PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 11, Pages 3411–3420
S 0002-9939(05)08265-1
Article electronically published on June 20, 2005

# ASYMPTOTIC DIRICHLET PROBLEM FOR THE SCHRÖDINGER OPERATOR VIA ROUGH ISOMETRY

YONG HAH LEE

(Communicated by Bennett Chow)

Abstract. We pose and solve the asymptotic Dirichlet problem for the Schrödinger operator via rough isometries on a certain class of Riemannian manifolds. With suitable potentials, we give the solvability of the problem for a naturally defined class of data functions.

## 1. Introduction

The asymptotic Dirichlet problem for an elliptic operator on a complete Riemannian manifold has long been an interesting topic of study to analysts and geometers. Let $\mathcal{L}$ be an elliptic operator on a Cartan-Hadamard manifold $M$, which means a complete simply connected manifold with nonpositive sectional curvature, and $f$ be a function in a class $\mathcal{F}$ on the asymptotic boundary $M(\infty)$ of the manifold. The asymptotic Dirichlet problem is to find a solution $u$ for the elliptic operator $\mathcal{L}$ such that $u = f$ on $M(\infty)$. For instance, the usual asymptotic Dirichlet problem for harmonic functions is the case when $\mathcal{L}$ is the Laplacian $\Delta$ on a Cartan-Hadamard manifold $M$ and $\mathcal{F}$ is the class of continuous functions on $M(\infty)$. The asymptotic Dirichlet problem is well understood in the works of Choi [5], Anderson [1], Sullivan [12], Schoen [2] and others, when $M$ is a Cartan-Hadamard manifold with sectional curvature restriction such as the pinching condition $-b^2 \leq K_M \leq -a^2 < 0$. The essence of all of their works is that the curvature assumption enables one to control the angle via the Toponogov comparison theorem and the convexity property near the asymptotic boundary. In [11], Schoen and Yau gave an interesting generalization that does not directly involve the curvature bound as follows: Let $(M, ds^2)$ be a Cartan-Hadamard manifold with sectional curvature $-b^2 \leq K_M \leq -a^2 < 0$, and let $d\tilde{s}^2$ be a new Riemannian metric on $M$ being uniformly equivalent to $ds^2$. If $(M, d\tilde{s}^2)$ has the bounded sectional curvature and the positive injectivity radius, then for any continuous function $f$ on $M(\infty)$, there exists a harmonic function $u$ on $(M, d\tilde{s}^2)$ such that $u = f$ on $M(\infty)$, where $M(\infty)$ denotes the set of the asymptotic classes of geodesic rays. Later, Cheng [4] proved the solvability of the asymptotic Dirichlet problem under more general assumptions than those of Schoen and Yau [11] as follows: Let $(M, ds^2)$ be a Cartan-Hadamard manifold with the first

Received by the editors September 25, 2001.
2000 *Mathematics Subject Classification.* Primary 58J05; Secondary 35J10.
*Key words and phrases.* Asymptotic Dirichlet problem, Schrödinger operator, rough isometry.
This work was supported by Korea Research Foundation Grant (KRF-2002-070-C00010).

Dirichlet eigenvalue $\lambda_1(M) > 0$. Assume that the local pinching condition holds on $(M, ds^2)$, i.e., there exist a point $o \in M$ and a constant $C \geq 1$ such that at any $x \in (M, ds^2)$,

$$|K(\sigma)| \leq C|K(\sigma')|, \tag{1.1}$$

where $\sigma$, $\sigma'$ are plane sections at $x$ containing the tangent vector of the geodesic joining $o$ to $x$ and $K(\sigma)$, $K(\sigma')$ are the sectional curvatures of the plane sections $\sigma$, $\sigma'$, respectively. Then the asymptotic Dirichlet problem on $(M, d\tilde{s}^s)$ is solvable, whenever a new metric $d\tilde{s}^2$ on $M$ is uniformly equivalent to $ds^2$. Recently, Choi, Kim and the present author [6] developed a new approach for a larger class of complete Riemannian manifolds. Let $M$ be a Cartan-Hadamard manifold satisfying the local volume-doubling condition and the local Poincaré inequality, and let $N$ be a complete Riemannian manifold satisfying the local volume-doubling condition. Suppose the first Dirichlet eigenvalue $\lambda_1(M) > 0$ and the local pinching condition (1.1) holds on $M$. Let $\varphi : M \to N$ be a rough isometry, expounded below. Then for each $f \in \mathcal{F}_\varphi$, there exists a solution $h \in C^\infty(N)$ such that

$$\begin{cases} \Delta h &= 0 \quad \text{on} \quad N; \\ (h - f)(\varphi(x)) &\to 0 \quad \text{as} \quad x \to \infty, \end{cases} \tag{1.2}$$

where $\mathcal{F}_\varphi$ denotes a function class on $N$ whose element $f$ satisfies the following conditions:

(i) $f \circ \varphi$ can be extended to $M \cup M(\infty)$ in such a way that $f \circ \varphi$ is continuous at every point of $M(\infty)$;

(ii) for given $\epsilon > 0$, there exists $T > 0$ such that $|f(x) - f(y)| < \epsilon$ if $y \in B_c(x)$ and $d(o, x) \geq T$ for some fixed point $o \in N$, where $c$ depends only on the rough isometry $\varphi$ and the geometry of $M$ and $N$.

In the above, statement (i) means that for any $v \in M(\infty)$, $f \circ \varphi(x)$ converges to a number $A_v$ as $x$ converges to $v$, i.e., for any $\epsilon > 0$, there exists a truncated cone neighborhood $K(v, \delta, R) = \{x \in M : \angle_o(v, \overrightarrow{ox}) < \delta,\ d(o, x) > R\}$ of $v$ such that $|f(x) - A_v| < \epsilon$ whenever $x \in K(v, \delta, R)$, where $\angle_o(v, \overrightarrow{ox})$ denotes the angle at $o$ between $v$ and the ray starting from $o$ and passing through $x$. In fact, both (i) and (ii) are valid in the case when $N$ itself is a Cartan-Hadamard manifold and $f$ is a continuous function on $N(\infty)$. It has to be emphasized that even the topology of $N$ differs from that of $M$ not to mention the nonpositivity of the curvature. So, as in the case of Schoen and Yau and of Cheng, there is no clear way in defining $N(\infty)$, the asymptotic boundary of $N$. This therefore forces one to state the continuity of data at the boundary in terms of $M(\infty)$ as above.

In line with the approach of [6], we solve the asymptotic Dirichlet problem for the Schrödinger operator in the following setting: We say that a function $u$ on a complete Riemannian manifold $M$ is a solution of the Schrödinger equation if $u$ satisfies the equation

$$(\Delta - V)u = 0$$

in the distribution sense, where $V$ is a nonnegative function on $M$. For the sake of convenience, throughout this paper, we assume that every potential $V$ is continuous. From this assumption, one can achieve the continuity of solutions of the Schrödinger equation. More generally, such a result can be extended to potentials in the local Kato class. (See [8].) In the above setting, we get the following result.

**Theorem 1.1.** *Let $M$ be a Cartan-Hadamard manifold satisfying the local volume-doubling condition and the local Poincaré inequality, and let $N$ be an $n$-dimensional complete Riemannian manifold satisfying the local volume-doubling condition. Suppose that the first Dirichlet eigenvalue $\lambda_1(M) > 0$ and $M$ satisfies the local pinching condition* (1)*. Let $\varphi : M \to N$ be a rough isometry. Then for each $f \in \mathcal{F}_\varphi$, there exists a unique solution $\tilde{u}$ on $N$ such that*

$$\begin{cases} (\Delta - V)\tilde{u} &= 0 \quad \text{on } N; \\ (\tilde{u} - f)(z) &\to 0 \quad \text{as } z \to \infty, \end{cases} \tag{1.3}$$

*where $V$ is a nonnegative function in $L^q(N)$ for any $n \le q < \infty$.*

In contrast to the result in [6], we obtain the uniqueness of the solution. Furthermore, the convergence in (1.3) also implies the convergence via a rough isometry as in (1.2); hence this result improves the result of [6] even in the case when $V \equiv 0$.

An important concept that is used in this paper is the rough isometry, which is a more general one than the bi-Lipschitz map. A map $\varphi : X \to Y$, not necessarily continuous, between two metric spaces $X$ and $Y$ is called a rough isometry if the following condition holds:

($R$) for some $\kappa > 0$, the $\kappa$-neighborhood of the image $\varphi(X)$ covers $Y$; there exist constants $a \ge 1$ and $b \ge 0$ such that for any $x_1, x_2$ in $X$,

$$a^{-1}d(x_1, x_2) - b \le d(\varphi(x_1), \varphi(x_2)) \le ad(x_1, x_2) + b,$$

where $d$ denotes the distances of $X$ and $Y$ induced from their metrics, respectively.

In particular, being rough isometric is an equivalence relation. (See [10].) However, since $\varphi$ is not assumed to be continuous, two roughly isometric metric spaces may have completely different topologies. For example, an infinite cylinder is roughly isometric to an infinite cylinder with infinitely many identical handles attached at equal distance going off to infinity. Because of the crudity of rough isometries, we need to add some local conditions on each manifold in order to deploy our theory via rough isometries between manifolds as follows: When we say that a map $\varphi : M \to N$ is a rough isometry between complete Riemannian manifolds $M$ and $N$, it means that $\varphi$ satisfies the following local volume comparison condition, together with ($R$):

($C$) there exists a constant $C \ge 1$ such that for any $x$ in $M$,

$$C^{-1}\mathrm{vol}B_1(x) \le \mathrm{vol}B_1(\varphi(x)) \le C\mathrm{vol}B_1(x);$$

each of $M$ and $N$ satisfies the local volume-doubling condition:

($D$) there exists a constant $C_r < \infty$ depending only on $r > 0$ such that for any $x$ in $M$ (in $N$, respectively)

$$\mathrm{vol}B_{2r}(x) \le C_r\mathrm{vol}B_r(x);$$

and the following local Poincaré inequality holds on $M$:

($P$) there exists a constant $C_r < \infty$ depending only on $r > 0$ such that for any $x$ in $M$ and for any $f \in C^\infty(B_r(x))$,

$$\int_{B_r(x)} |f - \overline{f}| \le C_r \int_{B_r(x)} |\nabla f|,$$

where $\overline{f} = (\mathrm{vol}B_r(x))^{-1} \int_{B_r(x)} f$.

Note that these local conditions are valid on a complete Riemannian manifold with the Ricci curvature bounded below and the positive injectivity radius. (See [3], [10] and [7].)

Another concept needed in this paper is the net, which is a useful tool in combinatorially approximating a Riemannian manifold. Let $d$ be the distance function on $M$. A subset $P$ of $M$ is called $\tau$-separated for some $\tau > 0$ if $d(p, p') \geq \tau$ for any distinct points $p$ and $p'$ of $P$. Let $P$ be a maximal, with respect to the order relation of inclusion, $\tau$-separated subset of $M$. Then we can define a net structure $\mathcal{N} = \{N_p : p \in P\}$ by setting $N_p = \{p' \in P : \tau \leq d(p, p') < 3\tau\}$. In particular, a maximal $\tau$-separated subset $P$ of $M$ with the net structure is called the $\tau$-net in $M$. A net $P$ is said to be uniform if there exists a constant $\lambda$ such that $\sharp N_p \leq \lambda < \infty$ for all $p \in P$, where $\sharp S$ denotes the cardinality of the set $S$. The local volume-doubling condition $(D)$ guarantees that a $\tau$-net $P$ on $M$ is uniform, and this uniformness plays a crucial role in proving the rough isometric invariance of some analytic properties.

## 2. Main result

Let $f \in \mathcal{F}_\varphi$. Then our problem is to find a solution $\tilde{u}$ for a given Schrödinger operator $\Delta - V$ on $N$ such that $(\tilde{u} - f)(z) \to 0$ as $z \to \infty$. By the result of [6], there exists a harmonic function $h$ such that $(h - f)(\varphi(x)) \to 0$ as $x \to \infty$. We will substitute the data $f$ with the harmonic function $h$, i.e., we find a solution $\tilde{u}$ such that $(\tilde{u} - h)(z) \to 0$ as $z \to \infty$. Then we have only to prove that $(h - f)(z) \to 0$ as $z \to \infty$.

Let us introduce several functions needed in this section. For each $f \in \mathcal{F}_\varphi$, $f \circ \varphi$ can be extended to a function $\tilde{f} \in C^\infty(M) \cap C^0(M \cup M(\infty))$ in such a way that $\tilde{f}|_{M(\infty)} = f \circ \varphi|_{M(\infty)}$. We may define $\tilde{f}$ to be radially constant outside a compact subset of $M$. On the other hand, the local pinching condition (1) imposes that $|\nabla \tilde{f}| \in L^s(M)$ for sufficiently large $s \geq 2$. (See Theorem 3.1 of [4].) Let $P$ and $Q$ be the $\tau$-net and $\nu$-net of $M$ and $N$, respectively. Define a function $\tilde{f}_\tau$ on $P$ by

$$\tilde{f}_\tau(p) = \Big( \frac{1}{\mathrm{vol} B_{4\tau}(p)} \int_{B_{4\tau}(p)} \tilde{f}^s \Big)^{1/s}$$

for $p \in P$. Define a new function $g : N \to \mathbb{R}$ by $g(x) = \sum_{q \in Q} (\tilde{f}_\tau \circ \varphi^{-1})(q) \eta_q(x)$, where $\varphi^{-1} : Q \to P$ is an inverse rough isometry of $\varphi$ such that $d((\varphi \circ \varphi^{-1})(q), q) \leq \kappa$ for each $q \in Q$, and $\eta_q(x)$ is a partition of unity defined as follows: Let $\xi_q$ be a Lipschitz function given by

$$\xi_q(x) = \begin{cases} 1 - \frac{2}{3\nu} d(x, q), & x \in B_{3\nu/2}(q); \\ 0\,, & \text{otherwise.} \end{cases}$$

Define $\eta_q(x) = \xi_q(x) / \sum_{q' \in Q} \xi_{q'}(x)$. It is easy to check that there exist constants $C_1 > 0$ and $C_2 > 0$ such that $\sum_{q' \in Q} \xi_{q'}(x) \geq C_1 > 0$ and $|\nabla \eta_q|(x) \leq C_2$ for all $x \in N$ and for all $q \in Q$, where $\nabla \eta_q$ is a weak derivative.

By using the local condition $(P)$ and the uniformness of nets, we can compare the integral of the gradient of the newly defined function $g$ with that of the surrogate function $\tilde{f}$ on $M$ as follows:

**Lemma 2.1.** *For sufficiently large $s \geq 2$, we have the following:*

$$\int_N |\nabla g|^s \, \leq C \int_M |\nabla \tilde{f}|^s \, < \, \infty.$$

By [6], one may obtain the following additional results:

$$\int_N |h-g|^s \;\le\; C\int_N |\nabla g|^s \quad \text{and} \quad \sup_N |h| \;\le\; \sup_N |g| \,,$$

where $C$ depends only on $s$ and $\lambda_1(N)$.

For technical reasons and convenience' sake, we reset our problem as follows: For a given data $f \in \mathcal{F}_\varphi$, there exist nonnegative functions $f_+, f_- \in \mathcal{F}_\varphi$ such that $f = f_+ - f_-$. Thus there exist harmonic functions $h_1, h_2$ on $N$ such that $(h_1 - f_+)(\varphi(x)) \to 0$ and $(h_2 - f_-)(\varphi(x)) \to 0$ as $x \to \infty$. If there exist solutions $\tilde{u}_1$ and $\tilde{u}_2$ for the Schrödinger operator $\Delta - V$ on $N$ such that $(\tilde{u}_1 - h_1)(z) \to 0$ and $(\tilde{u}_2 - h_2)(z) \to 0$, respectively, as $z \to \infty$, then $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$ is the desired one. Therefore, we have only to solve the problem for the nonnegative data $f \in \mathcal{F}_\varphi$, hence a positive harmonic function $h$ on $N$ such that $(h - f)(\varphi(x)) \to 0$ as $x \to \infty$.

We solve the following Dirichlet problem, in the weak sense, on $B_R(o)$:

$$\begin{cases} \Delta u_R - V u_R &= -Vh \quad \text{on } B_R(o); \\ u_R &= 0 \quad\;\;\;\;\;\, \text{on } \partial B_R(o)\,, \end{cases} \tag{2.1}$$

where $V$ is a nonnegative function on $N$. Define a functional $E$ by

$$E(v) = \int_{B_R(o)} \frac{1}{2}\,|\nabla v|^2 + \frac{1}{2}\,V v^2 - Vhv$$

for all $v \in H_0^{1,2}(B_R(o))$. Since

$$E(v) \;\ge \int_{B_R(o)} -\frac{1}{2}\,Vh^2,$$

we can take a minimizer $u_R$ of this functional $E(v)$, whenever $V \in L^q_{loc}(N)$ for $1 \le q \le \infty$. For this solution $u_R$, using the maximum principle, we have $0 < u_R \le h$ on $B_R(o)$, whenever $V \not\equiv 0$, and the following result:

**Lemma 2.2.** *Let $N$ be a complete Riemannian manifold with the first Dirichlet eigenvalue $\mu_1(N) > 0$ for the Schrödinger operator $\Delta - V$, i.e.,*

$$\inf_{v \in C_0^\infty(N)} \frac{\int_N |\nabla v|^2 + V v^2}{\int_N v^2} > 0.$$

*Then there exists a constant $C > 0$ such that for the solution $u_R$ of equation* (2.1) *and for any $1 < q < \infty$,*

$$\int_{B_R(o)} u_R^q \;\le\; C \int_{B_R(o)} V^q \,,$$

*where $C$ depends only on $q$, $\mu_1(N)$, and* $\sup_N |h|$.

*Proof.* Put $u = u_R$. Since $u|_{\partial B_R(o)} = 0$, we have

$$\int_{B_R(o)} \nabla u^{q-1} \cdot \nabla u \;+\; \int_{B_R(o)} V u^q \;= \int_{B_R(o)} u^{q-1} V h;$$

hence

$$(q-1)\int_{B_R(o)} u^{q-2} |\nabla u|^2 \;+\; \int_{B_R(o)} V u^q \;= \int_{B_R(o)} u^{q-1} V h.$$

From $|\nabla u^{q/2}|^2 = (q/2)^2 u^{q-2}|\nabla u|^2$, we get

$$\begin{aligned}\frac{4(q-1)}{q^2}\,\mu_1(N)\int_{B_R(o)} u^q &\leq \frac{4(q-1)}{q^2}\int_{B_R(o)} |\nabla u^{q/2}|^2 + \int_{B_R(o)} V u^q \\ &= \int_{B_R(o)} u^{q-1} V h.\end{aligned}$$

By the Hölder inequality, we have the consequence. □

We have a monotone decreasing sequence of solutions $\{\tilde{u}_R\}$ such that $(\Delta - V)\tilde{u}_R = 0$ on $B_R(o)$ and $\tilde{u}_R = h$ on $\partial B_R(o)$. Therefore, there exists a limit function $\tilde{u}$ such that $(\Delta - V)\tilde{u} = 0$ on $N$. Put $u = h - \tilde{u}$. Then $u$ is a solution of the following equation:

$$(\Delta - V)u = -Vh \quad \text{on} \quad N. \tag{2.2}$$

Applying the standard Moser iteration, we get the following lemma:

**Lemma 2.3.** *Let $N$ be given in Lemma 2.2 with the Sobolev constant $S_1(N) > 0$. Suppose that $u$ is given above and $V$ is a nonnegative function in $L^q(N)$ where $n \leq q < \infty$. Then $u(z) \to 0$ as $z \to \infty$.*

*Proof.* Let $\eta$ be a Lipschitz function with compact support in a ball $B_{r_0}(z)$ for some fixed $r_0 > 0$. Multiplying $\eta^2 u^{p-1}$ on both sides of (2.2), where $p > 1$, and integrating by parts, we get

$$\begin{aligned}\int \eta^2 u^{p-2}|\nabla u|^2 &\leq \frac{2}{p-1}\int \eta u^{p-1}|\nabla u||\nabla\eta| + \frac{1}{p-1}\int \eta^2 u^{p-1} V h \\ &\leq \frac{1}{2}\int \eta^2 u^{p-2}|\nabla u|^2 + \frac{2}{(p-1)^2}\int u^p|\nabla\eta|^2 + \frac{1}{p-1}\int \eta^2 u^{p-1} V h.\end{aligned}$$

Thus

$$\int \eta^2 |\nabla u^{p/2}|^2 \leq \frac{p^2}{(p-1)^2}\int u^p|\nabla\eta|^2 + \frac{p^2}{2(p-1)}\int \eta^2 u^{p-1} V h\,.$$

On the other hand, since $\alpha|\nabla(\eta^2 u^p)| \leq 2\alpha^2\eta^2 u^p + u^p|\nabla\eta|^2 + \eta^2|\nabla u^{p/2}|^2$ for any $\alpha > 0$, we have

$$\begin{aligned}\alpha\int |\nabla(\eta^2 u^p)| &\leq 2\alpha^2\int \eta^2 u^p + \int u^p|\nabla\eta|^2 + \int \eta^2|\nabla u^{p/2}|^2 \\ &\leq 2\alpha^2\int \eta^2 u^p + 2p^2\int u^p|\nabla\eta|^2 + p^2\int \eta^2 u^{p-1} V h\,.\end{aligned}$$

Now choose a Lipschitz function $\eta$ given by

$$\eta(x) = \begin{cases} 1\,, & x \in B_r(z), \\ (s - d(x,z))/(s-r), & x \in B_s(z) \setminus B_r(z), \\ 0\,, & \text{otherwise}, \end{cases}$$

where $0 < r < s < r_0$. Set $\alpha = 1/(s-r)$. Then

$$S_1\Big(\int_{B_r(z)} u^{p\theta}\Big)^{1/\theta} \leq \frac{4p^2}{s-r}\int_{B_s(z)} u^p + p^2\int_{B_s(z)} u^{p-1} V h\,,$$

where $S_1 = S_1(N)$ and $\theta = n/(n-1)$.

For the sake of convenience, we will use the following norm notation: For a fixed point $z \in N$, put

$$||f||_{p,r} = \Big(\int_{B_r(z)} |f|^p\Big)^{1/p} \quad \text{and} \quad ||f||_{\infty,r} = \sup_{B_r(z)} |f|,$$

where $1 < p < \infty$ and $0 < r < r_0$. With this notation, we have

$$S_1\, ||u||^p_{p\theta,r} \le \frac{4p^2}{s-r}\, ||u||^p_{p,s} + p^2 \sup_N |h|\ ||V||_{q,s}\, ||u||^{p-1}_{(p-1)\mu,s}\,, \tag{2.3}$$

where $\mu = q/(q-1)$. Note that $p \le (p-1)\mu < p\theta$ for any $p \ge q > 1$.

Applying (2.3) to each $p \ge q$, we get

$$\begin{aligned} & S_1\, ||u||^p_{p\theta,r} \\ \le\ & \frac{4p^2}{s-r}\,\mathrm{vol}B_s(z)^{(p-q)/((p-1)q)}\, ||u||^{\,p}_{(p-1)\mu,s} + p^2 \sup_N |h|\ ||V||_{q,s}\, ||u||^{\,p-1}_{(p-1)\mu,s} \\ \le\ & \Big\{\frac{4p^2}{s-r} \sup_N |u|\, \mathrm{vol}B_s(z)^{1/q} + p^2 \sup_N |h|\ ||V||_{q,s}\Big\}\, ||u||^{\,p-1}_{(p-1)\mu,s} \\ \le\ & \frac{C_1p^2}{s-r}\,\mathrm{vol}B_s(z)^{1/q}\, ||u||^{\,p-1}_{(p-1)\mu,s}\,, \end{aligned}$$

where $C_1$ depends only on $\sup_N |h|$, $\sup_N |u|$ and $||V||_{q,s}$. Hence

$$||u||_{p\theta,r} \le \Big(\frac{C_2p^2}{s-r}\Big)^{1/p} ||u||^{1-1/p}_{(p-1)\mu,s}\ , \tag{2.4}$$

where $C_2 = C_1\mathrm{vol}B_s(z)^{1/q}/S_1$.

Choose sequences $\{p_i\}$ and $\{r_i\}$ such that

$$p_0 = q,\ \ p_i = 1 + (\theta/\mu) + (\theta/\mu)^2 + \cdots + (\theta/\mu)^{i-1} + q(\theta/\mu)^i$$

and

$$r_0 = s,\ \ r_i = s - s/2^2 - s/2^3 - \cdots - s/2^{i+1}$$

for $i = 1, 2, \cdots$, where $0 < s < r_0/2$. Applying (2.4) to the sequences $\{p_i\}$ and $\{r_i\}$,

$$||u||_{p_{i+1}\theta,r_{i+1}} \le \Big(\frac{C_2p_{i+1}}{r_i - r_{i+1}}\Big)^{1/p_{i+1}} ||u||^{1-1/p_{i+1}}_{p_i\theta,r_i}\ ,$$

i.e.,

$$||u||_{p_{i+1}\theta,r_{i+1}} \le \Big(\frac{C_2}{s}\Big)^{C_3(\mu/\theta)^{i+1}} \Big(\frac{2\theta}{\mu}\Big)^{C_3(\mu/\theta)^{i+1}} ||u||^{1-1/p_{i+1}}_{p_i\theta,r_i}\ ,$$

where $C_3 > 0$ depends on only $n$ and $q$. Iterating this inequality,

$$\begin{aligned} & ||u||_{p_{i+1}\theta,r_{i+1}} \\ \le\ & \Big(\frac{C_2}{s}\Big)^{C_3(\mu/\theta)^{i+1}} \Big(\frac{2\theta}{\mu}\Big)^{C_3(i+1)(\mu/\theta)^{i+1}} ||u||^{1-1/p_{i+1}}_{p_i\theta,r_i} \\ \le\ & \Big(\frac{C_2}{s}\Big)^{C_3\{(\mu/\theta)^{i+1}+(\mu/\theta)^i\}} \Big(\frac{2\theta}{\mu}\Big)^{C_3\{(i+1)(\mu/\theta)^{i+1}+i(\mu/\theta)^i\}} ||u||^{(1-1/p_{i+1})(1-1/p_i)}_{p_{i-1}\theta,r_{i-1}} \\ \le\ & \cdots \\ \le\ & \Big(\frac{C_2}{s}\Big)^{C_3\sum_{j=1}^{i+1}(\mu/\theta)^j} \Big(\frac{2\theta}{\mu}\Big)^{C_3\sum_{j=1}^{i+1} j(\mu/\theta)^j} ||u||^{\Pi_{j=1}^{i+1}(1-1/p_j)}_{p_0\theta,r_0}\,. \end{aligned}$$

Therefore, letting $i \to \infty$, there exist $\sigma > 0$ and $\delta > 0$ such that

$$||u||_{\infty,s/2} \le C_4\Big(\frac{\mathrm{vol}B_s(z)^{1/q}}{s}\Big)^{\sigma}||u||^{\delta}_{q\theta,s}\,. \tag{2.5}$$

On the other hand, applying (2.3) to $p = q$, we get

$$\begin{aligned} S_1\,||u||^q_{q\theta,s} &\le \frac{4q^2}{s}\,||u||^q_{q,2s} + q^2 \sup_N |h|\;||V||_{q,2s}\,||u||^{q/\mu}_{q,2s} \\ &\le \Big\{\frac{4q^2}{s}\sup_N |u|\,\mathrm{vol}B_{2s}(z)^{1/q} + q^2\sup_N |h|\;||V||_{q,2s}\Big\}\,||u||^{q/\mu}_{q,2s}\,, \end{aligned}$$

i.e.,

$$||u||_{q\theta,s} \le C_5\Big(\frac{\mathrm{vol}B_{2s}(z)^{1/q}}{s}\Big)^{1/q}||u||^{1/\mu}_{q,2s}\,, \tag{2.6}$$

where $C_5$ depends only on $\sup_N|h|$, $\sup_N|u|$ and $||V||_{q,2s}$.

Combining (2.5) and (2.6), we have

$$||u||_{\infty,s} \le C_6\Big(\frac{\mathrm{vol}B_{2s}(z)^{1/q}}{s}\Big)^{\sigma+\delta/q}||u||^{\delta/\mu}_{q,2s}\,, \tag{2.7}$$

where $C_6$ depends only on $\sup_N|h|$, $\sup_N|u|$ and $||V||_{q,2s}$. Note that for each point $z \in N$, we can select a suitable $0 < s < r_0/2$ in such a way that $s^{-1}\mathrm{vol}B_{2s}(z)^{1/q}$ is bounded above by a constant. Therefore from Lemma 2.2 and (2.7), for given $\epsilon > 0$, there exists a sufficiently large $R > 0$ such that $||u||_{\infty,s} < \epsilon$ whenever $d(o,z) \ge R$. □

*Proof of Theorem* 1.1. Since $S_1(M) > 0$, $\lambda_1(M) > 0$ and $N$ is roughly isometric to $M$, we have $S_1(N) > 0$ and $\lambda_1(N) > 0$; hence $\mu_1(N) > 0$. (See [10] or [7].) Combining Lemma 2.2 and Lemma 2.3, and Lemma 4.6 of [6], we get a solution $\tilde{u}$ such that

$$(\tilde{u} - h)(z) \to 0 \ \text{ and } \ (h - g)(z) \to 0 \ \text{ as } \ z \to \infty.$$

Therefore, we have only to prove that

$$(g(z) - f(z)) \to 0 \quad \text{as} \quad z \to \infty.$$

Note that

$$\begin{aligned} |g(z) - f(z)| &= \left|\sum_{q\in Q}\eta_q(z)\left(\frac{1}{\mathrm{vol}B_{4\tau}(\varphi^{-1}(q))}\int_{B_{4\tau}(\varphi^{-1}(q))}\overline{f}^s\right)^{1/s} - f(z)\right| \\ &\le \sum_{q\in Q}\eta_q(z)\left(\frac{1}{\mathrm{vol}B_{4\tau}(\varphi^{-1}(q))}\int_{B_{4\tau}(\varphi^{-1}(q))}|\overline{f}(y) - f(z)|^s dy\right)^{1/s}. \end{aligned}$$

By the compactness of $M \cup M(\infty)$, for given $\epsilon > 0$, there exist $R > 0$, $v_1, v_2, \cdots,$ $v_i \in M(\infty)$ and positive numbers $\delta_1, \delta_2, \cdots, \delta_i$ such that $M(\infty) \subset \bigcup_{j=1}^i K(v_j, \delta_j, R)$,

$$|\overline{f}(y) - \overline{f}(v_j)| < \epsilon \ \text{ and } \ |\overline{f}(v_j) - f\circ\varphi(y)| < \epsilon$$

if $y \in K(v_j, 2\delta_j, R)$ for each $j = 1, 2, \cdots, i$. Thus for sufficiently large $R > 0$, if $d(o', z) \ge R$, where $o'$ is a fixed point in $N$, then for some $v_j \in M(\infty)$, we have $y \in K(v_j, \delta_j, R)$, and hence $|\overline{f}(y) - f \circ \varphi(y)| < \epsilon$. Note that for each $y \in \bigcup_{q \in B_{3\nu/2}(z)} B_{4\tau}(\varphi^{-1}(q))$,

$$\begin{aligned} d(\varphi(y), z) &\leq d(\varphi(y), \varphi \circ \varphi^{-1}(q)) + d(\varphi \circ \varphi^{-1}(q), z) \\ &\leq ad(y, \varphi^{-1}(q)) + b + d(\varphi \circ \varphi^{-1}(q), q) + d(q, z) \\ &\leq 4a\tau + b + \kappa + 3\nu/2. \end{aligned}$$

By the second condition (ii) of the function class $\mathcal{F}_\varphi$, we get $|f \circ \varphi(y) - f(z)| < \epsilon$. Consequently, we obtain equation (1.3).

The uniqueness immediately follows from the comparison principle and the equation (1.3). □

Note that in the case of Cartan-Hadamard manifolds, the equation (1.3) implies the continuity of the solution near infinity in the usual sense. So, in the setting of the usual asymptotic Dirichlet problem, we have some interesting new results as follows.

**Corollary 2.4.** *Let $\varphi, M, N$ and $V$ be as in Theorem* 1.1. *Suppose that $N$ is also a Cartan-Hadamard manifold and that $\varphi$ can be extended to a map $\varphi : M \cup M(\infty) \to N \cup N(\infty)$ in such a way that it is continuous at every point of $M(\infty)$. Then the usual asymptotic Dirichlet problem for the Schrödinger operator $\Delta - V$ on $N$ is solvable, i.e., for any continuous function $f$ on $N(\infty)$, there exists a unique solution $u$ on $N \cup N(\infty)$ such that*

$$\begin{cases} (\Delta - V)\, \tilde{u} = 0 & \text{on } N; \\ \tilde{u} = f & \text{on } N(\infty), \end{cases}$$

*where $\Delta$ is the Laplacian of $N$.*

*Proof.* Let $f$ be a continuous function on $N \cup N(\infty)$. Since the rough isometry $\varphi : M \to N$ extends to a map $\varphi : M \cup M(\infty) \to N \cup N(\infty)$ in such a way that it is continuous at every point of $M(\infty)$, we have a continuous function $f \circ \varphi$ on $M(\infty)$. From this fact, together with the continuity of $f$ on $N(\infty)$, $f \in \mathcal{F}_\varphi$. Therefore, similarly arguing in the proof of Theorem 1.1, we have a solution $\tilde{u}$ for the Schrödinger operator such that $|\tilde{u}(z) - f(z)| \to 0$ as $z \to \infty$. □

**Corollary 2.5.** *Let $M$ be a Cartan-Hadamard manifold being roughly isometric to a Cartan-Hadamard manifold with the sectional curvature pinched between two negative constants. Then the usual asymptotic Dirichlet problem for the Schrödinger operator $\Delta - V$ on $M$, where $V$ is a nonnegative function in $L^q(N)$ for any $n \leq q < \infty$, is solvable.*

*Proof.* Using the result of Li and Wang [9], we can extend the given rough isometry to a map in such a way that it is a homeomorphism between asymptotic boundaries at infinity. The rest of the proof is the same as that in the proof of Corollary 2.4. □

Let $(M, ds^2)$ be a Cartan-Hadamard manifold and the identity map $\mathrm{id} : (M, ds^2) \to (M, d\tilde{s}^2)$ be a rough isometry. Then for any continuous function $f$ on $(M, ds^2)$, it is easy to check that $f \in \mathcal{F}_{\mathrm{id}}$. Therefore, our result is new even in this simple case. This is a direct generalization of Schoen and Yau's result [11], in which they solved the Dirichlet problem in the trivial case that $V \equiv 0$.

## References


1. M. T. Anderson, *The Dirichlet problem at infinity for manifolds of negative curvature*, J. Differential Geom. **18** (1983), 701–721. MR0730923 (85m:58178)
2. M. T. Anderson and R. Schoen, *Positive harmonic functions on complete manifolds of negative curvature*, Ann. of Math. **121** (1985), 429–461. MR0794369 (87a:58151)
3. P. Buser, *A note on the isoperimetric constant*, Ann. Sci. Ec. Norm. Sup. Paris **15** (1982), 213–230. MR0683635 (84e:58076)
4. S. Y. Cheng, *The Dirichlet problem at infinity for nonpositively curved manifolds*, Comm. Anal. Geom. **1** (1993), 101–112. MR1230275 (94h:53048)
5. H. I. Choi, *Asymptotic Dirichlet problems for harmonic functions on Riemannian manifolds*, Trans. Amer. Math. Soc. **281** (1984), 691–716. MR0722769 (85b:53040)
6. H. I. Choi, S. W. Kim and Y. H. Lee, *Rough isometry and the asymptotic Dirichlet problem*, Tôhoku Math. J. **50** (1998), 333–348. MR1638223 (99j:58211)
7. Th. Coulhon and L. Saloff-Coste, *Variétés riemanniennes isométriques à l'infini*, Rev. Mat. Iberoamericana **11** (1995), 687–726. MR1363211 (96m:53035)
8. A. A. Grigor'yan and W. Hansen, *Liouville property for Schrödinger operators*, Math. Ann. **312** (1998), 659–716. MR1660247 (2000a:58092)
9. P. Li and J. Wang, *On harmonic rough isometries into Hadamard space*, Asian J. Math. **2** (1998), 419–442. MR1724623 (2000j:53055)
10. M. Kanai, *Rough isometries, and combinatorial approximations of geometries of noncompact Riemannian manifolds*, J. Math. Soc. Japan **37** No.3 (1985), 391–413. MR0792983 (87d:53082)
11. R. Schoen and S. T. Yau, *Lectures on Differential Geometry*, International Press, Boston, 1994. MR1333601 (97d:53001)
12. D. Sullivan, *The Dirichlet problem at infinity for a negatively curved manifold*, J. Differential Geom. **18** (1983), 723–732. MR0730924 (85m:58177)


Department of Mathematics Education, Ewha Womans University, Seoul 120-750, Korea

*E-mail address*: yonghah@ewha.ac.kr