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MORITA EQUIVALENCE FOR QUANTUM HEISENBERG MANIFOLDS

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ABSTRACT. We discuss Morita equivalence within the family $\{D^c_{\mu\nu} : c \in \mathbb{Z},\ c > 0,\ \mu, \nu \in \mathbb{R}\}$ of quantum Heisenberg manifolds. Morita equivalence classes are described in terms of the parameters μ, ν and the rank of the free abelian group $G_{\mu\nu} = 2\mu\mathbb{Z} + 2\nu\mathbb{Z} + \mathbb{Z}$ associated to the C^*-algebra $D^c_{\mu\nu}$.

INTRODUCTION

Quantum Heisenberg manifolds $\{D^c_{\mu\nu} : c \in \mathbb{Z}, c > 0,\ \mu, \nu \in \mathbb{R}\}$ were constructed by Rieffel in [Rf4] as a quantization deformation of certain homogeneous spaces H/N_c, H being the Heisenberg group.

It was shown in [Ab1, 3.4] that $K_0(D^c_{\mu\nu}) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_c$, which implies that $D^c_{\mu\nu}$ and $D^{c'}_{\mu'\nu'}$ are not isomorphic unless $c = c'$. Besides, $D^c_{\mu\nu}$ and $D^c_{\mu'\nu'}$ are isomorphic when $(2\mu, 2\nu)$ and $(2\mu', 2\nu')$ belong to the same orbit under the usual action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{T}^2$ ([AE, Theorem 2.2]; see also [Ab2, 3.3]). The range of traces on $D^c_{\mu\nu}$ was discussed in [Ab2], where it was shown that the range of the homomorphism induced on $K_0(D^c_{\mu\nu})$ by any tracial state on $D^c_{\mu\nu}$ has range $G_{\mu\nu} = \mathbb{Z} + 2\mu\mathbb{Z} + 2\nu\mathbb{Z}$. As a consequence ([Ab2, 3.17]), the isomorphism condition stated above turns out to be necessary when the rank of $G_{\mu\nu}$ is either 1 or 3. Rieffel showed in [Rf4] that $D^c_{\mu\nu}$ is simple if and only if $\{1, \mu, \nu\}$ is linearly independent over the field of rational numbers (i.e rank $G_{\mu\nu} = 3$). It might be interesting to know whether in this case the classification can be made by means of the results of Elliott and Gong ([EG]).

The quantum Heisenberg manifold $D^c_{\mu\nu}$ was described in [AEE] as a crossed product by a Hilbert C^*-bimodule. In order to discuss Morita equivalence within this family, we adapt to this setting some of the techniques employed in the analogous discussion for non-commutative tori ([Rf3]) and Heisenberg C^*-algebras ([Pa2]). Thus we generalize in Section 1 Green's result (discussed by Rieffel in [Rf2, Situation 10]) on the Morita equivalence of the crossed products $C_0(M/K) \rtimes H$ and $C_0(M/H) \rtimes K$, for free and proper commuting actions on a locally compact space M. This result provides the main tool used to discuss Morita equivalence for quantum Heisenberg manifolds (Section 2).

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1. Morita equivalence of crossed products by certain Hilbert C^*-bimodules over commutative C^*-algebras

For a Hilbert C^*-bimodule X over a C^*-algebra A, the crossed product $A \rtimes X$ was introduced in [AEE] (see also [Pi]) as the universal C^*-algebra for which there exist a $*$-homomorphism $i_A : A \longrightarrow A \rtimes X$ and a continuous linear map $i_X : X \longrightarrow A \rtimes X$ such that

$$i_X(ax) = i_A(a)i_X(x),\ i_A(\langle x, y\rangle_L) = i_X(x)i_X(y)^*,$$
$$i_X(xa) = i_X(x)i_A(a),\ i_A(\langle x, y\rangle_R) = i_X(x)^*i_X(y).$$

The crossed product $A \rtimes X$ carries a dual action δ of S^1, defined by $\delta_z(i_A(a)) = i_A(a)$, $\delta_z(i_X(x)) = zi_X(x)$, for $a \in A$, $x \in X$ and $z \in S^1$. Moreover, if a C^*-algebra B carries an action δ of S^1 such that B is generated as a C^*-algebra by the fixed point subalgebra $B_0 = \{b \in B : \delta_z(b) = b\ \forall z \in S^1\}$ and the first spectral subspace $B_1 = \{b \in B : \delta_z(b) = zb\ \forall z \in S^1\}$, then B is isomorphic to $B_0 \rtimes B_1$ (where B_1 has the obvious Hilbert C^*-bimodule structure over B_0), via an isomorphism that takes the action δ into the dual action.

If X is an A-Hilbert C^*-bimodule and $\alpha \in \mathrm{Aut}(A)$, we denote by X_α the Hilbert C^*-bimodule over A obtained by leaving unchanged the left structure, and by setting

$$x \cdot_{X_\alpha} a := x\alpha(a), \langle x, y\rangle_R^{X_\alpha} := \alpha^{-1}(\langle x, y\rangle_R),$$

where the undecorated notation refers to the original right structure of X.

For $\alpha \in \mathrm{Aut}(A)$ and the usual A-Hilbert C^*-bimodule structure on A, the crossed product $A \rtimes A_\alpha$ is easily checked to be the usual crossed product $A \rtimes_\alpha \mathbb{Z}$.

Definition 1.1. Given a proper action α of $\mathbb{Z}$ on a locally compact Hausdorff space M and a unitary $u \in C_b(M)$, let $X^{\alpha,u}$ denote the set of functions $f \in C_b(M)$ satisfying $f = u\alpha(f)$, and such that the map $x \mapsto |f(x)|$, which is constant on α-orbits, belongs to $C_0(M/\alpha)$. Then $X^{\alpha,u}$ is a Hilbert C^*-bimodule over $C_0(M/\alpha)$ for pointwise multiplication on the left and the right, and inner products given by $\langle f, g\rangle_L = f\overline{g}$, $\langle f, g\rangle_R = \overline{f}g$.

Proposition 1.2. *Let α and β be free and proper commuting actions of $\mathbb{Z}$ on a locally compact Hausdorff space M, and let u be a unitary in $C_b(M)$. Then the C^*-algebras $C_0(M/\alpha) \rtimes X_\beta^{\alpha,u}$ and $C_0(M/\beta) \rtimes X_\alpha^{\beta,u^*}$ are Morita equivalent.*

Proof. Let $U : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathcal{U}(C_b(M))$ be given by

$$U(n,k) = \begin{cases} 1 & \text{if either } n = 0 \text{ or } k = 0, \\ \prod_{i \in S_k, j \in S_n} \alpha^i\beta^j(u), & \text{for } n, k > 0, \\ \prod_{i \in S_k, j \in S_n} \alpha^i\beta^j(u^*), & \text{for either } n \text{ or } k < 0, \text{ and } nk \neq 0, \end{cases}$$

where $S_l = \{0, 1, \dots l-1\}$ if $l > 0$ and $S_l = \{-1, -2, \dots l\}$ if $l < 0$. Straightforward computations show that $U(m+n, k) = U(m,k)\beta^m(U(n,k))$, and $U(n, k+l) = U(n,k)\alpha^k(U(n,l))$.

Consider the proper actions γ^α and γ^β of $\mathbb{Z}$ on $C_0(M) \rtimes_\beta \mathbb{Z}$ and $C_0(M) \rtimes_\alpha \mathbb{Z}$, respectively, given by

$$[\gamma_k^\alpha(\phi)](n) = U(n,k)\alpha^k[\phi(n)] \text{ and } [\gamma_n^\beta(\psi)](k) = U^*(n,k)\beta^n[\psi(k)],$$

for $\phi \in C_c(\mathbb{Z}, C_0(M)) \subset C_0(M) \rtimes_\beta \mathbb{Z}$ and $\psi \in C_c(\mathbb{Z}, C_0(M)) \subset C_0(M) \rtimes_\alpha \mathbb{Z}$.

These two actions correspond, respectively, to $\gamma^{\alpha,U}$ and γ^{β,U^*} in [Ab1, Propositions 1.2 and 2.1]. By virtue of [Ab1, Theorem 2.12], the generalized fixed-point algebras, in the sense of [Rf5, Definition 1.4], D^α and D^β of $C_0(M) \rtimes_\beta \mathbb{Z}$ and

$C_0(M) \rtimes_\alpha \mathbb{Z}$ under the actions γ^α and γ^β, respectively, are Morita equivalent. The result will then be proved once we show that $D^\alpha \cong C_0(M/\alpha) \rtimes X_\beta^{\alpha,u}$ and $D^\beta \cong C_0(M/\beta) \rtimes X_\alpha^{\beta,u^*}$.

Recall from [Ab1, Proposition 2.1] that D^α is defined to be the closed span in $\mathcal{M}(C_0(M) \rtimes_\beta \mathbb{Z})$ of the set $\{P_\alpha(\phi^* * \psi) : \phi, \psi \in C_c(\mathbb{Z} \times M)\}$, where

$$P_\alpha(\phi)(x,n) = \sum_{k\in\mathbb{Z}} [\gamma_k^\alpha(\phi)](x,n),$$

for $\phi \in C_c(\mathbb{Z} \times M) \subset C_0(M) \rtimes_\beta \mathbb{Z},\ x \in M$, and $n \in \mathbb{Z}$.

The C^*-algebra D^α can also be described ([Ab1, Proposition 2.8]) as the closure in $\mathcal{M}(C_0(M) \rtimes_\beta \mathbb{Z})$ of the $*$-subalgebra $C^\alpha = \{F \in C_c(\mathbb{Z}, C_b(M)) : \gamma^\alpha(F) = F$ and $\pi_\alpha(\text{supp } F(n))$ is precompact for all $n \in \mathbb{Z}\}$, where π_α denotes the canonical projection $\pi_\alpha : M \longrightarrow M/\alpha$.

Now, since C^α is contained in $C_b(M) \rtimes_\beta \mathbb{Z}$, which is closed in $\mathcal{M}(C_0(M) \rtimes_\beta \mathbb{Z})$, so is D^α. Moreover, the C^*-algebra D^α is invariant under the dual action $\hat{\beta}$ of $\mathbb{T}$ on $C_b(M) \rtimes_\beta \mathbb{Z}$:

$$\begin{aligned}[]
[\gamma^\alpha(\hat{\beta}_z F)](n,x) &= U(n,1)(x)(\hat{\beta}_z(F))(n,\alpha^{-1}x) \\
&= U(n,1)(x) z^n F(n,\alpha^{-1}x) \\
&= z^n F(n,x) \\
&= (\hat{\beta}_z F)(n,x),
\end{aligned}$$

for $F \in C^\alpha$, $x \in M$, $n \in \mathbb{Z}$, and $z \in \mathbb{T}$. Besides, supp $(\hat{\beta}_z(F)(n)) =$ supp $F(n)$ for all $n \in \mathbb{Z}$, so $\hat{\beta}_z(F) \in C^\alpha$ for all $z \in \mathbb{T}$.

We next show that the action $\hat{\beta}$ on D^α is semi-saturated. That is, that, as a C^*-algebra, D^α is generated by the fixed-point subalgebra D_0 and the first spectral subspace $D_1 = \{d \in D^\alpha : \hat{\beta}_z(d) = zd\ \forall z \in \mathbb{T}\}$ for the restriction of the dual action $\hat{\beta}$.

Since the maps $P_i : D^\alpha \longrightarrow D_i$ given by $P_i(a) = \int_{\mathbb{T}} z^{-i} \hat{\beta}_z(a)\, dz$ are surjective contractions, D_i is the closure of $P_i(C^\alpha)$. Now, for $i = 0, 1$, $C_i = C^\alpha \cap F_i$, and $D_i = D^\alpha \cap F_i$, where F_0 and F_1 are, respectively, the fixed-point subalgebra and the first spectral subspace of $C_b(M) \rtimes_\beta \mathbb{Z}$, which are known to be the δ_i-maps; that is, $F_i = \{F \in C_c(\mathbb{Z}, C_b(M)) : \text{supp } F = \{i\}\}$.

Note that

$$C^\alpha \cap F_0 = \{f\delta_0 : f \in C_b(M) : \pi_\alpha(\text{supp } f) \text{ is precompact and } f = \alpha(f)\}$$

can be identified with $C_c(M/\alpha)$ via $f\delta_0 \mapsto \tilde{f}$, where $\tilde{f} \circ \pi_\alpha = f$, and that this map extends to a $*$-isomorphism between D_0 and $C_0(M/\alpha)$.

Now, D_1 is a Hilbert C^*-bimodule over D_0 for

(1a) $$(f\delta_0) * (g\delta_1) = (fg)\delta_1 \text{ and } (g\delta_1) * (f\delta_0) = (g\beta(f))\delta_1,$$

(1b) $$\langle f\delta_1, g\delta_1 \rangle_L = (f\delta_1) * (g\delta_1)^* = (f\bar{g})\delta_0,$$

(1c) $$\langle f\delta_1, g\delta_1 \rangle_R = (f\delta_1)^* * (g\delta_1) = (\beta^{-1}(\bar{f}g))\delta_0.$$

Note that D_1 is full on the left (and on the right, by a similar argument) as a Hilbert C^*-bimodule over $C_0(M/\alpha)$. For $\langle D_1, D_1 \rangle_L$, the closed linear span in $C_0(M/\alpha) \cong D_0$ of the set $\{\langle f\delta_1, g\delta_1 \rangle_L : f\delta_1, g\delta_1 \in D_1\}$ is a closed ideal of $C_0(M/\alpha)$. Therefore, unless $\langle D_1, D_1 \rangle_L = C_0(M/\alpha)$, there exists $x_0 \in M$ such that $f(x_0) = 0$ for all $f\delta_1 \in D_1$.

Now, given $x_0 \in M$, we can choose ([Rf2, Situation 10]) a neighborhood U of x_0 such that $U \cap \alpha^k(U) = \emptyset$ for $k \neq 0$. Let $g \in C_c(M)^+$ be such that supp $g \subset U$ and $g(x_0) = 1$.

Then

$$[P_\alpha\big((g^{1/2}\delta_0)^* * (g^{1/2}\delta_1)\big)](x,n) = (P_\alpha(g\delta_1))(x,n) = \Big(\sum_k U(1,k)(x) g(\alpha^{-k}(x))\Big)\delta_1(n),$$

so $P_\alpha\big((g^{1/2}\delta_0)^* * (g^{1/2}\delta_1)\big) \in D_1$ and equals 1 at $(x_0, 1)$.

In order to prove that $C^\alpha \subset C^*(D_0, D_1)$, it suffices to show that $f\delta_k \in C^*(D_0, D_1)$ for $f\delta_k \in C^\alpha$, $k \in \mathbb{Z}$. Since C^α is closed under involution, we may assume that $k \geq 0$. We show this fact, which clearly holds for $k = 0$ and $k = 1$, by induction on k.

If $f\delta_k \in C^\alpha$ and $\epsilon > 0$, since $\pi_\alpha(\text{supp } f)$ is precompact in M/α, and D_1 is full over $C_0(M/\alpha)$, we can find $\phi_i, \psi_i \in D_1, i = 1, \ldots, p$, such that

$$\| \sum_i (\phi_i * \psi_i^*) * f\delta_1 - f\delta_1 \|_{D^\alpha} = \| \sum_i \langle \phi_i, \psi_i \rangle_L f - f \|_{C_b(M)} < \epsilon.$$

Now, since ϕ_i and $\psi_i^* * f$ belong to $C^*(D_0, D_1)$ for $i = 1, \ldots, p$, so does f. This shows that $D^\alpha = C^*(D_0, D_1)$ and, consequently, by [AEE, Theorem 3.1], that $D^\alpha \cong D_0 \rtimes D_1$.

It only remains to note now that $D_0 \rtimes D_1 \cong C_0(M/\alpha) \rtimes X_\beta^{\alpha,u}$. As noted above, D_0 is isomorphic to $C_0(M/\alpha)$. On the other hand, the map $f\delta_1 \mapsto f$ takes $C^\alpha \cap F_1$ to $X_\beta^{\alpha,u}$. By keeping track of the formulae in (1a)–(1c), one easily checks that the map is an isometry, so it extends to an isometry from D_1 to $X_\beta^{\alpha,u}$, which is onto because its image contains the dense set

$$X_0^{\alpha,u} = \{f \in X_\beta^{\alpha,u}\text{: the map } x \mapsto |f(x)| \text{ is compactly supported on } M/\alpha\}.$$

(Note that $X_0^{\alpha,u}$ is dense in $X_\beta^{\alpha,u}$, because, if $\{e_\lambda\}$ is an approximate identity for $C_c(M/\alpha)$, then $e_\lambda f$ converges to f for all $f \in X_\beta^{\alpha,u}$.)

This shows that D^α is isomorphic to $C_0(M/\alpha) \rtimes X_\beta^{\alpha,u}$. Analogously, D^β is isomorphic to $C_0(M/\beta) \rtimes X_\alpha^{\beta,u^*}$. □

2. Morita equivalence for quantum Heisenberg manifolds

In [AEE] (see also [AE, 2]) the quantum Heisenberg manifold $D_{\mu\nu}^c$ was shown to be the crossed product of $C(\mathbb{T}^2)$, the C^*-algebra of continuous functions on the torus, by the Hilbert C^*-bimodule $M_{\alpha_{\mu\nu}}^c$, where $\alpha_{\mu\nu}(x,y) = (x + 2\mu, y + 2\nu)$, and

$$M^c = \{f \in C_b(\mathbb{R} \times \mathbb{T}) : f(x+1, y) = e^{-2\pi i c y} f(x,y)\}$$

is the Hilbert C^*-bimodule obtained by letting $C(\mathbb{T}^2)$ act by pointwise product, and by defining the inner products $\langle f, g\rangle_L = f\overline{g}$, $\langle f, g\rangle_R = \overline{f}g$.

Remark 2.1. The C^*-algebras $D_{\mu\nu}^c$ and $D_{\mu'\nu'}^c$ are isomorphic when the projections of $(2\mu, 2\nu)$ and $(2\mu', 2\nu')$ on the torus are in the same orbit under the usual action of $\mathrm{GL}_2(\mathbb{Z})$ ([AE, Theorem 2.2], see also [Ab2, Remark 3.3]).

Proposition 2.2. *Let $\mu \neq 0$. Then $D_{\mu\nu}^c$ and $D_{\frac{1}{4\mu}, \frac{\nu}{2\mu}}^c$ are Morita equivalent.*

Proof. We follow the lines of [Rf3, 1.1] and apply Proposition 1.2 to the following setting: α and β consist of translation on $\mathbb{R}\times\mathbb{T}$ by $(\frac{1}{2\mu},0)$ and $(1,2\nu)$, respectively, and $u\in C_b(\mathbb{R}\times\mathbb{T})$ is given by $u(x,y)=e(-cy)$, where $\mathbb{T}$ is viewed as $\mathbb{R}/\mathbb{Z}$ and, for a real number h, $e(h)=e^{2\pi ih}$.

Then, by Proposition 1.2, $C((\mathbb{R}\times\mathbb{T})/\alpha)\rtimes X_\beta^{\alpha,u}$ and $C((\mathbb{R}\times\mathbb{T})/\beta)\rtimes X_\alpha^{\beta,u^*}$ are Morita equivalent, where

$$X^{\alpha,u}=\{F\in C_b(\mathbb{R}\times\mathbb{T}):\ F(x-\frac{1}{2\mu},y)=e(cy)F(x,y)\}\text{ and}$$

$$\begin{aligned}X^{\beta,u^*}&=\{F\in C_b(\mathbb{R}\times\mathbb{T}):\ F(x-1,y-2\nu)=e(-cy)F(x,y)\}\\&=\{F\in C_b(\mathbb{R}\times\mathbb{T}):\ F(x+1,y+2\nu)=e(c(y+2\nu))F(x,y)\}.\end{aligned}$$

Let $H_\alpha: C(\mathbb{T}^2)\longrightarrow C((\mathbb{R}\times\mathbb{T})/\alpha)$ and $H_\beta: C(\mathbb{T}^2)\longrightarrow C((\mathbb{R}\times\mathbb{T})/\beta)$ be the isomorphisms given by

$$(H_\alpha\phi)(x,y)=\phi(2\mu x,y),\ (H_\beta\phi)(x,y)=\phi(x,2\nu x-y),$$

and, for $(\mu',\nu')=(\frac{1}{4\mu},\frac{\nu}{2\mu})$, set

$$J_\alpha: M^c_{\alpha_{\mu\nu}}\longrightarrow X_\beta^{\alpha,u}\text{ and } J_\beta: M^c_{\alpha_{\mu'\nu'}}\longrightarrow X_\alpha^{\beta,u^*},$$

$$(J_\alpha f)(x,y)=f(2\mu x,y),\ (J_\beta f)(x,y)=e(cx(x+1)\nu)f(x,2\nu x-y).$$

Note that

$$(J_\alpha f)(x-\frac{1}{2\mu},y)=f(2\mu x-1,y)=e(cy)(J_\alpha f)(x,y)$$

and

$$\begin{aligned}(J_\beta f)(x+1,y+2\nu)&=e(c(x+1)(x+2)\nu)f(x+1,2\nu x-y)\\&=e(c(x+1)(x+2)\nu)e(-c(2\nu x-y))f(x,2\nu x-y)\\&=e(c(y+2\nu))(J_\beta f)(x,y),\end{aligned}$$

so the definitions make sense.

For $i=\alpha,\beta$, it is easily checked that J_i is a bijection and that, for $\phi\in C(\mathbb{T}^2)$, $f,g\in M^c$:

$$J_i(\phi\cdot f)=H_i(\phi)\cdot J_i(f),\ J_i(f\cdot\phi)=J_i(f)\cdot H_i(\phi),$$
$$\langle J_if,J_ig\rangle_L=H_i(\langle f,g\rangle_L),\ \langle J_if,J_ig\rangle_R=H_i(\langle f,g\rangle_R).$$

This shows that $D^c_{\mu\nu}=C(\mathbb{T}^2)\rtimes M^c_{\alpha_{\mu\nu}}$ and $D^c_{\mu'\nu'}=C(\mathbb{T}^2)\rtimes M^c_{\alpha_{\mu'\nu'}}$ are isomorphic, respectively, to $C((\mathbb{R}\times\mathbb{T})/\alpha)\rtimes X_\beta^{\alpha,u}$ and $C((\mathbb{R}\times\mathbb{T})/\beta)\rtimes X_\alpha^{\beta,u^*}$, and they are, consequently, Morita equivalent to each other. □

Corollary 2.3. *Let $\mu\notin\mathbb{Q}$, and let $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in GL_2(\mathbb{Z})$. If*

$$2\mu'=\frac{2a\mu+b}{2c\mu+d}\text{ and }2\nu'=\frac{2\nu}{2c\mu+d},$$

then the quantum Heisenberg manifolds $D^c_{\mu\nu}$ and $D^c_{\mu'\nu'}$ are Morita equivalent.

Proof. It suffices to check the statement for $A_1=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $A_2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, since A_1 and A_2 generate $\mathrm{GL}_2(\mathbb{Z})$ ([Ku, Appendix B]), and $(\mu,\nu)\mapsto(\mu',\nu')$ defines an action of $\mathrm{GL}_2(\mathbb{Z})$ on $(\mathbb{R}\setminus\mathbb{Q})\times\mathbb{R}$. For $A=A_1$ we get isomorphic C^*-algebras by Remark 2.1. For $A=A_2$, we get $(\mu',\nu')=(\frac{1}{4\mu},\frac{\nu}{2\mu})$, and the result follows from Proposition 2.2. □

Proposition 2.4. *Let $\{1,\mu,\nu\}$ be linearly independent over $\mathbb{Q}$, and let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in GL_3(\mathbb{Z})$. If*

$$2\mu' = \frac{2a\mu + 2b\nu + c}{2g\mu + 2h\nu + i} \text{ and } 2\nu' = \frac{2d\mu + 2e\nu + f}{2g\mu + 2h\nu + i},$$

then the quantum Heisenberg manifolds $D^c_{\mu\nu}$ and $D^c_{\mu'\nu'}$ are Morita equivalent.

Proof. As in the proof of Theorem 1.7 in [Pa2], $A = A_1A_2A_3$, where

$$A_1 = \begin{pmatrix} A & B & C \\ D & E & F \\ 0 & 0 & 1 \end{pmatrix},\ A_2 \begin{pmatrix} G & 0 & H \\ 0 & 1 & 0 \\ I & 0 & J \end{pmatrix},\ A_3 = \begin{pmatrix} K & L & 0 \\ M & N & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $A_i \in \mathrm{GL}_3(\mathbb{Z})$, for $i = 1,2,3$.

Since the map $(\mu,\nu) \mapsto (\mu',\nu')$ defines an action of $\mathrm{GL}_3(\mathbb{Z})$ on the set $\{(\mu,\nu) \in \mathbb{R}^2 : \{1,\mu,\nu\}$ is linearly independent over $\mathbb{Q}\}$, it suffices to check the statement for A_i, $i = 1,2,3$.

For $A = A_1$ and $A = A_3$ the C^*-algebras $D^c_{\mu\nu}$ and $D^c_{\mu'\nu'}$ are isomorphic by Remark 2.1. Thus it suffices to show the result for $A = A_2$. The map $\begin{pmatrix} G & H \\ I & J \end{pmatrix} \mapsto \begin{pmatrix} G & 0 & H \\ 0 & 1 & 0 \\ I & 0 & J \end{pmatrix}$ is a group homomorphism from $\mathrm{GL}_2(\mathbb{Z})$ into $\mathrm{GL}_3(\mathbb{Z})$, and $\mathrm{GL}_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which implies that we only need to prove the statement for $A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. For A_1 we get $2\mu' = 2\mu + 1$, $2\nu' = 2\nu$, so $D^c_{\mu\nu}$ and $D^c_{\mu'\nu'}$ are isomorphic by Remark 2.1. Proposition 2.2 takes care of the case $A = A_2$, since then we have $(\mu',\nu') = (\frac{1}{4\mu}, \frac{\nu}{2\mu})$. □

Notation 2.5. We denote by $G_{\mu\nu}$ the subgroup of $\mathbb{R}$ generated by $\{1, 2\mu, 2\nu\}$. It was shown in [Ab2, Theorem 3.16] that the homomorphism induced on $K_0(D^c_{\mu\nu})$ by any tracial state on $D^c_{\mu\nu}$ has range $G_{\mu\nu}$.

Remark 2.6. If rank $G_{\mu\nu} = 2$, then there exist an irrational number ν' and integers $p,q \in \mathbb{Z}$, $p \neq 0$, $(p,q) = 1$, such that $D^c_{\mu\nu}$ and $D^c_{\frac{p}{2q},\nu'}$ are isomorphic.

Proof. We proceed as in [Pa1, Proposition 1.5]. Let $\mu_0 = 2\mu$, $\nu_0 = 2\nu$. Since the group generated by $\{1,\mu_0,\nu_0\}$ has rank 2, either μ_0 or ν_0 is irrational. We may assume that ν_0 is irrational, because, by Remark 2.1, $D^c_{\mu\nu}$ and $D^c_{\nu\mu}$ are isomorphic. Besides, there exist $M,N,P \in \mathbb{Z}$, with $N \neq 0$ such that $M + N\mu_0 + P\nu_0 = 0$, so we have $\mu_0 = \frac{k}{l}\nu_0 + \frac{m}{n}$, with $(k,l) = 1$. If $k = 0$, then $\mu_0 \in \mathbb{Q}$, and we are done. Otherwise take $a,b \in \mathbb{Z}$ such that $ak + bl = 1$, so that $\begin{pmatrix} -l & k \\ a & b \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$, and set

$$(\mu_0',\nu_0') = \begin{pmatrix} -l & k \\ a & b \end{pmatrix} (\mu_0,\nu_0).$$

Then

$$\mu_0' = -l(\frac{k}{l}\nu_0 + \frac{m}{n}) + k\nu_0 = \frac{-lm}{n} \in \mathbb{Q}$$

and

$$\nu_0' = a(\frac{k}{l}\nu_0 + \frac{m}{n}) + b\nu_0 = \frac{1}{l}\nu_0 + \frac{am}{n} \notin \mathbb{Q}.$$

We now take $\nu' = \nu_0'/2$ and $p/q = \mu_0'$, in lowest terms. By Remark 2.1 $D^c_{\mu\nu}$ and $D^c_{\frac{p}{2q},\nu'}$ are isomorphic. $\square$

Proposition 2.7. *Let p and q be non-zero integers such that $(p,q)=1$, and let $\nu \in \mathbb{R}$. Then $D^c_{\frac{p}{2q},\nu}$ is Morita equivalent to $D^c_{0,q\nu}$.*

Proof. By Remark 2.1 we may assume that p and q are positive. By applying Proposition 2.2 to $(\mu,\nu)=(q/2,\nu)$, we get that $D^c_{0,\nu} \cong D^c_{q/2,\nu}$ is Morita equivalent to $D^c_{\frac{1}{2q},\frac{\nu}{q}}$, thus proving the proposition for $p=1$. For $p>1$, let $r_0=q$, $r_1=p$, and, if $r_{i+1}\neq 1$, define r_{i+2} by $r_i = m_{i+1}r_{i+1}+r_{i+2}$, where $0\le r_{i+2}<r_{i+1}$, and $m_{i+1}\in\mathbb{Z}$.

Actually, $r_{i+2}>0$; otherwise r_{i+1} divides r_i, and it follows that r_{i+1} divides r_j for all $j\le i$. In particular, r_{i+1} divides both p and q, which contradicts the fact that $r_{i+1}\neq 1$. Now, since $r_{i+1}<r_i$, there is an index i_0 for which $r_{i_0}=1$.

On the other hand, it follows from Proposition 2.2 that, for any real number κ, $D^c_{\frac{r_i}{2r_{i-1}},\kappa}$ is Morita equivalent to $D^c_{\frac{r_{i-1}}{2r_i},\kappa\frac{r_{i-1}}{r_i}}$, which in turn is isomorphic to $D^c_{\frac{r_{i+1}}{2r_i},\kappa\frac{r_{i-1}}{r_i}}$.

Thus $D^c_{\frac{p}{2q},\nu} = D^c_{\frac{r_1}{2r_0},\nu}$ is Morita equivalent to $D^c_{\frac{r_j}{2r_{j-1}},\frac{q\nu}{r_{j-1}}}$ for any $j\le i_0$. In particular, for $j=i_0$, we have that $D^c_{\frac{p}{2q},\nu}$ is Morita equivalent to $D^c_{\frac{1}{2r_{i_0-1}},\frac{\nu q}{r_{i_0-1}}}$, which, as shown above, is Morita equivalent to $D^c_{0,\nu q}$. $\square$

Theorem 2.8. *Two quantum Heisenberg manifolds $D^c_{\mu\nu}$, $D^{c'}_{\mu'\nu'}$ are Morita equivalent if and only if $c=c'$ and there exists a positive real number r such that*

$$\mathbb{Z}+2\mu\mathbb{Z}+2\nu\mathbb{Z} = r(\mathbb{Z}+2\mu'\mathbb{Z}+2\nu'\mathbb{Z}).$$

In particular, the rank of the free abelian group $G_{\mu\nu}=\mathbb{Z}+2\mu\mathbb{Z}+2\nu\mathbb{Z}$ is the same for Morita equivalent quantum Heisenberg manifolds, and:

(1) *If rank $G_{\mu\nu}=1=$ rank $G_{\mu'\nu'}$, then $D^c_{\mu\nu}$ is Morita equivalent to $D^c_{\mu',\nu'}$. In particular, $D^c_{\mu\nu}$ is Morita equivalent to the commutative Heisenberg manifold $D^c_{0,0}$.*

(2) *If rank $G_{\mu\nu}=2=$ rank $G_{\mu'\nu'}$, let $\{\alpha,\frac{p}{q}\}$ and $\{\alpha',\frac{p'}{q'}\}$ be bases of $G_{\mu\nu}$ and $G_{\mu'\nu'}$, respectively, where α and α' are irrational numbers and $p,p',q,q'\in\mathbb{Z}$, $(p,q)=(p'q')=1$. Then $D^c_{\mu,\nu}$ and $D^c_{\mu',\nu'}$ are Morita equivalent if and only if there exists $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in GL_2(\mathbb{Z})$ such that*

$$q'\alpha' = \frac{aq\alpha'+b}{cq\alpha'+d}.$$

In particular, $D^c_{\frac{p}{2q},\nu}$ is Morita equivalent to $D^c_{0,q\nu}$.

(3) *If rank $G_{\mu\nu}=3=$ rank $G_{\mu'\nu'}$, then $D^c_{\mu\nu}$ and $D^c_{\mu'\nu'}$ are Morita equivalent if and only if there exists $\begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\in GL_3(\mathbb{Z})$ such that*

$$2\mu' = \frac{2a\mu+2b\nu+c}{2g\mu+2h\nu+i} \text{ and } 2\nu' = \frac{2d\mu+2e\nu+f}{2g\mu+2h\nu+i}.$$

Proof. It was shown in [Ab1, 3.4] that $K_0(D^c_{\mu\nu})=\mathbb{Z}^3\oplus\mathbb{Z}_c$, which implies that $D^c_{\mu\nu}$ and $D^{c'}_{\mu'\nu'}$ are not Morita equivalent for $c\neq c'$.

Besides ([Ab2, Theorem 3.16]), all tracial states on $D^c_{\mu\nu}$ induce the same homomorphism on $K_0(D^c_{\mu\nu})$, whose range is the group $G_{\mu\nu} = 2\mu\mathbb{Z} + 2\nu\mathbb{Z} + \mathbb{Z}$. Since ([Rf1, 2.2]) there is a bijection between finite traces of Morita equivalent unital C^*-algebras, we must have $G_{\mu\nu} = rG_{\mu'\nu'}$ for some positive real number r when $D^c_{\mu\nu}$ and $D^c_{\mu'\nu'}$ are Morita equivalent. An immediate consequence of this fact is that the rank of $G_{\mu\nu}$ is invariant under Morita equivalence.

If rank $G_{\mu\nu} = 1$, then, by [Ab2, Remark 3.5], $D^c_{\mu\nu}$ is isomorphic to $D^c_{0,\frac{1}{2p}}$ for some non-zero integer p, so $D^c_{\mu\nu}$ is isomorphic to $D^c_{\frac{1}{2p},0}$ by Remark 2.1. Now, by Proposition 2.7, $D^c_{\frac{1}{2p},0}$ is Morita equivalent $D^c_{0,0}$.

If rank $G_{\mu\nu} = 2 =$ rank $G_{\mu'\nu'}$ and $G_{\mu\nu} = rG_{\mu'\nu'}$ for some positive r, let $\{\alpha, \frac{p}{q}\}$ and $\{\alpha', \frac{p'}{q'}\}$ be bases of $G_{\mu\nu}$ and $G_{\mu'\nu'}$, respectively, where $\alpha,\ \alpha'$ are irrational numbers, and p, p', q, q' are integers, with $(p, q) = (p', q') = 1$. Since $\mathbb{Z} \subset G_{\mu\nu}$ $(G_{\mu'\nu'})$ we have that p $(p') = \pm 1$ and, by Remark 2.1, we may assume $p = p' = 1$. Then we have that $\alpha\mathbb{Z}+1/q\mathbb{Z} = r(\alpha'\mathbb{Z}+1/q'\mathbb{Z})$, which implies that $\alpha q\mathbb{Z}+\mathbb{Z} = (rq/q')(\alpha' q'\mathbb{Z}+\mathbb{Z})$. A standard argument shows that

$$q\alpha = \frac{aq'\alpha' + b}{cq'\alpha' + d} \text{ for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

Therefore $D^c_{\frac{q\alpha}{2},0}$ and $D^c_{\frac{q'\alpha'}{2},0}$ are Morita equivalent by Corollary 2.3.

On the other hand, by Remark 2.6, $D^c_{\mu\nu}$ and $D^c_{\mu',\nu'}$ are isomorphic, respectively, to $D^c_{\frac{m}{2n},\beta}$ and $D^c_{\frac{m'}{2n'},\beta'}$, for some irrational numbers β and β' and integers m, m', n, n' such that $(m, n) = (m', n') = 1$. Therefore $\{2\beta, \frac{1}{n}\}$ and $\{2\beta', \frac{1}{n'}\}$ are bases of $G_{\mu\nu}$ and $G_{\mu'\nu'}$, respectively, and it follows from the argument above that $D^c_{n\beta,0}$ and $D^c_{n'\beta',0}$ are Morita equivalent. It only remains to note now that, by Proposition 2.7 and Remark 2.1, $D^c_{n\beta,0}$ and $D^c_{n'\beta',0}$ are Morita equivalent to $D^c_{\mu\nu}$ and $D^c_{\mu'\nu'}$, respectively.

Finally, if rank $G_{\mu\nu} = 3 =$ rank $G_{\mu'\nu'}$ and $G_{\mu\nu} = rG_{\mu'\nu'}$ for some positive r, then let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Z})$ be the transpose of the matrix that changes coordinates between the bases $\{2r\mu', 2r\nu', r\}$ and $\{2\mu, 2\nu, 1\}$ of $G_{\mu\nu}$. Then

$$2\mu' = \frac{2a\mu + 2b\nu + c}{2g\mu + 2h\nu + i} \text{ and } 2\nu' = \frac{2d\mu + 2e\nu + f}{2g\mu + 2h\nu + i},$$

which implies, by Proposition 2.4, that $D^c_{\mu\nu}$ and $D^c_{\mu'\nu'}$ are Morita equivalent. □

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