CONTINUOUS SELECTIONS AND FIXED POINTS OF MULTI-VALUED MAPPINGS ON NONCOMPACT OR NONMETRIZABLE SPACES

LAI-JIU LIN, NGAI-CHING WONG, AND ZENN-TSUEN YU

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Abstract. In this paper, we obtain several new continuous selection theorems for multi-valued mappings on completely regular spaces and fixed point theorems for multi-valued maps on nonmetrizable spaces. They, in particular, provide a partial solution of a conjecture of X. Wu.

1. Introduction

In [4, 5], Browder first used a continuous selection theorem to prove the Fan-Browder fixed point theorem. Later, Yannelis and N. D. Prabahakar [17], Ben-El-Mechaiekh [2, 3], Ding, Kim and Tan [8], Horvath [11], Wu [16, 15], Park [12, 13], and many others, established several continuous selection theorems with applications. We note that in all the continuous selection theorems studied by the above authors, the multi-valued maps are defined on a compact or paracompact space. In [18], Yu and Lin studied continuous selections of multi-valued mappings defined on noncompact spaces, but they assume some kind of coercivity conditions instead.

In this paper, we establish a continuous selection theorem for a multi-valued map defined on a completely regular topological space. We do not assume the compactness of its domain.

In the second part of this paper, we discuss collectively fixed points of lower semicontinuous multi-valued maps. Recently, many authors studied fixed point theorems of lower semicontinuous multi-valued maps; see for example [14, 6, 15, 1]. In particular, Wu established the following one.

Theorem 1.1 ([15]). Let \( X \) be a nonempty subset of a Hausdorff locally convex topological vector space, let \( D \) be a nonempty compact metrizable subset of \( X \), and let \( T : X \to 2^D \) be a multi-valued mapping with the following properties:

(a) \( T(x) \) is a nonempty closed convex set for each \( x \) in \( X \);
(b) \( T \) is lower semicontinuous.

Then there exists a point \( \bar{x} \) in \( D \) such that \( \bar{x} \in T(\bar{x}) \).

Wu conjectured in [15] that the conclusion of Theorem 1.1 remains true even if the metrizability condition of \( D \) is dropped. In this paper, we shall use the

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approximately continuous selection theorem of Deutsch and Kenderov [7] (see also [20]) to establish an approximate fixed point theorem for a sub-lower semicontinuous multi-valued map. This gives rise to a partial solution of the conjecture of Wu [15]. We shall also provide a simple proof of a Himmelberg type collectively fixed point theorem. We remark that our results differ from the approximate fixed point theorem recently established by Park [13].

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2. Preliminaries

Let $X$ and $Y$ be topological spaces. A multi-valued map $T : X \to 2^Y$ is a map from $X$ into the power set $2^Y$ of $Y$. Let $T^{-1} : Y \to 2^X$ be defined by the condition that $x \in T^{-1}y$ if and only if $y \in T(x)$. Recall that

(a) $T$ is said to be closed if its graph $G_r(T) = \{(x,y) : x \in X, y \in T(x)\}$ is closed in the product space $X \times Y$;
(b) $T$ is said to be upper semicontinuous (in short, u.s.c.) at $x$ if for every open set $V$ in $Y$ with $T(x) \subseteq V$, there exists a neighborhood $W(x)$ of $x$ such that $T(W(x)) \subseteq V$; $T$ is said to be u.s.c. on $X$ if $T$ is u.s.c. at every point of $X$;
(c) $T$ is said to be lower semicontinuous (in short, l.s.c.) at $x$ if for every open neighborhood $V(y)$ of every $y$ in $T(x)$, there exists a neighborhood $W(x)$ of $x$ such that $T(U) \cap V(y) \neq \emptyset$ for all $u \in W(x)$; $T$ is said to be l.s.c. on $X$ if $T$ is l.s.c. at every point of $X$;
(d) in case $Y$ is a topological linear space, $T$ is said to be sub-lower semicontinuous (see, e.g., [20]) at an $x \in X$ if for each neighborhood $V$ of $0$ in $Y$, there is a $z \in T(x)$ and a neighborhood $U(x)$ of $x$ in $X$ such that $z \in T(y) + V$ for each $y \in U(x)$; $T$ is said to be sub-lower semicontinuous on $X$ if $T$ is sub-lower semicontinuous at every point of $X$. It is plain that if $T$ is lower semicontinuous at $x$, then $T$ is sub-lower semicontinuous at $x$.

The following lemmas are needed in this paper.

**Lemma 2.1** (Deutsch and Kenderov [7]). Let $X$ be a paracompact topological space, let $Y$ be a locally convex topological linear space, and let $F : X \to 2^Y$. Then $F$ is sub-lower semicontinuous if and only if for each neighborhood $V$ of $0$ in $Y$, there is a continuous function $f : X \to Y$ such that $f(x) \in F(x) + V$ for each $x \in X$.

**Lemma 2.2** (Yuan [19]). Let $X$ be a topological space, let $Y$ be a nonempty subset of a topological vector space with a base $\mathcal{B}$ for the zero neighborhoods, and let $F : X \to 2^Y$. For each $V$ in $\mathcal{B}$, define $F_V : X \to 2^Y$ by

$$F_V(x) = (F(x) + V) \cap Y, \quad \forall x \in X.$$ 

Write $\bar{y} \in \overline{F(x)}$ if $(\bar{x}, \bar{y}) \in \overline{G_rF}$. Then for any $\bar{x}$ in $X$ and $\bar{y}$ in $Y$, we have

$$\bar{y} \in \overline{F(x)} \quad \text{whenever} \quad \bar{y} \in \bigcap_{V \in \mathcal{B}} \overline{F_V(x)}.$$ 

**Lemma 2.3** (Himmelberg [10]). Let $X$ be a nonempty convex subset of a locally convex topological vector space. Let $T : X \to 2^X$ be a u.s.c. multi-valued map
with nonempty closed convex values such that \( T(X) = \bigcup_{x \in X} T(x) \) is contained in a compact subset of \( X \). Then there exists an \( \bar{x} \) in \( X \) such that \( \bar{x} \in T(\bar{x}) \).

**Lemma 2.4** (Granas [9]; see also Ding, Kim and Tan [8]). Let \( D \) be a nonempty compact subset of a topological vector space. Then the convex hull \( \text{co} D \) of \( D \) is \( \sigma \)-compact and hence is paracompact.

### 3. Continuous selection theorems

Note that the set \( S^{-1}(y) = \{ x \in X : y \in S(x) \} \) below can have empty interior for some \( y \) in \( K \).

**Theorem 3.1.** Let \( X \) be a completely regular space and let \( K \) be a nonempty subset of a Hausdorff topological vector space \( E \). Assume a multi-valued function \( S : X \rightarrow 2^K \) satisfies the following conditions:

(a) For each \( x \) in \( X \), the set \( S(x) \) is convex.

(b) \( X = \bigcup \{ \text{int} S^{-1}(y) : y \in K \} \).

Then for any compact subset \( F \) of \( X \) there is an open dense subset \( U \) of \( X \) containing \( F \) such that \( S \) has a continuous selection \( f : U \rightarrow K \), that is, \( f(x) \in S(x) \) for all \( x \) in \( U \).

**Proof.** By assumption (b), there are finitely many points \( y_1, \ldots, y_n \) in \( K \) such that

\[
F \subseteq \text{int} S^{-1}(y_1) \cup \cdots \cup \text{int} S^{-1}(y_n).
\]

For each \( k = 1, \ldots, n \) and \( x \) in \( F \cap \text{int} S^{-1}(y_k) \), there is a continuous function \( g_x \) on \( X \) such that \( 0 \leq g_x \leq 1 \), \( g_x(x) = 1 \) and \( g_x \) vanishes outside \( \text{int} S^{-1}(y_k) \). By the compactness of \( F \), there are finitely many \( g_x \) such that for every point in \( F \) at least one of them assumes a value not less than 1/2. Summing them in an appropriate way, we will have nonnegative continuous functions \( g_1, \ldots, g_n \) on \( X \) such that \( g_k \) vanishes outside \( \text{int} S^{-1}(y_k) \), and \( \sum_{k=1}^n g_k(x) \geq 1/2 \) for all \( x \) in \( F \). Let \( V = \{ x \in X : \sum_{k=1}^n g_k(x) > 1/3 \} \). Set \( f_j(x) = g_j(x) / \sum_{k=1}^n g_k(x) \) on \( V \), and \( f_j(x) = 3g_j(x) \) on \( X \setminus V \). Define a continuous function \( f_V : X \rightarrow E \) by

\[
f_V(x) = \sum_{k=1}^n f_k(x) y_k, \quad \forall x \in X.
\]

For each \( x \) in \( V \) and for each \( k \) with \( f_k(x) \neq 0 \), we have \( x \in \text{int} S^{-1}(y_k) \). Hence, \( y_k \in S(x) \). Consequently, \( f_V(x) \in \text{co}(S(x)) = S(x) \subseteq K \) for all \( x \) in \( V \). In other words, the restriction of \( f_V \) to \( V \) gives rise to a continuous selection of \( S \) on the open set \( V \) which contains \( F \).

Denote by

\[
\mathcal{W} = \{ (f_W, W) : \text{where } W \text{ is an open subset of } X \text{ containing } F \text{ and } f_W : W \rightarrow K \text{ gives rise to a continuous selection of } S \text{ on } W \}.
\]

Then \( \mathcal{W} \) is not empty as \( (f_V, V) \in \mathcal{W} \). Order \( \mathcal{W} \) by extension and we get a nonempty partially ordered set. In other words, \( (f_W, W) \leq (f_V, V) \) if \( W \subseteq V \) and \( f_V|_W = f_W \). Applying Zorn’s Lemma, we get a maximal element \( (f_U, U) \) of \( \mathcal{W} \).

The last step is to verify that \( U \) is dense in \( X \). Suppose not and there were an \( x \) in \( X \) outside the closure of \( U \). Let \( x \in \text{int} S^{-1}(y) \) for some \( y \) in \( K \). By setting \( f|_W \equiv y \), we get a continuous selection of \( S \) on an open neighborhood \( W \) of \( x \).
disjoint from $U$ by restriction. Then the union $f_{U \cup W} : U \cup W \rightarrow K$ defined in a
natural way provides a contradiction to the maximality of $(f_U, U)$. \hfill \square

We call a topological space $X$ residually paracompact if for every open dense subset $U$ of $X$ the complement $X \setminus U$ is paracompact.

**Theorem 3.2.** In addition to conditions (a) and (b) in Theorem 3.1, if we assume further that

(c) $X$ is residually paracompact,

then there is a continuous function $f : X \rightarrow K$ such that $f(x) \in S(x)$ for all $x$ in $X$.

**Proof.** It follows from Theorem 3.1 that there is a continuous function $f_U : U \rightarrow K$
defined on an open dense subset $U$ of $X$ with $f_U(x) \in S(x)$ for all $x$ in $U$. For each $z$ in $X \setminus U$, there is a $y$ in $K$ such that $z \in \text{int} S^{-1}(y)$ by condition (b). By setting $f_{W_z} \equiv y$ we get a continuous selection of $S$ on an open neighborhood $W_z$ of $z$. The paracompactness of $X \setminus U$ ensures it has a locally finite covering by open sets in $X$, each of which is contained in some $W_z$. Adding one more open set $U$, we have a locally finite open covering of $X$. This provides us with a family $\{g_\lambda\}_\lambda$ of nonzero continuous functions from $X$ into $[0, 1]$ dominated by the open sets $W_z$ and $U$ such that $g_\lambda(x) = 0$ for all but finitely many $\lambda$'s and $\sum \lambda g_\lambda(x) = 1$ for all $x$ in $X$. If $g_\lambda$ vanishes outside $U$, we set $f_\lambda = f_U$. Otherwise, we fix a choice of $z$ such that $g_\lambda$ vanishes outside $W_z$, and set $f_\lambda = f_{W_z}$. Define $f : X \rightarrow K$ by

$$f(x) = \sum \lambda g_\lambda(x)f_\lambda(x), \quad \forall x \in X.$$ 

For each $x$ in $X$, only finitely many $g_\lambda(x)$'s are nonzero in the sum, and the nonzero terms give rise to a convex combination of points in the convex set $S(x)$. Thus $f(x) \in S(x)$ for all $x$ in $X$. \hfill \square

It is easy to see that the following corollary follows from Theorem 3.1.

**Corollary 3.3.** The conclusion of Theorem 3.1 remains true if conditions (a) and (b) are replaced by

(a)' for each $x$ in $X$, the set $S(x)$ is a nonempty convex set;

(b)' for each $y$ in $K$, the set $S^{-1}(y)$ is open.

**Remark 3.4.** Corollary 3.3 implies Theorem 3.1

**Proof.** Let $T : X \rightarrow 2^K$ be defined by

$$T(x) = \{y \in K : x \in \text{int} S^{-1}(y)\}.$$ 

Then $T^{-1}(y) = \text{int} S^{-1}(y)$ is open for each $y$ in $K$. By (b), for each $x$ in $X$, there exists $y$ in $K$ such that $x \in \text{int} S^{-1}(y)$. Therefore $y \in T(x) \neq \emptyset$ for each $x$ in $X$. Let $H : X \rightarrow 2^K$ be defined by $H(x) = \text{co} T(x)$. Then $H(x)$ is nonempty for each $x$ in $X$, and $H^{-1}(y)$ is open for each $y$ in $K$. By Corollary 3.3, there is an open dense subset $U$ of $X$, containing any but a fixed compact set $D$, and there is a continuous function $f : U \rightarrow K$ such that $f(x) \in H(x) = \text{co} T(x) \subset S(x)$ for all $x$ in $U$. \hfill \square
4. Fixed point theorems

**Theorem 4.1.** For each $i$ in a nonempty index set $I$, let $X_i$ be a nonempty convex subset of a Hausdorff locally convex topological vector space $E_i$, and let $D_i$ be a compact subset of $X_i$. Let $X = \prod_{i \in I} X_i$ be the product space. Let $F_i : X \to 2^{D_i}$ be sub-lower semicontinuous with nonempty convex values. Then for every neighborhood $V_i$ of 0 in $E_i$, there exists a point $\bar{x}_V = (x_{V_i})$ in $D = \prod_{i \in I} D_i$ such that $(\bar{x}_V + V_i) \cap F_i(\bar{x}_V) \neq \emptyset$ for all $i$ in $I$.

**Proof.** Let $V_i$ be a neighborhood of zero in $E_i$ for each $i$ in $I$. Fix any $i$ in $I$. There exists an absolutely convex neighborhood $W_i$ of 0 such that $W_i \subset V_i$. Note that $D$ is a compact subset of $X$. By Lemma 2.3, $co D$ is a paracompact subset of $X$. Since $F_i : X \to 2^{D_i}$ is a sub-lower semicontinuous multi-valued map with nonempty convex values, by Lemma 2.1 there exists a continuous function $f_i : co D \to D_i$ such that

$$f_i(x) \in (F_i(x) + W_i) \cap D_i \quad \text{for each } x \in co D.$$ 

Define $f : co D \to D$ by $f(x) = \prod_{i \in I} f_i(x)$ for $x$ in $co D$. By the Himmelberg fixed point theorem (Lemma 2.3), there exists an $\bar{x}_V = (\bar{x}_{V_i})_{i \in I}$ in $co D$ such that $\bar{x}_V = f(\bar{x}_V) = \prod_{i \in I} f_i(\bar{x}_{V_i})$. That is, $\bar{x}_{V_i} = F_i(\bar{x}_V) \in (F_i(\bar{x}_V) + W_i) \cap D_i$. Thus, $\bar{x}_V \in D_i$ and $(\bar{x}_V + W_i) \cap F_i(\bar{x}_V) \neq \emptyset$ for all $i$ in $I$. Since $W_i \subset V_i$, we have $(\bar{x}_i + V_i) \cap F_i(\bar{x}_i) \neq \emptyset$ for all $i$ in $I$. $\square$

**Theorem 4.2.** Suppose in Theorem 4.1 we assume further that for each $x = (\bar{x}_i)_{i \in I} \in X$, its coordinates $x_i \in \overline{F_i(x)} \setminus F_i(x)$ for all $i$ in $I$. Then there exists a point $\bar{x} = (\bar{x}_i)_{i \in I} \in D = \prod_{i \in I} D_i$ such that $\bar{x}_i \in F_i(\bar{x})$ for each $i$ in $I$.

**Proof.** For each $i$ in $I$, let $B_i$ be the collection of all absolutely convex open neighborhoods of zero in $E_i$ and $B = \bigcap_{i \in I} B_i$. Given any $V = \prod_{i \in I} V_i$ in $B$, let $Q_V = \{x \in D : x_i \in \overline{F_i(x)} \setminus F_i(x) \text{ for all } i \text{ in } I\}$. Then $Q_V$ is a nonempty closed subset of $D$ for each $V$ in $B$ by Theorem 4.1. Let $\{V^{(1)}, \ldots, V^{(n)}\}$ be any finite subset of $B$. Write $V^{(i)} = \prod_{j \in I} V_j^{(i)}$, where $V_j^{(i)} \subset B_j$ for each $i = 1, \ldots, n$. Let $V = \prod_{i \in I} \left(\bigcap_{j = 1}^n V_j^{(i)}\right) \in B$. Clearly, $\emptyset \neq Q_V \subseteq \bigcap_{i = 1}^n Q_{V^{(i)}}$.

Since $Q_V \subset D$ and $D$ is compact, $\bigcap_{V \in B} Q_V \neq \emptyset$. Let $\bar{x} = (\bar{x}_i)_{i \in I} \in \bigcap_{V \in B} Q_V$. Then $\bar{x}_i \in \overline{F_i(\bar{x})}$ for all $i$ in $I$ and all $V_i$ in $B$, i.e., $\bar{x}_i \in \bigcap_{V_i \in B_i} \overline{F_i(x)}$ for all $i$ in $I$. It follows from Lemma 2.2 that $\bar{x}_i \in \overline{F_i(x)}$ for all $i$ in $I$. By assumption, $\bar{x}_i \in F_i(\bar{x})$ for all $i$ in $I$. $\square$

We remark that if $F_i$ is closed, then $x_i \not\in \overline{F_i(x)} \setminus F_i(x)$ for each $x = (x_i)_{i \in I}$ in $X$. As a special case of Theorem 4.2, we have the following collectively Himmelberg type fixed point theorem.

**Corollary 4.3.** For each $i$ in a nonempty index set $I$, let $X_i$ be a nonempty convex subset of a locally convex topological vector space $E_i$, let $D_i$ be a nonempty compact subset of $X_i$, and let $f_i : X = \prod_{i \in I} X_i \to D_i$ be a continuous function. Then there exists $\bar{x} = (\bar{x}_i)_{i \in I} \in D = \prod_{i \in I} D_i$ such that $\bar{x}_i = f_i(\bar{x})$ for each $i$ in $I$.

If the index set $I$ is a singleton, then Theorem 4.2 reduces to the following corollary, which provides a partial solution to a conjecture of Wu.

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Corollary 4.4. Let $X$ be a nonempty convex subset of a locally convex topological vector space $E$, let $D$ be a nonempty compact subset of $X$, and let $F : X \to 2^D$ be sub-lower semicontinuous with nonempty convex values. Suppose $x \notin F(x) \setminus x$ for each $x$ in $X$. Then there exists a point $\bar{x}$ in $D$ such that $\bar{x} \in F(\bar{x})$.

By Theorem 4.1 we have the following almost fixed point theorem.

Corollary 4.5. The conclusions of Theorems 4.1 and 4.2 remain valid if the condition "$F_i : X \to 2^{D_i}$ is sub-lower semicontinuous for each $i$ in $I$" is replaced by "$F_i^{-1}(y)$ is open for each $y$ in $D_i$ and each $i$ in $I$.”

Finally we remark that in case $I$ is a singleton, Theorem 4.1 provides a different result from [12, Theorem 3].

References


Department of Mathematics, National Changhua University of Education, Changhua, 50058, Taiwan
E-mail address: maljlin@math.ncue.edu.tw

Department of Applied Mathematics, National Sun Yat-sen University, and National Center for Theoretical Sciences, Kaohsiung, 80424, Taiwan
E-mail address: wong@math.nsysu.edu.tw

Department of Electrical Engineering, Nan-Kai Institute of Technology, Nantour 542, Taiwan