

EXAMPLES OF RATIONAL HOMOTOPY TYPES OF BLOW-UPS

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(Communicated by Paul Goerss)

ABSTRACT. We give an example of two homotopic embeddings $j_0, j_1: V \hookrightarrow W$ of manifolds with isomorphic complex normal bundles but such that the blow-ups of W along j_0 and along j_1 have different rational homotopy types.

1. INTRODUCTION

Let $j: V \hookrightarrow W$ be an embedding of closed smooth manifolds and suppose that its normal bundle ξ is equipped with a structure of the complex vector bundle. The *blow-up* of W along j is a new manifold \widetilde{W} which is roughly obtained by replacing the submanifold V in W by the projective bundle $P\xi$ (see Definition 2.1 below or [3, Chapter 4, Section 6] or [8, end of Section 6.2] for more details). In [6] we have studied this blow-up construction from the rational homotopy theory viewpoint. There we proved the following

Theorem 1.1. *If $\dim W \geq 2 \dim V + 4$ and if V and W are simply connected, then the rational homotopy type of the blow-up \widetilde{W} of W along j depends only on the rational homotopy class of the map j and on the rational Chern classes of ξ .*

Actually in that paper we described explicitly the rational homotopy type of \widetilde{W} . Moreover the simple-connectivity hypotheses were also weaker than stated here.

A natural question is whether the conclusion of Theorem 1.1 still holds without the “stability” hypothesis, $\dim W \geq 2 \dim V + 4$. The goal of this note is to provide an explicit counterexample to such a generalization.

To state our main result we first describe two embeddings $j_0, j_1: V \hookrightarrow W$ which were previously studied in [7]. Let $W = S^{15}$ and $V = S^2 \times S^7$. The embedding j_0 is defined as the composite

$$j_0: V = S^2 \times S^7 \hookrightarrow \mathbb{R}^3 \times \mathbb{R}^8 = \mathbb{R}^{11} \subset \mathbb{R}^{11} \cup \{\infty\} \cong S^{11} \xrightarrow{\epsilon} S^{15} = W,$$

where the inclusions of the spheres in the euclidian spaces are the standard ones, $\mathbb{R}^{11} \cup \{\infty\} \cong S^{11}$ is the stereographic homeomorphism, and ϵ is the inclusion of a subequator. To describe the other embedding j_1 consider the Hopf fibration

$$S^7 \rightarrow S^{15} \xrightarrow{\pi} S^8.$$

Received by the editors August 21, 2003 and, in revised form, February 20, 2004.

2000 *Mathematics Subject Classification.* Primary 55P62, 14F35, 57R40.

Key words and phrases. Rational homotopy theory, blow-up.

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Let $\epsilon': S^2 \hookrightarrow S^8$ be the inclusion of a subequator. The restriction of π to $\epsilon'(S^2)$ is a trivial S^7 -bundle, so we can consider the inclusion

$$j_1: V = S^2 \times S^7 \cong \pi^{-1}(S^2) \hookrightarrow S^{15} = W.$$

These two embeddings are clearly homotopic since $[S^2 \times S^7, S^{15}] = \{*\}$. It is easy to check that their normal bundles are trivial real bundles of rank 6. On the other hand j_0 and j_1 are not isotopic because the complement of $j_1(V)$ in W is

$$(1.1) \quad W \setminus j_1(V) = \pi^{-1}(S^8 \setminus \epsilon'(S^2)) \simeq \pi^{-1}(S^5) \cong S^5 \times S^7,$$

but, since $j_0(V)$ is included in the equator of S^{15} , the complement $W \setminus j_0(V)$ has the homotopy type of a suspension. In particular this complement has the rational homotopy type of a wedge of spheres whose Betti numbers are the same, by Alexander duality, as the Betti numbers of $W \setminus j_1(V)$. Therefore

$$(1.2) \quad W \setminus j_0(V) \simeq_{\mathbb{Q}} S^5 \vee S^7 \vee S^{12}.$$

Now we can state the main result of this paper (the notion of formal space used in the statement is defined in Section 2):

Proposition 1.2. *Let $j_0, j_1: V \hookrightarrow W$ be the two embeddings described above with their normal bundles equipped with the structure of a trivial complex bundle of rank 3. Then the blow-ups \widetilde{W}_0 and \widetilde{W}_1 of W along, respectively, j_0 and j_1 have different rational homotopy types. More precisely the cohomology algebras $H^*(\widetilde{W}_0; \mathbb{Q})$ and $H^*(\widetilde{W}_1; \mathbb{Q})$ are isomorphic but*

- (i) *all Massey products are trivial in $H^*(\widetilde{W}_0; \mathbb{Q})$ but not in $H^*(\widetilde{W}_1; \mathbb{Q})$;*
- (ii) *\widetilde{W}_0 is formal and \widetilde{W}_1 is not formal;*
- (iii) *$\pi_8(\widetilde{W}_0) \otimes \mathbb{Q} \cong \mathbb{Q}$ and $\pi_8(\widetilde{W}_1) \otimes \mathbb{Q} = 0$.*

The proof of this proposition is the content of the last section.

2. THE RATIONAL HOMOTOPY TYPE OF A BLOW-UP

First we review the blow-up construction. Let $j: V \hookrightarrow W$ be an embedding of closed manifolds with a complex normal bundle ξ of rank k . Fix a hermitian metric on ξ and consider the associated sphere bundle, $\pi_0: S\xi \rightarrow V$, and projective bundle, $\pi': P\xi \rightarrow V$. Since $P\xi$ is the orbit space of $S\xi$ by the S^1 -action, we get an induced projection map $q: S\xi \rightarrow P\xi$ over V . The disk bundle $D\xi$ is diffeomorphic to some tubular neighborhood T of V in W and $S\xi \cong \partial T$. Let $B := \overline{W \setminus T}$ be the closure of the complement of T in W and consider the inclusion

$$l: S\xi \cong \partial T = \partial B \hookrightarrow B.$$

Definition 2.1. The *blow-up*, \widetilde{W} , of W along j is the pushout of the diagram

$$B \xleftarrow{l} S\xi \xrightarrow{q} P\xi.$$

It turns out that \widetilde{W} is again a closed manifold.

Next we review very briefly Sullivan's theory [9] for describing the rational homotopy type of spaces by certain algebraic objects, namely *commutative differential graded algebras* or CDGA's for short. For more details on this theory we refer the

reader to [2]. We denote by $A_{PL}: \text{Top} \rightarrow \text{CDGA}$ the contravariant functor of polynomial differential forms and if $f: X \rightarrow Y$ is a map we write $f^* := A_{PL}(f)$. By a *CDGA model* of a space X we mean a chain of quasi-isomorphisms of CDGA's

$$(2.1) \quad (A, d_A) \xleftarrow{\cong} (A_1, d_1) \xrightarrow{\cong} \cdots \xleftarrow{\cong} (A_r, d_r) \xrightarrow{\cong} A_{PL}(X).$$

More generally a *CDGA model* of a diagram of spaces is a diagram of CDGA's connected through a chain of quasi-isomorphisms to the diagram obtained by applying A_{PL} to the given diagram of spaces (see [6, Definition 2.9] for a more precise definition). A *Sullivan model* of a space X is a quasi-isomorphism of CDGA's

$$\rho: (\wedge Z, d) \xrightarrow{\cong} A_{PL}(X)$$

where $(\wedge Z, d)$ is a Sullivan algebra in the sense of [2, §12]. Of course a Sullivan model is a special case of a CDGA model. The main result of Sullivan's theory is that the rational homotopy type of a simply-connected space of finite type is determined by any of its CDGA models. A space X is called *formal* if $(H^*(X), 0)$ is a CDGA model of X .

Now we will describe the rational homotopy type of the blow-up. Let (Q, d_Q) be a CDGA model of the manifold V . Let $\gamma_i \in Q^{2i} \cap \ker d_Q$ be representatives of the Chern classes $c_i(\xi) \in H^{2i}(V; \mathbb{Q}) \cong H^{2i}(Q, d_Q)$ and assume that $\gamma_0 = 1$. Consider the relative Sullivan algebras ([2, §14])

$$\left(Q \otimes \wedge(x_2, z_{2k-1}), Dx = 0, Dz = \sum_{i=0}^k \gamma_i \otimes x^{k-i} \right)$$

and

$$(Q \otimes \wedge(z_{2k-1}), \bar{D}z = \gamma_k).$$

Let $\rho: (\wedge Z, d) \xrightarrow{\cong} A_{PL}(B)$ be a Sullivan model of the complement $B = \overline{W} \setminus \bar{T}$.

Lemma 2.2. *With the notation of this section there exists a CDGA map $\kappa: (\wedge Z, d) \rightarrow (Q \otimes \wedge z, \bar{D})$ such that the diagram*

$$(Q, d_Q) \xrightarrow{\iota} (Q \otimes \wedge(x, z), D) \xrightarrow{\text{proj}} (Q \otimes \wedge z, \bar{D}) \xleftarrow{\kappa} (\wedge Z, d)$$

is a CDGA-model of the diagram

$$V \xleftarrow{\pi'} P\xi \xleftarrow{q} S\xi \xrightarrow{l} B.$$

Proof. By definition of the Chern classes [5, Chapter 17, Section 2], there is an isomorphism of $H^*(V)$ -algebras

$$(2.2) \quad H^*(P\xi) \cong (H^*(V) \otimes \mathbb{Q}[x_2]) / \left(\sum_{i=0}^k c_i(\xi) \otimes x^{k-i} \right),$$

where x corresponds to the canonical class $a_\xi := -c_1(\lambda_\xi) \in H^2(P\xi)$ where λ_ξ is the tautological complex line bundle over $P\xi$. Let $\theta \in A_{PL}^2(P\xi) \cap \ker d$ be a cocycle representing a_ξ . It is easy to check that $q^*(a_\xi) = 0$ in $H^2(S\xi)$ because $q^*\lambda_\xi$ is a trivial line bundle (see [6, Lemma 5.1]). Therefore there exists $\hat{\theta} \in A_{PL}^1(S\xi)$ such that $d\hat{\theta} = q^*(\theta)$.

Since (Q, d_Q) is a model of V , by [6, Lemma 2.6] there exists a cofibrant CDGA (\hat{Q}, \hat{d}_Q) and surjective quasi-isomorphisms β and β' ,

$$(Q, d_Q) \xleftarrow[\simeq]{\beta} (\hat{Q}, \hat{d}_Q) \xrightarrow[\simeq]{\beta'} A_{PL}(V).$$

Let $\hat{\gamma}_0 = 1$ and since β is surjective there exist cocycles $\hat{\gamma}_i \in \hat{Q}^{2i} \cap \ker \hat{d}_Q$ for $i \geq 1$ such that $\beta(\hat{\gamma}_i) = \gamma_i$.

By (2.2) the cocycle

$$(2.3) \quad \sum_{i=0}^k \pi'^*(\beta'(\hat{\gamma}_i)) \cdot \theta^{k-i} \in A_{PL}^{2k}(P\xi)$$

represents the zero cohomology class. Therefore there exists $\zeta \in A_{PL}^{2k-1}(P\xi)$ such that $d\zeta$ is the cocycle of (2.3).

Define the following commutative diagram of CDGA's:

$$(2.4) \quad \begin{array}{ccccc} (\hat{Q}, \hat{d}_Q) & \xrightarrow{\quad} & (\hat{Q} \otimes \wedge(x_2, z_{2k-1}), \hat{D}) & \xrightarrow{\quad} & (\hat{Q} \otimes \wedge(x_2, z_{2k-1}, \bar{x}_1), \hat{D}) \\ \downarrow \beta' & & \downarrow \lambda' & & \downarrow \mu' \\ A_{PL}(V) & \xrightarrow{\pi'^*} & A_{PL}(P\xi) & \xrightarrow{q^*} & A_{PL}(S\xi) \end{array}$$

by $\hat{D}(x) = 0$, $\hat{D}(z) = \sum_{i=0}^k \hat{\gamma}_i \otimes x^{k-i}$, $\hat{D}(\bar{x}) = x$, $\lambda'(x) = \theta$, $\lambda'(z) = \zeta$, $\mu'(\bar{x}) = \bar{\theta}$. From (2.2) we deduce that λ' is a quasi-isomorphism. We also have the obvious projection map

$$\text{proj}: (\hat{Q} \otimes \wedge(x, z, \bar{x}), \hat{D}) \xrightarrow{\simeq} (\hat{Q} \otimes \wedge z, \bar{D}(z) = \hat{\gamma}_k).$$

Since $[\hat{\gamma}_k] = c_k(\xi)$ is the Euler class of the sphere bundle $S\xi$, we deduce that μ' is also a quasi-isomorphism.

By taking the pushout of the top line of diagram (2.4) along the quasi-isomorphism $\beta: \hat{Q} \rightarrow Q$ we get that

$$Q \xrightarrow{\quad} (Q \otimes \wedge(x, z), D) \xrightarrow{\text{proj}} (Q \otimes \wedge z, \bar{D})$$

is a CDGA model of the bottom line of diagram (2.4).

The CDGA model $(Q \otimes \wedge z, \bar{D})$ of $S\xi$ can be represented by two surjective quasi-isomorphisms

$$(Q \otimes \wedge z, \bar{D}) \xleftarrow[\simeq]{\sigma} (\hat{A}, \hat{d}_A) \xrightarrow[\simeq]{\sigma'} A_{PL}(S\xi).$$

We can lift the composite $l^*\rho: (\wedge Z, d) \rightarrow A_{PL}(S\xi)$ along the surjective quasi-isomorphism σ' , and composing this lift with σ gives a map $\kappa: (\wedge Z, d) \rightarrow (Q \otimes \wedge z, \bar{D})$. The quasi-isomorphisms σ and σ' can be chosen so that we get the stated equivalence of line diagrams. This finishes the proof of the lemma. \square

Now we show how the model of the diagram in the previous lemma can be used to build a CDGA model of the blow-up. Let

$$A_0 \xrightarrow{\phi_0} A \xleftarrow{\phi_1} A_1$$

be a diagram of CDGA's which we denote by (ϕ_0, ϕ_1) . One can factor ϕ_0 into a quasi-isomorphism followed by a surjection,

$$A_0 \xrightarrow{\cong} A'_0 \xrightarrow{\phi'_0} A .$$

The pullback of the diagram (ϕ'_0, ϕ_1) is called the *homotopy pullback* of the diagram (ϕ_0, ϕ_1) (see [1, Section 10] or [6, Lemma 2.5]). Homotopy pullbacks are preserved (up to quasi-isomorphism) by quasi-isomorphisms of diagrams. Moreover a Mayer-Vietoris argument shows that a CDGA model of the homotopy pushout of a diagram of spaces $X_0 \xleftarrow{f_0} X \xrightarrow{f_1} X_1$ is given by the homotopy pullback of any CDGA model of the diagram $A_{PL}(X_0) \xrightarrow{f_0^*} A_{PL}(X) \xleftarrow{f_1^*} A_{PL}(X_1)$ (see [4, Section 15.15]).

Lemma 2.3. *If*

$$(2.5) \quad (Q \otimes \wedge(x, z), D) \xrightarrow{\text{proj}} (Q \otimes \wedge z, \bar{D}) \xleftarrow{\bar{\kappa}} (A, d) ,$$

is a CDGA model of

$$P\xi \xleftarrow{q} S\xi \xrightarrow{l} B$$

then the pullback of (2.5) is a CDGA model of the blow-up \widetilde{W} .

Proof. This follows from the discussion above on homotopy pullbacks of CDGA. See [6, Lemma 2.7 and 6.5] for more details. \square

3. PROOF OF THE MAIN PROPOSITION

This section is completely devoted to the proof of Proposition 1.2. For this we will compute CDGA models of the blow-ups \widetilde{W}_0 and \widetilde{W}_1 . In this section we keep the notation of the previous section.

A product of spheres is formal, therefore a CDGA model of $V = S^2 \times S^7$ is given by

$$(Q, d_Q) = (H^*(V; \mathbb{Q}), 0) = (\wedge(u_2, v_7)/(u^2), 0) .$$

Since ξ is trivial we can take $\gamma_0 = 1, \gamma_1 = \gamma_2 = \gamma_3 = 0$ to represent the Chern classes. Denote by B_0 (resp. B_1) the closure of the complement of a tubular neighborhood of $j_0(V)$ (resp. $j_1(V)$) in W .

First we compute a CDGA model of \widetilde{W}_1 . By (1.1) $B_1 \simeq S^5 \times S^7$ so it admits the Sullivan model $(\wedge(a_5, b_7), 0)$. Let

$$\kappa_1 : (\wedge(a_5, b_7), 0) \rightarrow (Q \otimes \wedge z_5, \bar{D} = 0) = (\wedge(u_2, v_7, z_5)/(u^2), 0)$$

be a model of $l_1 : S\xi \rightarrow B_1$ that satisfies the conclusion of Lemma 2.2. This map κ_1 is determined by the rational numbers α, β, β' such that

$$\begin{aligned} \kappa_1(a) &= \alpha \cdot z, \\ \kappa_1(b) &= \beta \cdot v + \beta' \cdot uz. \end{aligned}$$

Since the pushout of the diagram

$$V \xleftarrow{\pi'^q} S\xi \xrightarrow{l_1} B_1$$

is the sphere $W = S^{15}$, a cohomology computation using the Mayer-Vietoris Theorem implies that $\alpha \neq 0, \beta' \neq 0$, and $\beta \neq 0$ (because $\kappa_1(ab) = (\alpha\beta) \cdot vz$). By

replacing the generator a by some multiple we can assume that $\alpha = 1$. Similarly we can assume that $\beta' = 1$ and $\beta = 1$ by replacing b and v by some multiples. Thus $\kappa_1(a) = z$ and $\kappa_1(b) = v + uz$. According to Lemma 2.3 a model of \widetilde{W}_1 is given by the pullback \mathcal{B}_1 of the diagram

$$(Q \otimes \wedge(x, z), D) \xrightarrow{\text{proj}} (Q \otimes \wedge z, \bar{D}) \xleftarrow{\kappa_1} (\wedge(a_5, b_7), 0).$$

Since κ_1 is injective this pullback is isomorphic to a sub-CDGA of $(Q \otimes \wedge(x, z), D)$, namely

$$\mathcal{B}_1 \cong (\mathbb{Q} \cdot \{1, z, v + uz, vz\} \oplus x \cdot (\wedge(u, v, x, z)/(u^2)), D).$$

The cohomology algebra of \mathcal{B}_1 is given by the following table where $y = [ux]$, $t = [vx]$, and $w = [uvx]$:

0	2	4	6	9	11	13	15
1	x	x^2, y	xy	t	xt, w	$yt = xw$	$xyt = x^2w$

In other words $H^*(\widetilde{W}_1; \mathbb{Q})$ is isomorphic to the algebra

$$(3.1) \quad H := \frac{\wedge(x_2, y_4, t_9, w_{11})}{(x^3, y^2, x^2y, x^2t, yw, tw, xw - yt)}.$$

It is straightforward to compute the first generators of a minimal Sullivan model of \mathcal{B}_1 :

$$\begin{aligned} &(\wedge(x_2, y_4, z_5, b_7, e_7, \text{generators of degrees } \geq 11); \\ & \quad dx = 0, dy = 0, dz = x^3, db = x^2y, de = y^2, \dots), \end{aligned}$$

where the generators x, y, z, b , and e are sent to $x, ux, z, v + uz$, and 0 in \mathcal{B}_1 , respectively. Since there is no generator in degree 8 in the Sullivan model we deduce that $\pi_8(\widetilde{W}_1) \otimes \mathbb{Q} = 0$. The cohomology class t of degree 9 is represented in the Sullivan model by the cocycle $xb - zy$ which corresponds to the non-trivial Massey product $\langle x|x^2|y \rangle$. Since \widetilde{W}_1 admits a non-trivial Massey product, it is not a formal space.

Now we compute a CDGA model of \widetilde{W}_0 . By (1.2) $B_0 \simeq_{\mathbb{Q}} S^5 \vee S^7 \vee S^{12}$ which has the Sullivan model

$$\begin{aligned} &(\wedge Z_0, d_0) := (\wedge(a_5, b_7, r_{11}, c_{12}, \text{generators of degrees } \geq 15); \\ & \quad da = db = dc = 0, dr = ab, \dots). \end{aligned}$$

By Lemma 2.2 there is a map

$$\kappa_0: (\wedge Z_0, d_0) \rightarrow (Q \otimes \wedge z, \bar{D})$$

that is a model of $l_0: S\xi \rightarrow B_0$. Again a Mayer-Vietoris argument on the pushout $S^{15} = V \cup_{S\xi} B_0$ gives that, after a suitable change of generators, $\kappa_0(a) = z$, $\kappa_0(b) = uz$, $\kappa_0(c) = vz$, and κ_0 sends the other generators of the Sullivan model to 0. Therefore we can replace that model κ_0 by a smaller model

$$\bar{\kappa}_0: (H^*(B_0), 0) = (\mathbb{Q} \cdot \{1, a_5, b_7, c_{12}\}, 0) \rightarrow \left(\frac{\wedge(u_2, v_7, z_5)}{(u^2)}, \bar{D} = 0 \right)$$

with $\bar{\kappa}_0(a) = z$, $\bar{\kappa}_0(b) = uz$, and $\bar{\kappa}_0(c) = vz$. Lemma 2.3 implies that a CDGA model of \widetilde{W}_0 is given by the pullback of proj and $\bar{\kappa}_0$ which is

$$\mathcal{B}_0 \cong (\mathbb{Q} \cdot \{1, z, uz, vz\} \oplus x \cdot (\wedge(u, v, x, z)/(u^2)), D).$$

Recall the algebra H of (3.1). The following obvious projection map sending z , uz , and vz to zero, ux to y , vx to t , vwx to w , and x to x ,

$$\mathcal{B}_0 \xrightarrow{\cong} (H, 0)$$

is a quasi-isomorphism. This implies that \widetilde{W}_0 is formal, hence it does not admit any non-trivial Massey product. By computing the first generators of a minimal Sullivan model $(\wedge U, d)$ of \mathcal{B}_0 we find that $\dim U^8 = 1$, hence $\pi_8(\widetilde{W}_0) \otimes \mathbb{Q} \cong \mathbb{Q}$. This finishes the proof of Proposition 1.2.

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