PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 12, Pages 3441–3445
S 0002-9939(05)07663-X
Article electronically published on July 13, 2005

BOUNDS FOR THE INDEX OF THE CENTRE IN CAPABLE GROUPS

K. PODOSKI AND B. SZEGEDY

(Communicated by Jonathan I. Hall)

ABSTRACT. A group H is called capable if it is isomorphic to $G/\mathbf{Z}(G)$ for some group G. Let H be a capable group. I. M. Isaacs (2001) showed that if H is finite, then the index of the centre is bounded above by some function of $|H'|$. We show that if $|H'| < \infty$, then $|H : Z(H)| \leq |H'|^{c \log_2 |H'|}$ with some constant c and this bound is essentially best possible. We complete a result of Isaacs, showing that if H' is a cyclic group, then $|H : \mathbf{Z}(H)| \leq |H'|^2$.

1. Introduction

Let G be an arbitrary group. According to a classical theorem of Schur, if $|G : \mathbf{Z}(G)| < \infty$, then $|G'| < \infty$. An easy argument based on the ultra product method shows that there is a bound for the order of the derived subgroup in terms of the index of the centre. The best bound was given by Wiegold [7] showing that if $|G : \mathbf{Z}(G)| = n$, then $|G'| \leq n^{\frac{1}{2} log_2 n}$. Infinite extraspecial groups show that the converse of the theorem of Schur does not hold in general. However, P. Hall (see [6], p.423) observed that if $|G'| < \infty$, then $|G : \mathbf{Z_2}(G)|$ is bounded above in terms of $|G'|$ (where $\mathbf{Z_2}(G)$ denotes the second member of the upper central series of G). The first explicit bound was given by I. D. Macdonald [3]. Improving this bound we proved in [5] that

$$|G : \mathbf{Z_2}(G)| \leq |G'|^{c \log_2 |G'|},$$

and our examples show that this estimate is sharp apart from the value of the constant c.

A group H is said to be capable if there exists some group G such that $G/\mathbf{Z}(G)$ is isomorphic to H. I. M. Isaacs [2] proved that if H is a capable group and $|H'| = n$, then $|H : \mathbf{Z}(H)|$ is bounded above by some function f of n, or equivalently, if G is an arbitrary group and $|G' : G' \cap \mathbf{Z}(G)| = n$, then $|G : \mathbf{Z_2}(G)| \leq f(n)$. However, he has not given an explicit function $f(n)$. In our present paper we give the essentially best possible bound.

Theorem 1. *If G is a group (not necessarily finite) and $|G' : G' \cap \mathbf{Z}(G)| = n$, then $|G : \mathbf{Z_2}(G)| \leq n^{c \log_2 n}$ with $c = 2$.*

Received by the editors March 10, 2003 and, in revised form, January 6, 2004.
2000 *Mathematics Subject Classification.* Primary 20E34, 20D60, 20D15, 20D25.
This research was partially supported by the Hungarian National Research Foundation (OTKA), grant no. T038059.

Using this result for $H = G/\mathbf{Z}(G)$ we obtain the following.

Corollary 2. *If H is a capable group and $|H'| = n$, then*

$$|H : \mathbf{Z}(H)| \leq n^{c \log_2 n}$$

with $c = 2$.

Actually, the preceding result can be regarded as a converse of Wiegold's theorem. The sequence of groups G_n we constructed in [5] shows that these estimates are sharp apart from the value of the constant c. The proof of Theorem 1 shows that the value of the constant c is at most 2. We also mention that H. Heineken [1] constructed capable groups H for all odd prime numbers p and for all natural numbers n such that $|H'| = |\mathbf{Z}(H)| = p^n$ and $|H : \mathbf{Z}(H)| = p^{2n+\binom{n}{2}}$. Since these are the best known examples, we think that the constant c can be further improved. Although the above examples do not work for $p = 2$, similar estimates motivate us to think that perhaps $c = \frac{1}{2}$ is the best constant.

Question 3. *Is it true that if H is a capable group and $|H'| = n$, then $|H : \mathbf{Z}(H)| \leq n^{\frac{1}{2}\log_2 n + c_2}$ for some constant c_2?*

For groups with infinite derived subgroup a similar argument yields:

Theorem 4. *If G is a group and $|G' : G' \cap \mathbf{Z}(G)| = \kappa$ is an infinite cardinal, then $|G : \mathbf{Z_2}(G)| \leq 2^\kappa$.*

Corollary 5. *If H is a capable group and $|H'| = \kappa$ is an infinite cardinal, then $|H : \mathbf{Z}(H)| \leq 2^\kappa$.*

Remark 6. Related to infinite groups, similar results are included in [4] and [5]. For each infinite cardinal κ we constructed a group G such that $|G'| = \kappa$, $\mathbf{Z}(G) = 1$ and $|G| = 2^\kappa$ (see [5]). It follows that the previous estimates are sharp.

The second part of our paper deals with groups with cyclic derived subgroups.

For a capable group H, I. M. Isaacs [2] proved that if H is finite, H' is cyclic and all elements of order 4 in H' are central in H, then $|H : \mathbf{Z}(H)| \leq |H'|^2$. In the present paper we prove that the assumption about elements of order 4 can be omitted.

Theorem 7. *If H is a finite capable group and H' is cyclic, then $|H : \mathbf{Z}(H)| \leq |H'|^2$.*

For an arbitrary group G, we prove the following estimate.

Theorem 8. *If G is a finite group with G' cyclic of order n, then $|G : \mathbf{Z_2}(G)| \leq n\varphi(n)$, where φ is Euler's totient function.*

The previous estimate is sharp for the holomorph of a cyclic group.

2. Groups with arbitrary derived subgroups

In this section we prove Theorem 1 and Theorem 4.

Lemma 9. *Let H be a subgroup of G generated by k elements and $|G'| = n$. Then $|G : C_G(H)| \leq n^k$.*

Proof. Let $x_1, x_2, \dots, x_k$ be a generating system of H. Let us denote the conjugacy class of x_i in G by $Cl(x_i)$. Then

$$|G : C_G(H)| \leq \prod_{i=1}^{k} |G : C_G(x_i)| = \prod_{i=1}^{k} |Cl(x_i)| \leq |G'|^k = n^k.$$

□

Lemma 10. *Let G be an arbitrary group and $C < G$ be a proper subgroup. Then $G' = [G - C, G]$.*

Proof. It is enough to generate the commutators $\{[c, g] \mid c \in C; g \in G\}$. Let x be an arbitrary element of $G - C$. Then

$$[c, g] = [x, c^{-1}gc]^{-1}[cx, g] \in [G - C, G].$$

□

Lemma 11. *Let $Z = G' \cap \mathbf{Z}(G)$, and let U, V be subgroups of G such that $Z \leq U, V \leq G'$. Then there exist elements y, z of G with the following properties.*

(1) *If $Z \lneqq U$, then $U \cap C_G(y) \lneqq U$.*
(2) *If $V \lneqq G'$, then $V \lneqq \langle V, [y, z]\rangle$.*

Proof. Set $C = C_G(U)$. Suppose that $Z \lneqq U$. Now, $C \lneqq G$; thus $U \cap C_G(y) \lneqq U$ for all $y \in G - C$. Lemma 10 yields that $G' = [G - C, G]$. Consequently, if $V \lneqq G'$, then we can choose $y \in G - C$ and $z \in G$ such that $V \lneqq \langle V, [y, z]\rangle$. In the case of $Z = U$ and $V \lneqq G'$, then we can choose arbitrary $[y, z] \notin V$. □

Lemma 12. *Let $Z = G' \cap \mathbf{Z}(G)$, and suppose that $|G' : Z| = n$. Let T be a subgroup with $G' \leq T \leq G$ having the following properties.*

(1) *$G' = T'Z$.*
(2) *$G' \cap \mathbf{Z}(T) = Z$.*
(3) *T/Z can be generated by k elements.*

Then there exists $M \leq G$ such that $[M, G, G] = 1$ and $|G : M| \leq n^k$.

Proof. Let $M/Z = C_{G/Z}(T/Z)$. Then by Lemma 9, $|G : M| \leq n^k$. Now $[T, M, G] = 1$, and in particular, $[T, M, T] = 1$, so that $[T', M] = 1$ by the Three Subgroup Lemma, and hence $[G', M] = 1$. Now $[G, T, M] \leq [G', M] = 1$. Applying the Three Subgroup Lemma again, we obtain that $[M, G, T] = 1$. Consequently $[M, G] \leq G' \cap \mathbf{Z}(T) = Z$, and thus $[M, G, G] = 1$. □

Remark 13. The statement of Lemma 12 is also true if n and k are infinite cardinals.

Lemma 14. *Let G be a finite group and $|G' : Z| = n$. Then there exists T as in Lemma 12 with $k \leq 2\log_2 n$.*

Proof. We define the elements y_{i+1}, z_{i+1} $(0 \leq i \leq l - 1)$ recursively by applying Lemma 11 for $V_i = \langle Z, [y_1, z_1], [y_2, z_2], \dots, [y_i, z_i]\rangle$ and $U_i = C_{G'}(V_i)$. Now we have that

$$Z = V_0 \leq V_1 \leq V_2 \leq \cdots \leq V_l = G'$$

and

$$G' = U_0 \geq U_1 \geq U_2 \geq \cdots \geq U_l = Z,$$

where l is the smallest integer such that $V_l = G'$ and $U_l = Z$. It is clear that $l \leq \log_2 n$. Now $T = \langle Z, y_1, z_1, y_2, z_2, \cdots, y_l, z_l\rangle$ has the required properties. □

Proof of Theorem 1. It follows immediately from Lemma 12 and Lemma 14 that there exists a subgroup M of G such that $|G : M| \le n^{2\log_2 n}$ and $M \le \mathbf{Z_2}(G)$. $\square$

Proof of Theorem 4. First, we choose a subgroup T_1 such that $T_1'Z = G'$ and $|T_1| \le \kappa$. Let Q be a coset representative system for Z in $G' \setminus \mathbf{Z}(G)$. We choose elements y_q for all $q \in Q$ such that $y_q \notin C_G(q)$. The set $T_2 = \{y_q \mid q \in Q\}$ has cardinality κ and clearly $C_{G'}(Y) = Z$. Let $T = \langle T_1, T_2 \rangle$. Then $|T| = \kappa$, and the same argument as in Lemma 12 completes the proof. $\square$

3. Groups with cyclic derived subgroups

In this section we focus our attention on groups with cyclic derived subgroups.

Lemma 15. *Let G be a group, and write $Z = G' \cap \mathbf{Z}(G)$. Assume that G' is a p-group and G'/Z is cyclic of order n. Then there exists a subgroup $M \le G$ such that $[M, G, G] = 1$ and $|G : M| \le n^2$.*

Proof. Let $x \in G' - Z$ such that $x^p \in \mathbf{Z}(G)$. Set $C = C_G(x)$. It follows that $C_G(y) \cap G' = \mathbf{Z}(G) \cap G'$ for all $y \in G - C$. Using Lemma 10 we can find $a \in G - C$ and $b \in G$ such that $\langle Z, [a, b]\rangle = G'$. Let $T = \langle Z, a, b\rangle$, and note that T satisfies the three conditions of Lemma 12 with $k = 2$. $\square$

Proof of Theorem 7. We reduce to the case where G' is a p-group. For each prime divisor p of $|G'|$ let N_p be the normal p-complement of G' and work in the factor group G/N_p. This factor group satisfies the hypotheses with n replaced by a divisor of n_p, the p-part of n. Using the preceding lemma, we know that there exists a subgroup $M_p \le G$ such that $[M_p, G, G] \le N_p$ and $|G : M_p| \le (n_p)^2$. Let $M = \bigcap M_p$. Then $[M, G, G] \le \bigcap N_p = 1$ and $|G : M| \le \prod (n_p)^2 = n^2$. $\square$

Proof of Theorem 8. Using the multiplicativity of Euler's φ function, as in the previous proof, we can reduce to the case where G' is a p-group. If $G' \cap \mathbf{Z}(G) > 1$, then by Theorem 7, the index of the second center is at most $(n/p)^2 < n\varphi(n)$. In the case of $p = 2$ the unique element of order 2 in G' is central in G, thus $G' \cap \mathbf{Z}(G) > 1$. We can assume therefore that $G' \cap \mathbf{Z}(G) = 1$ and in particular $p > 2$. Now let $D = C_G(G')$, and note that $[G, D, D] = 1$. Therefore $D' \le \mathbf{Z}(G)$ by the Three Subgroup Lemma. Then $D' \le G' \cap \mathbf{Z}(G) = 1$; so D is abelian. It is obvious that $G/D \le Aut(G')$. Since G' is a cyclic p-group and $p > 2$, we have that G/D is cyclic of order dividing $\varphi(n)$. If x generates G modulo D, let $C = C_D(x)$. Then C centralizes $D\langle x\rangle = G$, and hence $C \le \mathbf{Z}(G)$. Consequently $|D : C| = |[D, x]| \le n$, and we deduce that $|G : \mathbf{Z}(G)| \le |G : D||D : C| \le n\varphi(n)$. $\square$

Remark 16. I. M. Isaacs [2] proved that if H is a capable nilpotent group with cyclic derived subgroup and all elements of order 4 are central in H, then $|H : Z(H)| = |H'|^2$. In this result the assumption about elements of order 4 cannot be omitted as the example of the dihedral group D of order 2^n ($n \ge 3$) shows. It is a capable group, D' is a cyclic group of order 2^{n-2} and $|D : Z(D)| = 2^{n-1}$.

Acknowledgments

The authors are very grateful to P. P. Pálfy and L. Pyber and especially to the referee for helpful comments.

References

[1] H. Heineken, Nilpotent groups of class two that can appear as central quotient groups, *Rend. Sem. Mat. Univ. Padova* 84 (1990), 241–248. MR1101296 (92c:20068)

[2] I. M. Isaacs, Derived subgroups and centers of capable groups, *Proc. Amer. Math. Soc.* 129 (2001), 2853–2859. MR1840087 (2002c:20035)

[3] I. D. Macdonald, Some explicit bounds in groups with finite derived groups, *Proc. London Math. Soc* (3) 11 (1961), 23–56. MR0124433 (23:A1745)

[4] K. Podoski, Groups covered by an infinite number of Abelian subgroups, *Combinatorica* 21 (3) (2001), 413–416. MR1848059 (2002e:20061)

[5] K. Podoski, B. Szegedy, Bounds in groups with finite Abelian coverings or with finite derived groups, *J. Group Theory* 5 (2002), 443–452. MR1931369 (2003i:20052)

[6] Derek J. S. Robinson, A Course in the Theory of Groups, Springer–Verlag, New York (1982). MR0648604 (84k:20001)

[7] J. Wiegold, Multiplicators and groups with finite central factor-groups, *Math. Z.* 89 (1965), 345–347. MR0179262 (31:3510)

Department of Algebra and Number Theory, Eötvös University, Pázmány Péter sétány 1/c, H-1117 Budapest, Hungary

E-mail address: pcharles@cs.elte.hu

Current address: Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, POB 127, H-1364 Budapest, Hungary

E-mail address: pcharles@renyi.hu

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, POB 127, H-1364 Budapest, Hungary

E-mail address: szegedy@renyi.hu