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A GENERAL MULTIPLICITY THEOREM FOR CERTAIN NONLINEAR EQUATIONS IN HILBERT SPACES

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Abstract. In this paper, we prove the following general result. Let X be a real Hilbert space and $J : X \to \mathbf{R}$ a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that

$$\limsup_{\|x\|\to+\infty} \frac{J(x)}{\|x\|^2} \leq 0 \ .$$

Then, for each $r \in \,]\inf_X J, \sup_X J[$ for which the set $J^{-1}([r,+\infty[)$ is not convex and for each convex set $S \subseteq X$ dense in X, there exist $x_0 \in S \cap J^{-1}(]-\infty, r[)$ and $\lambda > 0$ such that the equation

$$x = \lambda J'(x) + x_0$$

has at least three solutions.

The aim of the present paper is to prove the following result.

Theorem 1. *Let X be a real Hilbert space and let $J : X \to \mathbf{R}$ be a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that*

$$\limsup_{\|x\|\to+\infty} \frac{J(x)}{\|x\|^2} \leq 0 \ . \tag{1}$$

Then, for each $r \in \,]\inf_X J, \sup_X J[$ and each $x_0 \in J^{-1}(]-\infty, r[)$, at least one of the following assertions holds:

(*a*) *There exists $\lambda > 0$ such that the equation*

$$x = \lambda J'(x) + x_0$$

has at least three solutions.

(*b*) *There exists a unique $y \in J^{-1}([r,+\infty[)$ such that*

$$\|x_0 - y\| = \mathrm{dist}(x_0, J^{-1}([r,+\infty[)) = \mathrm{dist}(x_0, J^{-1}(r)) \ .$$

Among the most significant consequences of Theorem 1, there is the general multiplicity theorem announced in the title of the paper. It reads as follows.

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Theorem 2. *Let X be a real Hilbert space and let $J : X \to \mathbf{R}$ be a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that*

$$\limsup_{\|x\|\to+\infty} \frac{J(x)}{\|x\|^2} \leq 0 \ .$$

Then, for each $r \in \,]\inf_X J, \sup_X J[$ for which the set $J^{-1}([r,+\infty[)$ is not convex and for each convex set $S \subseteq X$ dense in X, there exist $x_0 \in S \cap J^{-1}(]-\infty, r[)$ and $\lambda > 0$ such that the equation

$$x = \lambda J'(x) + x_0$$

has at least three solutions.

To derive Theorem 2 from Theorem 1, we use a very recent result by I. G. Tsar'kov ([5]). We state it below in a form which is enough for our purposes.

Theorem A ([5, Corollary 2])**.** *Let X be a real Hilbert space and $C \subset X$ a sequentially weakly closed and nonconvex set.*

Then, for each convex set $S \subseteq X$ dense in X, there exists $x_0 \in S \setminus C$ such that the set $\{y \in C : \|x_0 - y\| = \operatorname{dist}(x_0, C)\}$ has at least two points.

In practice, when $\dim(X) = \infty$, Theorem A is a more precise version of the celebrated, classical result of Efimov and Stechkin on Chebyshev sets ([1]) (see also [6] for a proof based on convex analysis methods).

Now, the way of drawing Theorem 2 from Theorem 1 is transparent. Let us formalize it.

Proof of Theorem 2. Let $r \in \,]\inf_X J, \sup_X J[$ be such that the set $J^{-1}([r,+\infty[)$ is not convex and let $S \subseteq X$ be a convex set dense in X. Since J' is compact, the functional J turns out to be sequentially weakly continuous ([7], Corollary 41.9). Hence, the set $J^{-1}([r,+\infty[)$ is sequentially weakly closed (possibly not weakly closed). Consequently, by Theorem A, there exists $x_0 \in S \cap J^{-1}(]-\infty, r[)$ such that (b) of Theorem 1 does not hold. Hence, (a) of the same theorem holds, and that is the conclusion. □

We are going to prove Theorem 1. For the reader's convenience, we now recall the statements of two well-known results which are the main tools used later. We recall that a real function φ on a convex subset C of a vector space is said to be quasi-concave if, for every $r \in \mathbf{R}$, the set $\{x \in C : \varphi(x) > r\}$ is convex. We also recall that a Gâteaux differentiable functional J on a real Banach space X is said to satisfy the Palais-Smale condition if each sequence $\{x_n\}$ in X such that $\sup_{n\in\mathbf{N}} |J(x_n)| < +\infty$ and $\lim_{n\to+\infty} \|J'(x_n)\|_{X^*} = 0$ admits a strongly converging subsequence.

Theorem B ([4, Theorem 1 and Remark 1])**.** *Let X be a topological space, I a real interval, and $f : X \times I \to \mathbf{R}$ a function satisfying the following conditions:*

(c_1) *for every $x \in X$, the function $f(x, \cdot)$ is quasi-concave and continuous;*

(c_2) *for every $\lambda \in I$, the function $f(\cdot, \lambda)$ is lower semicontinuous and each of its local minima is a global minimum;*

(c_3) *there exist $\rho > \sup_{\lambda\in I} \inf_{x\in X} f(x, \lambda)$ and $\lambda_0 \in I$ such that the set*

$$\{x \in X : f(x, \lambda_0) \leq \rho\}$$

is compact.

Then

$$\sup_{\lambda \in I} \inf_{x \in X} f(x, \lambda) = \inf_{x \in X} \sup_{\lambda \in I} f(x, \lambda) \ .$$

Theorem C ([2, Corollary 1])**.** *Let X be a real Banach space and let $J : X \to \mathbf{R}$ be a continuously Gâteaux differentiable functional satisfying the Palais-Smale condition and having at least two local minima.*

Then J has at least three critical points.

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let $r \in \,]\inf_X J, \sup_X J[$ and $x_0 \in J^{-1}(]-\infty, r[)$. Assume that assertion (a) does not hold. So, let us suppose that, for each $\lambda > 0$, the equation

$$(E_\lambda) \qquad x = \lambda J'(x) + x_0$$

has at most two solutions. Now, define the function $f : X \times [0, +\infty[\to \mathbf{R}$ by setting

$$f(x, \lambda) = \frac{1}{2}\|x - x_0\|^2 + \lambda(r - J(x))$$

for all $(x, \lambda) \in X \times [0, +\infty[$. Let us check that f satisfies the hypotheses of Theorem B, the space X being endowed with the weak topology. It is clear that (c_1) and (c_3) (with $\lambda_0 = 0$) are satisfied. So, fix $\lambda \in [0, +\infty[$. As we have already observed, the functional J is sequentially weakly continuous. Hence, the functional $f(\cdot, \lambda)$ is sequentially weakly lower semicontinuous. Fix $\epsilon > 0$ so that $\frac{1}{2} - \epsilon\lambda > 0$. By (1), there is $\delta > 0$ such that

$$\sup_{\|x\| > \delta} \frac{J(x)}{\|x\|^2} < \epsilon \ .$$

Thus, we have

$$f(x, \lambda) > \left(\frac{1}{2} - \epsilon\lambda\right) \|x\|^2 - \|x_0\|\|x\| + \frac{1}{2}\|x_0\|^2 + \lambda r$$

for all $x \in X$, with $\|x\| > \delta$. Hence, we get

$$\lim_{\|x\| \to +\infty} f(x, \lambda) = +\infty \ .$$

From this, by the reflexivity of X, by the Eberlein-Smulyan theorem and by a classical result ([7], Example 38.25) we infer that $f(\cdot, \lambda)$ is weakly lower semicontinuous, has a global minimum and satisfies the Palais-Smale condition. On the other hand, the critical points of $f(\cdot, \lambda)$ are exactly the solutions of (E_λ). Hence, by assumption, $f(\cdot, \lambda)$ has at most two critical points. Then, thanks to Theorem C, $f(\cdot, \lambda)$ has exactly one global minimum and no other local minimum in the strong topology, and so, *a fortiori*, in the weak topology. Hence, condition (c_2) is satisfied. Therefore, Theorem B ensures that

$$(2) \qquad \sup_{\lambda \geq 0} \inf_{x \in X} f(x, \lambda) = \inf_{x \in X} \sup_{\lambda \geq 0} f(x, \lambda) \ .$$

Clearly, one has

$$(3) \qquad \inf_{x \in X} \sup_{\lambda \geq 0} f(x, \lambda) = \frac{1}{2} \inf_{x \in J^{-1}([r, +\infty[)} \|x - x_0\|^2 \ .$$

Furthermore, observe that, since $J^{-1}([r,+\infty[)$ is sequentially weakly closed, there exists $y \in J^{-1}([r,+\infty[)$ such that

$$\|x_0 - y\| = \mathrm{dist}(x_0, J^{-1}([r,+\infty[)) \ .$$

We claim that $y \in J^{-1}(r)$. Indeed, if $J(y) > r$, since J is continuous and $J(x_0) < r$, there would exist a point z in the line segment joining x_0 and y such that $J(z) = r$. So, we would have $\|x_0 - z\| < \mathrm{dist}(x_0, J^{-1}([r,+\infty[))$, an absurdity. In particular, this implies that

$$\mathrm{dist}(x_0, J^{-1}([r,+\infty[)) = \mathrm{dist}(x_0, J^{-1}(r)) \ .$$

Now, observe that the function $\inf_{x\in X} f(x,\cdot)$ is upper semicontinuous in $[0,+\infty[$ and that $\lim_{\lambda\to+\infty} \inf_{x\in X} f(x,\lambda) = -\infty$, since $r < \sup_X J$. Hence, there is $\lambda^* \geq 0$ such that

$$\inf_{x\in X} f(x,\lambda^*) = \sup_{\lambda\geq 0} \inf_{x\in X} f(x,\lambda) \ .$$

So, from (2) and (3), we get

$$\inf_{x\in X}\left(\frac{1}{2}\|x-x_0\|^2 - \lambda^* J(x)\right) = \inf_{x\in J^{-1}(r)}\left(\frac{1}{2}\|x-x_0\|^2 - \lambda^* J(x)\right) \ .$$

From this, we infer that $\lambda^* > 0$, since $J(x_0) < r$, and that each global minimum of the restriction of the functional $x \to \frac{1}{2}\|x-x_0\|^2 - \lambda^* J(x)$ to $J^{-1}(r)$ is a global minimum of the same functional on X. But, as we have seen above, for each $\lambda > 0$, the functional $x \to \frac{1}{2}\|x-x_0\|^2 - \lambda J(x)$ has exactly one global minimum in X. On the other hand, a point $y \in J^{-1}(r)$ is a global minimum for the restriction of the functional $x \to \frac{1}{2}\|x-x_0\|^2 - \lambda^* J(x)$ to $J^{-1}(r)$ if and only if $\|y - x_0\| = \mathrm{dist}(x_0, J^{-1}(r))$, and so (b) follows. □

Remark 1. The conclusion of Theorem 1 can be false when (1) is not satisfied. To see this, take, for instance, $X = \mathbf{R}$, $J(x) = x^3 - x$, $r = 0$ and $x_0 = \frac{1}{2}$. We also believe that some more sophisticated examples should show that the assumption about the compactness of J' cannot be omitted.

Further, observe that, applying Theorem 1 to both J and $-J$, we get the following result.

Theorem 3. *Let X be a real Hilbert space and $J : X \to \mathbf{R}$ a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that*

$$\lim_{\|x\|\to+\infty} \frac{J(x)}{\|x\|^2} = 0 \ .$$

Then, for each $r \in \,]\inf_X J, \sup_X J[$ and each $x_0 \in X \setminus J^{-1}(r)$, at least one of the following assertions holds:

(i) *There exists $\lambda \in \mathbf{R}$ such that the equation*

$$x = \lambda J'(x) + x_0$$

has at least three solutions.

(ii) *There exists a unique $y \in J^{-1}(r)$ such that*

$$\|x_0 - y\| = \mathrm{dist}(x_0, J^{-1}(r)) \ .$$

Reasoning as in the proof of Theorem 2, we obtain the following consequence of Theorem 3.

Theorem 4. *Let X be a real Hilbert space and $J : X \to \mathbf{R}$ a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that*

$$\lim_{\|x\|\to+\infty} \frac{J(x)}{\|x\|^2} = 0 \ .$$

Then, for each $r \in \,]\inf_X J, \sup_X J[$ for which the set $J^{-1}(r)$ is not convex and for each convex set $S \subseteq X$ dense in X, there exist $x_0 \in S \setminus J^{-1}(r)$ and $\lambda \in \mathbf{R}$ such that the equation

$$x = \lambda J'(x) + x_0$$

has at least three solutions.

We conclude by presenting an application of Theorem 2 to a two-point boundary value problem.

As usual, let $W_0^{1,2}(]0,1[)$ denote the closure of $C_0^\infty(]0,1[)$ in $W^{1,2}(]0,1[)$, the space of all $u \in L^2(]0,1[)$ having distributional derivative $u' \in L^2(]0,1[)$. We consider $W_0^{1,2}(]0,1[)$ endowed with the inner product

$$\langle u, v\rangle = \int_0^1 u'(t)v'(t)dt \ .$$

For each continuous function $f : \mathbf{R} \to \mathbf{R}$, define the functional $J_f : W_0^{1,2}(]0,1[) \to \mathbf{R}$ by putting

$$J_f(u) = \int_0^1 F(u(t))dt$$

for all $u \in W_0^{1,2}(]0,1[)$, where

$$F(\xi) = \int_0^\xi f(s)ds \ .$$

We then have

Theorem 5. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous nonconstant and nondecreasing function satisfying*

$$\lim_{|\xi|\to+\infty} \frac{F(\xi)}{\xi^2} = 0 \ . \tag{4}$$

Then, for each $r \in \,]\inf_{\mathbf{R}} F, \sup_{\mathbf{R}} F[$ and for each convex set $S \subseteq C_0^\infty(]0,1[)$ dense in $W_0^{1,2}(]0,1[)$, there exist $w \in S \cap J_f^{-1}(]-\infty, r[)$ and $\lambda > 0$ such that the problem

$$\begin{cases} -u'' = \lambda f(u) - w''(t) & \text{in } [0,1], \\ u(0) = u(1) = 0 \end{cases}$$

has at least three (classical) solutions.

Proof. By classical results ([3]), J_f is a continuously Gâteaux differentiable functional on $W_0^{1,2}(]0,1[)$ with compact derivative, and one has

$$\langle J_f'(u), v\rangle = \int_0^1 f(u(t))v(t)dt$$

for all $u, v \in W_0^{1,2}(]0,1[)$. Further, from (4), it readily follows that

$$\lim_{\|u\|\to+\infty} \frac{J_f(u)}{\|u\|^2} = 0 \ .$$

So, J_f satisfies the assumption of Theorem 2. Fix $r \in \,]\inf_{\mathbf{R}} F, \sup_{\mathbf{R}} F[$ (note that r is in the interior of the range of J_f (see the argument below)). Now, let us show that the set $J_f^{-1}(r)$ is not convex. First, we note that, for each $a \in \mathbf{R}$, there exists $u \in W_0^{1,2}(]0,1[)$ such that $u\left(\frac{1}{2}\right) = a$ and $\int_0^1 F(u(t))dt = r$. Indeed, set

$$A = \left\{ u \in W_0^{1,2}(]0,1[) : u\left(\frac{1}{2}\right) = a \right\} .$$

Fix r_1, r_2 satisfying $\inf_{\mathbf{R}} F < r_1 < r < r_2 < \sup_{\mathbf{R}} F$, and pick ξ_1, ξ_2 so that $F(\xi_1) = r_1$, $F(\xi_2) = r_2$. Next, choose $\epsilon > 0$ such that

$$r_1(1-4\epsilon) + 4\epsilon \sup_{[-\rho,\rho]} |F| < r < r_2(1-4\epsilon) - 4\epsilon \sup_{[-\rho,\rho]} |F| ,$$

where $\rho = \max\{|\xi_1|, |\xi_2|, |a|\}$. Finally, fix two functions $u_1, u_2 \in A$ so that

$$\max\left\{ \sup_{[0,1]} |u_1|, \sup_{[0,1]} |u_2| \right\} \leq \rho, \ u_1(t) = \xi_1, \ u_2(t) = \xi_2$$

for all $t \in [\epsilon, \frac{1}{2} - \epsilon] \cup [\frac{1}{2} + \epsilon, 1 - \epsilon]$. Then, we have

$$\int_0^1 F(u_1(t))dt \leq r_1(1-4\epsilon) + 4\epsilon \sup_{[-\rho,\rho]} |F| < r$$

as well as

$$\int_0^1 F(u_2(t))dt \geq r_2(1-4\epsilon) - 4\epsilon \sup_{[-\rho,\rho]} |F| > r .$$

Since A is connected (being convex) and the functional $u \to \int_0^1 F(u(t))dt$ is continuous, there is $u \in A$ such that $\int_0^1 F(u(t))dt = r$, as claimed. Now, since f is not constant, we can fix $a, b \in \mathbf{R}$ so that $f(a) \neq f(b)$. According to the previous claim, there are $u, v \in W_0^{1,2}(]0,1[)$ such that $u\left(\frac{1}{2}\right) = a$, $v\left(\frac{1}{2}\right) = b$ and $\int_0^1 F(u(t))dt = \int_0^1 F(v(t))dt = r$. Finally, we claim that, for some $\mu \in \,]0,1[$, we have

$$\int_0^1 F(u(t) + \mu(v(t) - u(t)))dt \neq r .$$

Arguing by contradiction, assume the contrary. Hence, the derivative of the function $\mu \to \int_0^1 F(u(t) + \mu(v(t) - u(t)))dt$ is zero in $[0,1]$. That is,

$$\int_0^1 f(u(t) + \mu(v(t) - u(t)))(v(t) - u(t))dt = 0$$

for all $\mu \in [0,1]$. From this, it clearly follows that

$$\int_0^1 (f(v(t)) - f(u(t)))(v(t) - u(t))dt = 0 .$$

Then, since f is nondecreasing, we infer that

$$(f(v(t)) - f(u(t)))(v(t) - u(t)) = 0$$

for all $t \in [0,1]$. So, since $u\left(\frac{1}{2}\right) \neq v\left(\frac{1}{2}\right)$, we get

$$f\left(u\left(\frac{1}{2}\right)\right) = f\left(v\left(\frac{1}{2}\right)\right) ,$$

a contradiction. Now, observe that J_f is convex, since f is nondecreasing. Consequently, $J^{-1}(]-\infty, r])$ is convex. Then, since $J^{-1}(r)$ is not convex, $J^{-1}([r, +\infty[)$

is not convex either. Now, let $S \subseteq C_0^\infty(]0,1[)$ be a convex set dense in $W_0^{1,2}(]0,1[)$. Theorem 2 ensures the existence of $w \in S \cap J_f^{-1}(]-\infty, r[)$ and $\lambda > 0$ such that the equation

$$v = \lambda J_f'(v) + w$$

has at least three solutions in $W_0^{1,2}(]0,1[)$. Note that v is one of them if and only if

$$\begin{aligned}\int_0^1 v'(t)\omega'(t)dt &= \lambda \int_0^1 f(v(t))\omega(t)dt + \int_0^1 w'(t)\omega'(t)dt \\ &= \int_0^1 (\lambda f(v(t)) - w''(t))\omega(t)dt\end{aligned}$$

for all $\omega \in W_0^{1,2}(]0,1[)$. This clearly implies that $v \in C^2([0,1])$, with

$$-v''(t) = \lambda f(v(t)) - w''(t)$$

for all $t \in [0,1]$. Consequently, the function v is a classical solution of the problem

$$\begin{cases} -u'' = \lambda f(u) - w''(t) & \text{in } [0,1], \\ u(0) = u(1) = 0 \ . \end{cases}$$

So, this problem has at least three solutions, and the proof is complete. □

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