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LOCAL AUTOMORPHISMS AND DERIVATIONS ON CERTAIN C^*-ALGEBRAS

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ABSTRACT. It is shown that continuous 2-local derivations on AF C^*-algebras are derivations and surjective 2-local *-automorphisms on prime C^*-algebras or on C^*-algebras such that the identity element is properly infinite are *-automorphisms.

A mapping ϕ of an algebra $\mathcal{A}$ into itself is called a local derivation (respectively, local automorphism) if for every $A \in \mathcal{A}$ there exists a derivation (respectively, automorphism) ϕ_A of $\mathcal{A}$, depending on A, such that $\phi(A) = \phi_A(A)$. These notions were introduced by Kadison [Kad] and Larson and Sourour [LaSo]. In fact, their definitions were stronger. They have assumed that these mappings are also linear. Larson and Sourour proved that every local derivation on $B(X)$, the algebra of all bounded linear operators on a Banach space X, is a derivation, and provided that X is infinite-dimensional, every surjective linear local automorphism of $B(X)$ is an automorphism. In [BrSe], they proved that the surjectivity assumption in the last result can be dropped if X is a separable Hilbert space.

It is easy to see that if we drop the assumption of linearity of the local maps, then the corresponding statements are no longer true. However, in [KoSl], they obtained the following result: If $\mathcal{A}$ is a unital Banach algebra and if $\phi : \mathcal{A} \to \mathbf{C}$ is a map (no linearity is assumed) having the property that $\phi(I) = 1$ and for every $A, B \in \mathcal{A}$, there exists a multiplicative linear functional $\phi_{A,B}$ on $\mathcal{A}$ such that $\phi(A) = \phi_{A,B}(A)$ and $\phi(B) = \phi_{A,B}(B)$, then ϕ is linear and multiplicative.

Motivated by the above considerations, Šemrl [Sem] introduced the following definition.

Definition. Let $\mathcal{A}$ be an algebra. A mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a 2-local derivation (respectively, 2-local automorphism) if for every $A, B \in \mathcal{A}$ there is a derivation (respectively, automorphism) $\phi_{A,B} : \mathcal{A} \to \mathcal{A}$, depending on A and B, such that $\phi(A) = \phi_{A,B}(A)$ and $\phi(B) = \phi_{A,B}(B)$.

Also, they showed the following result.

Theorem 1 ([Sem]). *Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Then every 2-local*

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automorphism $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ (no linearity, surjectivity or continuity of ϕ is assumed) is an automorphism and every 2-local derivation $\theta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ (no linearity or continuity of θ is assumed) is a derivation.

Molnár [Mol] and Xie and Lu [XiLu] showed that every 2-local automorphism on $M_n(\mathbf{C})$ is an automorphism and in [Kim2] they gave a short proof that every 2-local derivation on $M_n(\mathbf{C})$ is a derivation. It is the aim of this note to give a proof that continuous 2-local derivations on AF C^*-algebras are derivations and surjective 2-local *-automorphisms on prime C^*-algebras or on C^*-algebras such that the identity element is properly infinite are *-automorphisms. Here, the term 2-local *-automorphisms is used for 2-local automorphisms that preserve the *-operation. Recall that an approximately finite, or AF C^*-algebra is a unital C^*-algebra $\mathcal{A}$ which is an inductive limit of an increasing sequence of finite-dimensional C^*-algebras $\mathcal{A}_n,\ 1 \le n < \infty$, with unital embeddings $j_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$. An equivalent definition is to say that $\mathcal{A}$ is an AF C^*-algebra if it has an ascending sequence of finite-dimensional C^*-subalgebras whose closed union is $\mathcal{A}$. If each $\mathcal{A}_n$ is chosen to be a factor, then $\mathcal{A}$ is called a UHF C^*-algebra.

First, we consider 2-local derivations on M_n. The following theorem is seen in [Kim2, Theorem 3].

Theorem 2. *Let M_n be the $n \times n$ matrix algebra over $\mathbf{C}$ and $\phi : M_n \to M_n$ be a 2-local derivation. Then ϕ is a derivation.*

Proposition 3. *Let $\mathcal{A} = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ be a finite-dimensional C^*-algebra, where M_i is a $k(i) \times k(i)$ matrix algebra for some natural number $k(i)$. If $\phi : \mathcal{A} \to \mathcal{A}$ is a 2-local derivation, then ϕ is a derivation.*

Proof. Using the fact that every derivation of a von Neumann algebra is inner, for every $A, B \in \mathcal{A}$ there exists an element $T \in \mathcal{A}$ such that $\phi(A) = AT - TA$ and $\phi(B) = BT - TB$. Let $A = (A_1, 0, \ldots, 0), B = (B_1, 0, \ldots, 0)$ and $T = (T_1, \ldots, T_n)$ for $A_1, B_1 \in M_1$ and $T_i \in M_i$. Since

$$(A_1, 0, \ldots, 0)(T_1, \ldots, T_n) - (T_1, \ldots, T_n)(A_1, 0, \ldots, 0) = (A_1T_1 - T_1A_1, 0, \ldots, 0),$$

the restriction map $\phi_1 = \phi|_{M_1} : M_1 \to M_1$ is a 2-local derivation of M_1. By Theorem 2, ϕ_1 is a derivation. Let $\phi_i = \operatorname{ad} S_i$ for $i = 1, \ldots, n$. Since

$$\begin{aligned}\phi(A_1, \ldots, A_n) &= (A_1, \ldots, A_n)(T_1, \ldots, T_n) - (T_1, \ldots, T_n)(A_1, \ldots, A_n)\\ &= \phi_1(A_1) \oplus \ldots \oplus \phi_n(A_n)\\ &= (A_1S_1 - S_1A_1) \oplus \ldots \oplus (A_nS_n - S_nA_n)\\ &= (A_1, \ldots, A_n)(S_1, \ldots, S_n) - (S_1, \ldots, S_n)(A_1, \ldots, A_n),\end{aligned}$$

it follows that ϕ is a derivation. □

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. A conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ is a linear map Φ of $\mathcal{A}$ onto $\mathcal{B}$ such that $\Phi(B) = B$ for every $B \in \mathcal{B}$ and $\|\Phi(A)\| \le \|A\|$ for every $A \in \mathcal{A}$. It is well known that if Φ is a conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$, then it is a positive map and satisfies $\Phi(B_1AB_2) = B_1\Phi(A)B_2$ for every $A \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$.

Now we consider local derivations on AF C^*-algebras.

Theorem 4. *Let $\mathcal{A} = \overline{\bigcup \mathcal{A}_n}$ be an* AF *C^*-algebra and let $\phi : \mathcal{A} \to \mathcal{A}$ be a continuous 2-local derivation. Then ϕ is a derivation.*

Proof. We prove this by borrowing the idea of [Cri]. Fix n and let $k \geq n$. Take $A_n, B_n \in \mathcal{A}_n \subseteq \mathcal{A}_k$. Note that there is a sequence $\Phi_k : \mathcal{A} \to \mathcal{A}_k$ of conditional expectations with the property that $\Phi_k(A) \to A$ as $k \to \infty$ for every $A \in \mathcal{A}$. Let us consider the map $\Phi_k \circ \phi : \mathcal{A} \to \mathcal{A}_k$. We claim that $\Phi_k \circ \phi|_{\mathcal{A}_k} : \mathcal{A}_k \to \mathcal{A}_k$ is a 2-local derivation. To see this, for any $A, B \in \mathcal{A}_k$, there is a derivation $\alpha : \mathcal{A} \to \mathcal{A}$ such that $\phi(A) = \alpha(A)$ and $\phi(B) = \alpha(B)$. Then $\Phi_k \circ \phi(A) = \Phi_k(\alpha(A))$ and $\Phi_k \circ \phi(B) = \Phi_k(\alpha(B))$. But for $X, Y \in \mathcal{A}_k$, we have that

$$(\Phi_k \circ \alpha)(XY) = \Phi_k(X\alpha(Y) + \alpha(X)Y) = X(\Phi_k\alpha(Y)) + \Phi_k(\alpha(X))Y,$$

so that $\Phi_k \circ \alpha$ is a derivation. Let $\eta_k = \Phi_k \circ \phi|_{\mathcal{A}_k}$. Since η_k is a 2-local derivation, it is a derivation by Proposition 3. Hence, from $\eta_k(A_nB_n) = A_n\eta_k(B_n) + \eta_k(A_n)B_n$, it follows that

$$\phi(A_nB_n) = A_n\phi(B_n) + \phi(A_n)B_n$$

on $\bigcup \mathcal{A}_n$. Similarly, from $\eta_k(A_n + B_n) = \eta_k(A_n) + \eta_k(B_n)$, we have that ϕ is additive on $\bigcup \mathcal{A}_n$. Hence ϕ is a derivation on $\bigcup \mathcal{A}_n$ which is dense in $\mathcal{A}$. As ϕ is continuous we conclude that ϕ is a derivation. □

The automorphism group and hence the set of 2-local *-automorphisms of a C^*-algebra is often difficult to deal with. In the case of UHF C^*-algebras, certain simplifications occur. It should be mentioned that ideas from both [Mol] and [XiLu] are used in the proof of the following theorem.

Theorem 5. *Let $\mathcal{A} = \overline{\bigcup \mathcal{A}_n}$ be a* UHF *C^*-algebra and let $\phi : \mathcal{A} \to \mathcal{A}$ be a 2-local *-automorphism. Then ϕ is an injective *-homomorphism.*

Proof. By [Davi, Corollary IV.5.8], any *-automorphism α of a UHF C^*-algebra $\mathcal{A}$ is approximately inner, that is, there is a sequence α_k of inner automorphisms of $\mathcal{A}$ such that $\alpha(A) = \lim \alpha_k(A)$ for every $A \in \mathcal{A}$. Let $A, B \in \mathcal{A}$. Then there exists a set $\{U_k | k = 1, 2, \ldots\}$ of unitaries in $\mathcal{A}$ such that $\phi(A) = \lim U_kAU_k^*$ and $\phi(B)^* = \lim U_kB^*U_k^*$. Since $\phi(A)\phi(B)^* = \lim U_kAB^*U_k^*$, it follows that

$$\tau(\phi(A)\phi(B)^*) = \tau(AB^*) \ (A, B \in \mathcal{A}),$$

where τ is the faithful trace state of $\mathcal{A}$. Then we have for any $C \in \mathcal{A}$,

$$\tau[(\phi(A+B) - \phi(A) - \phi(B))\phi(C)^*] = \tau[((A+B) - A - B)C^*] = 0$$

by the linearity of τ and we obtain that

$$\tau[(\phi(A+B) - \phi(A) - \phi(B))(\phi(A+B) - \phi(A) - \phi(B))^*] = 0.$$

Consequently, it follows that ϕ is additive, and hence it is linear. Note that from the 2-local property, ϕ is zero-product preserving, that is, $AB = 0$ implies $\phi(A)\phi(B) = 0$. Also ϕ is square preserving, that is, $\phi(A^2) = \phi(A)^2$. Since ϕ is additive, linearizing the equation $\phi(A^2) = \phi(A)^2$, we get

$$\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A)$$

for every $A, B \in \mathcal{A}$. Let $A, B \in \bigcup \mathcal{A}_n$. We may assume that both A and B lie in some $\mathcal{A}_n$. Let $\{E_{ij}\}$ be the system of matrix units of $\mathcal{A}_n$. We show that $\phi(E_{ij}E_{kl}) = \phi(E_{ij})\phi(E_{kl})$ for every i, j, k and l.

Case 1. $j \neq k$. Then $E_{ij}E_{kl} = 0$. So, $\phi(E_{ij}E_{kl}) = \phi(E_{ij})\phi(E_{kl}) = 0$.

Case 2. $j = k, i \neq l$. Then $\phi(E_{ij}E_{jl}) = \phi(E_{ij}E_{jl} + E_{jl}E_{ij}) = \phi(E_{ij})\phi(E_{jl}) + \phi(E_{jl})\phi(E_{ij}) = \phi(E_{ij})\phi(E_{jl})$.

Case 3. $j = k, i = l, i \neq j$. Then we have

$$\phi(E_{ii})+\phi(E_{jj}) = \phi(E_{ii}+E_{jj}) = \phi(E_{ij}E_{ji}+E_{ji}E_{ij}) = \phi(E_{ij})\phi(E_{ji})+\phi(E_{ji})\phi(E_{ij}).$$

If we multiply $\phi(E_{ii})$ from the right, then we have

$$\phi(E_{ii}) = \phi(E_{ij})\phi(E_{ji})\phi(E_{ii}) = \phi(E_{ij})\phi(E_{ji})$$

by Case 2. Hence $\phi(E_{ij}E_{ji}) = \phi(E_{ii}) = \phi(E_{ij})\phi(E_{ji})$.

Case 4. $i = j = k = l$. Then $\phi({E_{ii}}^2) = \phi(E_{ii})^2$.

Now let $A = \sum a_{ij}E_{ij}$ and $B = \sum b_{kl}E_{kl}$. Then

$$\begin{aligned}\phi(A)\phi(B) &= \left[\sum a_{ij}\phi(E_{ij})\right]\left[\sum b_{kl}\phi(E_{kl})\right]\\ &= \sum a_{ij}b_{kl}\phi(E_{ij}E_{kl})\\ &= \sum a_{ij}b_{jl}\phi(E_{il})\\ &= \phi(AB).\end{aligned}$$

This shows that ϕ is multiplicative on $\bigcup \mathcal{A}_n$, which is dense in $\mathcal{A}$. Since ϕ is continuous from the 2-local property, it follows that ϕ is an injective *-homomorphism. □

Recall that a C^*-algebra $\mathcal{A}$ is called prime if whenever $A\mathcal{A}B = \{0\}$ for $A, B \in \mathcal{A}$, then either $A = 0$ or $B = 0$. Equivalently, every pair of nonzero closed ideals of $\mathcal{A}$ has nonempty intersection. For $n \geq 2$, the Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by isometries $S_1, \dots, S_n$ such that $S_1S_1^* + \ldots + S_nS_n^* = I$. The identity element I of a C^*-algebra $\mathcal{A}$ is called properly infinite if there are orthogonal projections P_1 and P_2 such that $I \sim P_1 \sim P_2$ and $P_1 + P_2 \leq I$. If we assume that 2-local *-automorphisms are surjective on some class of C^*-algebras including UHF C^*-algebras, then they are *-automorphisms. More precisely, we have the following theorem.

Theorem 6. *Let $\mathcal{A}$ be a unital prime C^*-algebra that has a nontrivial idempotent or a C^*-algebra such that I is properly infinite. If $\phi : \mathcal{A} \to \mathcal{A}$ is a surjective 2-local *-automorphism, then it is a *-automorphism.*

Proof. We use the result of Mazur and Ulam [MaUl] which states that if $\mathcal{V}$ is a real normed space and $T : \mathcal{V} \to \mathcal{V}$ is a bijective map which preserves the distance on $\mathcal{V}$ (i.e., T satisfies $\|T(X) - T(Y)\| = \|X - Y\|$ $(X, Y \in \mathcal{V})$), then T can be written in the form $T(X) = L(X) + X_0$ $(X \in \mathcal{V})$, where $L : \mathcal{V} \to \mathcal{V}$ is a real linear isometry and X_0 is a fixed element of $\mathcal{V}$. Since $\phi(0) = 0$, ϕ is additive and hence it is linear. As $\phi(A^2) = \phi(A)^2$ for $A \in \mathcal{A}$, it follows that ϕ is a Jordan *-automorphism.

(1). $\mathcal{A}$ is a prime C^*-algebra. Note that any Jordan automorphism onto a prime algebra is either an automorphism or an antiautomorphism by [Her, Theorem 3.1]. We claim that there exist $A_0, B_0 \in \mathcal{A}$ such that $A_0B_0 = 0$ but $B_0A_0 \neq 0$. To do this, assume that the claim is not true and let A, B be elements of $\mathcal{A}$ such that $AB = 0$. Then for any $X \in \mathcal{A}$ we have $ABX = 0$. So, $BXA = 0$. Since $\mathcal{A}$ is prime, we have that either $A = 0$ or $B = 0$. As $E(I - E) = 0$ for any idempotent E in $\mathcal{A}$, it follows that every nonzero idempotent is the identity element, from which the claim follows. Suppose, on the contrary, that ϕ is an antiautomorphism. Taking $A_0, B_0 \in \mathcal{A}$ as in the claim, we have $0 = \phi(A_0)\phi(B_0) = \phi(B_0A_0)$. This contradiction yields the conclusion.

(2). $\mathcal{A}$ is a C^*-algebra acting on $\mathcal{H}$ such that I is properly infinite. We prove this by borrowing the idea of [BaMo]. By [Sto, Theorem 3.3], there is a projection E in the center of the von Neumann algebra generated by $\mathcal{A}$ such that $\phi_1 : A \mapsto \phi(A)E$ is a *-homomorphism, $\phi_2 : A \mapsto \phi(A)(I-E)$ is a *-antihomomorphism and $\phi = \phi_1 + \phi_2$. Since $I \in \mathcal{A}$ is properly infinite, there are projections $P_1, P_2 \in \mathcal{A}$ such that $P_1 \sim I, P_2 \sim I$ and $P_1 + P_2 \leq I$. So, there are isometries U_1, U_2 such that $U_1U_1^* = P_1$ and $U_2U_2^* = P_2$. By the 2-local property, $\phi(U_i)$ $(i = 1, 2)$ are isometries. Hence, $\phi_2(U_i)$ $(i = 1, 2)$ are isometries on $(I - E)\mathcal{H}$, from which it follows that

$$I - E \geq \phi_2(U_1U_1^* + U_2U_2^*) = \phi_2(U_1)^*\phi_2(U_1) + \phi_2(U_2)^*\phi_2(U_2) = 2(I - E).$$

Consequently, $E = I$ and hence ϕ is a *-automorphism. □

Corollary 7. *Let $\mathcal{A}$ be a simple* AF *C^*-algebra or a C^*-algebra containing $\mathcal{O}_n$ as a C^*-subalgebra. If $\phi : \mathcal{A} \to \mathcal{A}$ is a surjective* 2*-local *-automorphism, then ϕ is a *-automorphism.*

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