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EXAMPLES OF RATIONAL HOMOTOPY TYPES OF BLOW-UPS

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Abstract. We give an example of two homotopic embeddings $j_0, j_1 : V \hookrightarrow W$ of manifolds with isomorphic complex normal bundles but such that the blow-ups of W along j_0 and along j_1 have different rational homotopy types.

1. Introduction

Let $j\colon V \hookrightarrow W$ be an embedding of closed smooth manifolds and suppose that its normal bundle ξ is equipped with a structure of the complex vector bundle. The *blow-up* of W along j is a new manifold $\widetilde{W}$ which is roughly obtained by replacing the submanifold V in W by the projective bundle $P\xi$ (see Definition 2.1 below or [3, Chapter 4, Section 6] or [8, end of Section 6.2] for more details). In [6] we have studied this blow-up construction from the rational homotopy theory viewpoint. There we proved the following

Theorem 1.1. *If* $\dim W \geq 2\dim V + 4$ *and if* V *and* W *are simply connected, then the rational homotopy type of the blow-up* $\widetilde{W}$ *of* W *along* j *depends only on the rational homotopy class of the map* j *and on the rational Chern classes of* ξ.

Actually in that paper we described explicitly the rational homotopy type of $\widetilde{W}$. Moreover the simple-connectivity hypotheses were also weaker than stated here.

A natural question is whether the conclusion of Theorem 1.1 still holds without the "stability" hypothesis, $\dim W \geq 2\dim V + 4$. The goal of this note is to provide an explicit counterexample to such a generalization.

To state our main result we first describe two embeddings $j_0, j_1 : V \hookrightarrow W$ which were previously studied in [7]. Let $W = S^{15}$ and $V = S^2 \times S^7$. The embedding j_0 is defined as the composite

$$j_0\colon V = S^2 \times S^7 \hookrightarrow \mathbb{R}^3 \times \mathbb{R}^8 = \mathbb{R}^{11} \subset \mathbb{R}^{11} \cup \{\infty\} \cong S^{11} \overset{\epsilon}{\hookrightarrow} S^{15} = W,$$

where the inclusions of the spheres in the euclidian spaces are the standard ones, $\mathbb{R}^{11} \cup \{\infty\} \cong S^{11}$ is the stereographic homeomorphism, and ϵ is the inclusion of a subequator. To describe the other embedding j_1 consider the Hopf fibration

$$S^7 \to S^{15} \xrightarrow{\pi} S^8.$$

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Let $\epsilon' \colon S^2 \hookrightarrow S^8$ be the inclusion of a subequator. The restriction of π to $\epsilon'(S^2)$ is a trivial S^7-bundle, so we can consider the inclusion

$$j_1 \colon V = S^2 \times S^7 \cong \pi^{-1}(S^2) \hookrightarrow S^{15} = W.$$

These two embeddings are clearly homotopic since $[S^2 \times S^7, S^{15}] = \{*\}$. It is easy to check that their normal bundles are trivial real bundles of rank 6. On the other hand j_0 and j_1 are not isotopic because the complement of $j_1(V)$ in W is

$$W \setminus j_1(V) = \pi^{-1}(S^8 \setminus \epsilon'(S^2)) \simeq \pi^{-1}(S^5) \cong S^5 \times S^7, \tag{1.1}$$

but, since $j_0(V)$ is included in the equator of S^{15}, the complement $W \setminus j_0(V)$ has the homotopy type of a suspension. In particular this complement has the rational homotopy type of a wedge of spheres whose Betti numbers are the same, by Alexander duality, as the Betti numbers of $W \setminus j_1(V)$. Therefore

$$W \setminus j_0(V) \simeq_{\mathbb{Q}} S^5 \vee S^7 \vee S^{12}. \tag{1.2}$$

Now we can state the main result of this paper (the notion of formal space used in the statement is defined in Section 2):

Proposition 1.2. *Let $j_0, j_1 \colon V \hookrightarrow W$ be the two embeddings described above with their normal bundles equipped with the structure of a trivial complex bundle of rank 3. Then the blow-ups $\widetilde{W}_0$ and $\widetilde{W}_1$ of W along, respectively, j_0 and j_1 have different rational homotopy types. More precisely the cohomology algebras $H^*(\widetilde{W}_0;\mathbb{Q})$ and $H^*(\widetilde{W}_1;\mathbb{Q})$ are isomorphic but*

(i) *all Massey products are trivial in $H^*(\widetilde{W}_0;\mathbb{Q})$ but not in $H^*(\widetilde{W}_1;\mathbb{Q})$;*
(ii) *$\widetilde{W}_0$ is formal and $\widetilde{W}_1$ is not formal;*
(iii) *$\pi_8(\widetilde{W}_0) \otimes \mathbb{Q} \cong \mathbb{Q}$ and $\pi_8(\widetilde{W}_1) \otimes \mathbb{Q} = 0$.*

The proof of this proposition is the content of the last section.

2. The rational homotopy type of a blow-up

First we review the blow-up construction. Let $j \colon V \hookrightarrow W$ be an embedding of closed manifolds with a complex normal bundle ξ of rank k. Fix a hermitian metric on ξ and consider the associated sphere bundle, $\pi_0 \colon S\xi \to V$, and projective bundle, $\pi' \colon P\xi \to V$. Since $P\xi$ is the orbit space of $S\xi$ by the S^1-action, we get an induced projection map $q \colon S\xi \to P\xi$ over V. The disk bundle $D\xi$ is diffeomorphic to some tubular neighborhood T of V in W and $S\xi \cong \partial T$. Let $B := \overline{W \setminus T}$ be the closure of the complement of T in W and consider the inclusion

$$l \colon S\xi \cong \partial T = \partial B \hookrightarrow B.$$

Definition 2.1. The *blow-up*, $\widetilde{W}$, of W along j is the pushout of the diagram

$$B \xleftarrow{\ l\ } S\xi \xrightarrow{\ q\ } P\xi\ .$$

It turns out that $\widetilde{W}$ is again a closed manifold.

Next we review very briefly Sullivan's theory [9] for describing the rational homotopy type of spaces by certain algebraic objects, namely *commutative differential graded algebras* or CDGA's for short. For more details on this theory we refer the

reader to [2]. We denote by $A_{PL}\colon \mathrm{Top} \to \mathrm{CDGA}$ the contravariant functor of polynomial differential forms and if $f\colon X \to Y$ is a map we write $f^* := A_{PL}(f)$. By a *CDGA model* of a space X we mean a chain of quasi-isomorphisms of CDGA's

$$(A, d_A) \xleftarrow{\simeq} (A_1, d_1) \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} (A_r, d_r) \xrightarrow{\simeq} A_{PL}(X). \tag{2.1}$$

More generally a *CDGA model* of a diagram of spaces is a diagram of CDGA's connected through a chain of quasi-isomorphisms to the diagram obtained by applying A_{PL} to the given diagram of spaces (see [6, Definition 2.9] for a more precise definition). A *Sullivan model* of a space X is a quasi-isomorphism of CDGA's

$$\rho\colon (\wedge Z, d) \xrightarrow{\simeq} A_{PL}(X)$$

where $(\wedge Z, d)$ is a Sullivan algebra in the sense of [2, §12]. Of course a Sullivan model is a special case of a CDGA model. The main result of Sullivan's theory is that the rational homotopy type of a simply-connected space of finite type is determined by any of its CDGA models. A space X is called *formal* if $(H^*(X), 0)$ is a CDGA model of X.

Now we will describe the rational homotopy type of the blow-up. Let (Q, d_Q) be a CDGA model of the manifold V. Let $\gamma_i \in Q^{2i} \cap \ker d_Q$ be representatives of the Chern classes $c_i(\xi) \in H^{2i}(V; \mathbb{Q}) \cong H^{2i}(Q, d_Q)$ and assume that $\gamma_0 = 1$. Consider the relative Sullivan algebras ([2, §14])

$$\left(Q \otimes \wedge(x_2, z_{2k-1}), Dx = 0, Dz = \sum_{i=0}^{k} \gamma_i \otimes x^{k-i}\right)$$

and

$$(Q \otimes \wedge(z_{2k-1}), \bar{D}z = \gamma_k).$$

Let $\rho\colon (\wedge Z, d) \xrightarrow{\simeq} A_{PL}(B)$ be a Sullivan model of the complement $B = \overline{W \setminus T}$.

Lemma 2.2. *With the notation of this section there exists a CDGA map* $\kappa\colon (\wedge Z, d) \to (Q \otimes \wedge z, \bar{D})$ *such that the diagram*

$$(Q, d_Q) \xrightarrow{\iota} (Q \otimes \wedge(x, z), D) \xrightarrow{\mathrm{proj}} (Q \otimes \wedge z, \bar{D}) \xleftarrow{\kappa} (\wedge Z, d)$$

is a CDGA-model of the diagram

$$V \xleftarrow{\pi'} P\xi \xleftarrow{q} S\xi \xrightarrow{l} B.$$

Proof. By definition of the Chern classes [5, Chapter 17, Section 2], there is an isomorphism of $H^*(V)$-algebras

$$H^*(P\xi) \cong (H^*(V) \otimes \mathbb{Q}[x_2]) \,/\, \left(\sum_{i=0}^{k} c_i(\xi) \otimes x^{k-i}\right), \tag{2.2}$$

where x corresponds to the canonical class $a_\xi := -c_1(\lambda_\xi) \in H^2(P\xi)$ where λ_ξ is the tautological complex line bundle over $P\xi$. Let $\theta \in A^2_{PL}(P\xi) \cap \ker d$ be a cocycle representing a_ξ. It is easy to check that $q^*(a_\xi) = 0$ in $H^2(S\xi)$ because $q^*\lambda_\xi$ is a trivial line bundle (see [6, Lemma 5.1]). Therefore there exists $\bar{\theta} \in A^1_{PL}(S\xi)$ such that $d\bar{\theta} = q^*(\theta)$.

Since (Q, d_Q) is a model of V, by [6, Lemma 2.6] there exists a cofibrant CDGA $(\hat{Q}, \hat{d}_Q)$ and surjective quasi-isomorphisms β and β',

$$(Q, d_Q) \xleftarrow[\simeq]{\beta} (\hat{Q}, \hat{d}_Q) \xrightarrow[\simeq]{\beta'} A_{PL}(V) \ .$$

Let $\hat{\gamma}_0 = 1$ and since β is surjective there exist cocycles $\hat{\gamma}_i \in \hat{Q}^{2i} \cap \ker \hat{d}_Q$ for $i \geq 1$ such that $\beta(\hat{\gamma}_i) = \gamma_i$.

By (2.2) the cocycle

$$\sum_{i=0}^{k} \pi'^*(\beta'(\hat{\gamma}_i)) \cdot \theta^{k-i} \ \in \ A_{PL}^{2k}(P\xi) \tag{2.3}$$

represents the zero cohomology class. Therefore there exists $\zeta \in A_{PL}^{2k-1}(P\xi)$ such that $d\zeta$ is the cocycle of (2.3).

Define the following commutative diagram of CDGA's:

$$\begin{tikzcd}
(\hat{Q}, \hat{d}_Q) \arrow[r, tail] \arrow[d, "\beta'"] & (\hat{Q} \otimes \wedge(x_2, z_{2k-1}), \hat{D}) \arrow[r, tail] \arrow[d, "\lambda'"] & (\hat{Q} \otimes \wedge(x_2, z_{2k-1}, \bar{x}_1), \hat{D}) \arrow[d, "\mu'"] \\
A_{PL}(V) \arrow[r, "\pi'^*"] & A_{PL}(P\xi) \arrow[r, "q^*"] & A_{PL}(S\xi)
\end{tikzcd} \tag{2.4}$$

by $\hat{D}(x) = 0$, $\hat{D}(z) = \sum_{i=0}^{k} \hat{\gamma}_i \otimes x^{k-i}$, $\hat{D}(\bar{x}) = x$, $\lambda'(x) = \theta$, $\lambda'(z) = \zeta$, $\mu'(\bar{x}) = \bar{\theta}$. From (2.2) we deduce that λ' is a quasi-isomorphism. We also have the obvious projection map

$$\text{proj}\colon (\hat{Q} \otimes \wedge(x, z, \bar{x}), \hat{D}) \xrightarrow{\simeq} (\hat{Q} \otimes \wedge z, \bar{\hat{D}}(z) = \hat{\gamma}_k).$$

Since $[\hat{\gamma}_k] = c_k(\xi)$ is the Euler class of the sphere bundle $S\xi$, we deduce that μ' is also a quasi-isomorphism.

By taking the pushout of the top line of diagram (2.4) along the quasi-isomorphism $\beta\colon \hat{Q} \to Q$ we get that

$$Q \ \rightarrowtail (Q \otimes \wedge(x, z), D) \xrightarrow{\text{proj}} (Q \otimes \wedge z, \bar{D})$$

is a CDGA model of the bottom line of diagram (2.4).

The CDGA model $(Q \otimes \wedge z, \bar{D})$ of $S\xi$ can be represented by two surjective quasi-isomorphisms

$$(Q \otimes \wedge z, \bar{D}) \xleftarrow[\simeq]{\sigma} (\hat{A}, \hat{d}_A) \xrightarrow[\simeq]{\sigma'} A_{PL}(S\xi).$$

We can lift the composite $l^*\rho\colon (\wedge Z, d) \to A_{PL}(S\xi)$ along the surjective quasi-isomorphism σ', and composing this lift with σ gives a map $\kappa\colon (\wedge Z, d) \to (Q \otimes \wedge z, \bar{D})$. The quasi-isomorphisms σ and σ' can be chosen so that we get the stated equivalence of line diagrams. This finishes the proof of the lemma. $\square$

Now we show how the model of the diagram in the previous lemma can be used to build a CDGA model of the blow-up. Let

$$A_0 \xrightarrow{\phi_0} A \xleftarrow{\phi_1} A_1$$

be a diagram of CDGA's which we denote by (ϕ_0, ϕ_1). One can factor ϕ_0 into a quasi-isomorphism followed by a surjection,

$$A_0 \xrightarrow{\simeq} A_0' \xrightarrow{\phi_0'} A \ .$$

The pullback of the diagram (ϕ_0', ϕ_1) is called the *homotopy pullback* of the diagram (ϕ_0, ϕ_1) (see [1, Section 10] or [6, Lemma 2.5]). Homotopy pullbacks are preserved (up to quasi-isomorphism) by quasi-isomorphisms of diagrams. Moreover a Mayer-Vietoris argument shows that a CDGA model of the homotopy pushout of a diagram of spaces $X_0 \xleftarrow{f_0} X \xrightarrow{f_1} X_1$ is given by the homotopy pullback of any CDGA model of the diagram $A_{PL}(X_0) \xrightarrow{f_0^*} A_{PL}(X) \xleftarrow{f_1^*} A_{PL}(X_1)$ (see [4, Section 15.15]).

Lemma 2.3. *If*

$$(Q \otimes \wedge(x,z), D) \xrightarrow{\text{proj}} (Q \otimes \wedge z, \bar{D}) \xleftarrow{\bar{\kappa}} (A, d) \ , \tag{2.5}$$

is a CDGA model of

$$P\xi \xleftarrow{q} S\xi \xrightarrow{l} B$$

then the pullback of (2.5) *is a CDGA model of the blow-up* $\widetilde{W}$.

Proof. This follows from the discussion above on homotopy pullbacks of CDGA. See [6, Lemma 2.7 and 6.5] for more details. □

3. Proof of the main proposition

This section is completely devoted to the proof of Proposition 1.2. For this we will compute CDGA models of the blow-ups $\widetilde{W}_0$ and $\widetilde{W}_1$. In this section we keep the notation of the previous section.

A product of spheres is formal, therefore a CDGA model of $V = S^2 \times S^7$ is given by

$$(Q, d_Q) = (H^*(V; \mathbb{Q}), 0) = \big(\wedge(u_2, v_7)/(u^2), 0\big) \, .$$

Since ξ is trivial we can take $\gamma_0 = 1$, $\gamma_1 = \gamma_2 = \gamma_3 = 0$ to represent the Chern classes. Denote by B_0 (resp. B_1) the closure of the complement of a tubular neighborhood of $j_0(V)$ (resp. $j_1(V)$) in W.

First we compute a CDGA model of $\widetilde{W}_1$. By (1.1) $B_1 \simeq S^5 \times S^7$ so it admits the Sullivan model $(\wedge(a_5, b_7), 0)$. Let

$$\kappa_1 \colon (\wedge(a_5, b_7), 0) \to (Q \otimes \wedge z_5, \bar{D} = 0) = \big(\wedge(u_2, v_7, z_5)/(u^2), 0\big)$$

be a model of $l_1 \colon S\xi \to B_1$ that satisfies the conclusion of Lemma 2.2. This map κ_1 is determined by the rational numbers α, β, β' such that

$$\begin{aligned} \kappa_1(a) &= \alpha \cdot z, \\ \kappa_1(b) &= \beta \cdot v + \beta' \cdot uz. \end{aligned}$$

Since the pushout of the diagram

$$V \xleftarrow{\pi' q} S\xi \xrightarrow{l_1} B_1$$

is the sphere $W = S^{15}$, a cohomology computation using the Mayer-Vietoris Theorem implies that $\alpha \neq 0$, $\beta' \neq 0$, and $\beta \neq 0$ (because $\kappa_1(ab) = (\alpha\beta) \cdot vz$). By

replacing the generator a by some multiple we can assume that $\alpha = 1$. Similarly we can assume that $\beta' = 1$ and $\beta = 1$ by replacing b and v by some multiples. Thus $\kappa_1(a) = z$ and $\kappa_1(b) = v + uz$. According to Lemma 2.3 a model of $\widetilde{W}_1$ is given by the pullback $\mathcal{B}_1$ of the diagram

$$(Q \otimes \wedge(x,z), D) \xrightarrow{\text{proj}} (Q \otimes \wedge z, \bar{D}) \xleftarrow{\kappa_1} (\wedge(a_5, b_7), 0)\ .$$

Since κ_1 is injective this pullback is isomorphic to a sub-CDGA of $(Q \otimes \wedge(x,z), D)$, namely

$$\mathcal{B}_1 \cong \left(\mathbb{Q} \cdot \{1, z, v + uz, vz\} \oplus x \cdot \left(\wedge(u,v,x,z)/(u^2)\right), D\right).$$

The cohomology algebra of $\mathcal{B}_1$ is given by the following table where $y = [ux]$, $t = [vx]$, and $w = [uvx]$:

0	2	4	6	9	11	13	15
1	x	x^2, y	xy	t	xt, w	$yt = xw$	$xyt = x^2w$

In other words $H^*(\widetilde{W}_1; \mathbb{Q})$ is isomorphic to the algebra

$$H := \frac{\wedge(x_2, y_4, t_9, w_{11})}{(x^3, y^2, x^2y, x^2t, yw, tw, xw - yt)}. \tag{3.1}$$

It is straightforward to compute the first generators of a minimal Sullivan model of $\mathcal{B}_1$:

$$(\wedge(x_2, y_4, z_5, b_7, e_7, \text{ generators of degrees } \geq 11);$$
$$dx = 0, dy = 0, dz = x^3, db = x^2y, de = y^2, \ldots),$$

where the generators x, y, z, b, and e are sent to x, ux, z, $v + uz$, and 0 in $\mathcal{B}_1$, respectively. Since there is no generator in degree 8 in the Sullivan model we deduce that $\pi_8(\widetilde{W}_1) \otimes \mathbb{Q} = 0$. The cohomology class t of degree 9 is represented in the Sullivan model by the cocycle $xb - zy$ which corresponds to the non-trivial Massey product $\langle x | x^2 | y \rangle$. Since $\widetilde{W}_1$ admits a non-trivial Massey product, it is not a formal space.

Now we compute a CDGA model of $\widetilde{W}_0$. By (1.2) $B_0 \simeq_{\mathbb{Q}} S^5 \vee S^7 \vee S^{12}$ which has the Sullivan model

$$(\wedge Z_0, d_0) := (\wedge(a_5, b_7, r_{11}, c_{12}, \text{ generators of degrees } \geq 15);$$
$$da = db = dc = 0, dr = ab, \ldots).$$

By Lemma 2.2 there is a map

$$\kappa_0 \colon (\wedge Z_0, d_0) \to (Q \otimes \wedge z, \bar{D})$$

that is a model of $l_0 \colon S\xi \to B_0$. Again a Mayer-Vietoris argument on the pushout $S^{15} = V \cup_{S\xi} B_0$ gives that, after a suitable change of generators, $\kappa_0(a) = z$, $\kappa_0(b) = uz$, $\kappa_0(c) = vz$, and κ_0 sends the other generators of the Sullivan model to 0. Therefore we can replace that model κ_0 by a smaller model

$$\bar{\kappa}_0 \colon (H^*(B_0), 0) = (\mathbb{Q} \cdot \{1, a_5, b_7, c_{12}\}, 0) \to \left(\frac{\wedge(u_2, v_7, z_5)}{(u^2)}, \bar{D} = 0\right)$$

with $\bar{\kappa}_0(a) = z$, $\bar{\kappa}_0(b) = uz$, and $\bar{\kappa}_0(c) = vz$. Lemma 2.3 implies that a CDGA model of $\widetilde{W}_0$ is given by the pullback of proj and $\bar{\kappa}_0$ which is

$$\mathcal{B}_0 \cong \left(\mathbb{Q} \cdot \{1, z, uz, vz\} \oplus x \cdot \left(\wedge(u,v,x,z)/(u^2)\right), D\right).$$

Recall the algebra H of (3.1). The following obvious projection map sending z, uz, and vz to zero, ux to y, vx to t, uvx to w, and x to x,

$$\mathcal{B}_0 \xrightarrow{\simeq} (H, 0)$$

is a quasi-isomorphism. This implies that $\widetilde{W}_0$ is formal, hence it does not admit any non-trivial Massey product. By computing the first generators of a minimal Sullivan model $(\wedge U, d)$ of $\mathcal{B}_0$ we find that $\dim U^8 = 1$, hence $\pi_8(\widetilde{W}_0) \otimes \mathbb{Q} \cong \mathbb{Q}$. This finishes the proof of Proposition 1.2.

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