cs} + B \qquad (s \in R_s),$$
$$n(t) = \delta e^{Ct} - \frac{\alpha}{\omega} \quad (t \in R_t), \quad k(q) = \omega\delta e^{Cq} + B \quad (q \in R_{s+t}),$$

where $\alpha, B, P, C, \omega, \delta$ are arbitrary constants with $C\omega\delta \neq 0$.

Note 2. If k is constant, then (1.3) is a particular case of $\sum f_j(s)g_j(t) = 0$, whose general solutions are well known (see e.g. [1, pp. 160–165]).

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Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

E-mail address: jdaczel@math.uwaterloo.ca