

NECESSARY CONDITIONS FOR SCHATTEN CLASS LOCALIZATION OPERATORS

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ABSTRACT. We study time-frequency localization operators of the form $A_a^{\varphi_1, \varphi_2}$, where a is the symbol of the operator and φ_1, φ_2 are the analysis and synthesis windows, respectively. It is shown in an earlier paper by the authors that a sufficient condition for $A_a^{\varphi_1, \varphi_2} \in S_p(L^2(\mathbb{R}^d))$, the Schatten class of order p , is that a belongs to the modulation space $M^{p, \infty}(\mathbb{R}^{2d})$ and the window functions to the modulation space M^1 . Here we prove a partial converse: if $A_a^{\varphi_1, \varphi_2} \in S_p(L^2(\mathbb{R}^d))$ for every pair of window functions $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^{2d})$ with a uniform norm estimate, then the corresponding symbol a must belong to the modulation space $M^{p, \infty}(\mathbb{R}^{2d})$. In this sense, modulation spaces are optimal for the study of localization operators. The main ingredients in our proofs are frame theory and Gabor frames. For $p = \infty$ and $p = 2$, we recapture earlier results, which were obtained by different methods.

1. INTRODUCTION

Localization operators, as studied by Daubechies [4] in 1988, serve as a tool to localize a signal simultaneously in time and frequency (or in phase space). Under different names, such as Anti-Wick operators, Gabor multipliers, Toeplitz operators or wave packets, they have become useful in PDE, in quantum mechanics and quantization, or in signal analysis and optics, and therefore they have been studied extensively. For a sample of diverse contributions and extended references, we refer to [1, 3, 4, 5, 8, 9, 11, 12, 19, 20].

In [3], localization operators are studied as a part of time-frequency analysis. Such an approach is suggested by the very definition of a localization operator by means of the short-time Fourier transform. Using the function spaces that are naturally associated to the short-time Fourier transform, the so-called modulation spaces, we have obtained the sharpest sufficient conditions for boundedness and Schatten class properties that are known so far [3]. For bounded operators and for Hilbert-Schmidt operators we have also shown that conditions involving modulation spaces are also necessary.

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In this note we answer the following question: *What can be said about the symbols of operators $A_a^{\varphi_1, \varphi_2}$ in the Schatten class $S_p(L^2(\mathbb{R}^d))$, when $p \neq 2$ or $p \neq \infty$?*

Our answer will demonstrate once more that modulation spaces are the appropriate symbol classes for the study of localization operators and that they are in a sense optimal.

The technical innovation in this paper is the use of frame theory, in particular Gabor frames, to study the Schatten class properties of localization operators.

Statement of results. The protagonists of time-frequency analysis are the time-frequency shifts (or phase space translations) defined by

$$(1) \quad M_\omega T_x f(t) = e^{2\pi i \omega t} f(t - x), \quad t, x, \omega \in \mathbb{R}^d.$$

For a fixed non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ (the Schwartz class), the short-time Fourier transform (for short STFT) of $f \in \mathcal{S}'(\mathbb{R}^d)$ (the space of tempered distributions), with respect to the window g , is given by

$$(2) \quad V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \omega t} dt.$$

A time-frequency localization operator $A_a^{\varphi_1, \varphi_2}$ with symbol a , analysis window function φ_1 , and synthesis window function φ_2 is defined formally by means of the STFT as

$$(3) \quad A_a^{\varphi_1, \varphi_2} f = \int_{\mathbb{R}^{2d}} a(x, \omega) V_{\varphi_1} f(x, \omega) M_\omega T_x \varphi_2 dx d\omega,$$

whenever the vector-valued integral makes sense. If $\varphi_1(t) = \varphi_2(t) = e^{-\pi t^2}$, then $A_a^{\varphi_1, \varphi_2}$ boils down to the classical Anti-Wick operator and the mapping $a \mapsto A_a^{\varphi_1, \varphi_2}$ is interpreted as a quantization rule [1, 17, 20].

In a weak sense, the definition of $A_a^{\varphi_1, \varphi_2}$ in (3) can be expressed by

$$(4) \quad \langle A_a^{\varphi_1, \varphi_2} f, g \rangle = \langle a V_{\varphi_1} f, V_{\varphi_2} g \rangle = \langle a, \overline{V_{\varphi_1} f} V_{\varphi_2} g \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$

Using (4) it is easy to see that if $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then $A_a^{\varphi_1, \varphi_2}$ is a well-defined continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$.

In [3] we have studied in detail the class of localization operators as a multilinear mapping

$$(5) \quad (a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1, \varphi_2},$$

and derived norm estimates when a, φ_1 and φ_2 belong to certain modulation spaces. Modulation space norms measure the time-frequency concentration (phase space distribution) of functions and distributions on the time-frequency plane (on phase space). The unweighted modulation spaces are defined as follows: Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, the *modulation space* $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L^{p,q}(\mathbb{R}^{2d})$ (mixed-norm spaces). The norm on $M^{p,q}$ is given by

$$\|f\|_{M^{p,q}} := \|V_g f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \omega)|^p dx \right)^{q/p} d\omega \right)^{1/q}.$$

If $p = q$, we write M^p instead of $M^{p,p}$.

The space $M^{p,q}(\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window g . Different non-zero windows $g \in M^1$ yield equivalent norms on $M^{p,q}$. This property will be crucial in the sequel, because we will choose a

suitable window g in estimates of the $M^{p,q}$ -norm. Within the class of modulation spaces, one finds several standard function spaces, for instance $M^2 = L^2$, M^1 coincides with Feichtinger’s algebra $S_0(\mathbb{R}^d)$, and using weighted versions, one also finds certain Sobolev spaces and the Shubin classes among the modulation spaces [3, 10]. For more background and the general theory of modulation spaces we refer to [6, 10].

Our main result about modulation spaces and localization operators can be formulated as follows.

Theorem 1. *Let $1 \leq p \leq \infty$.*

(a) *The mapping $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1, \varphi_2}$ is bounded from $M^{p, \infty}(\mathbb{R}^{2d}) \times M^1(\mathbb{R}^d) \times M^1(\mathbb{R}^d)$ into $S_p(L^2(\mathbb{R}^d))$ with a norm estimate*

$$\|A_a^{\varphi_1, \varphi_2}\|_{S_p} \leq B \|a\|_{M^{p, \infty}} \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}$$

for a suitable constant $B > 0$.

(b) *Conversely, assume that $A_a^{\varphi_1, \varphi_2} \in S_p(L^2(\mathbb{R}^d))$ for all windows $\varphi_1, \varphi_2 \in M^1$ and that there exists a constant $B > 0$ depending only on the symbol a such that*

$$(6) \quad \|A_a^{\varphi_1, \varphi_2}\|_{S_p} \leq B \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}, \quad \forall \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d).$$

Then $a \in M^{p, \infty}$.

While it seems hopeless to find a characterization for $A_a^{\varphi_1, \varphi_2} \in S_p$ with a fixed pair of windows φ_1, φ_2 , our main result shows that the condition $a \in M^{p, \infty}$ is optimal for a natural class of windows. Part (a) was already proved in [3] and was shown to include the known conditions in [2, 8, 20]. Part (b) was known only for the cases $p = 2$ and $p = \infty$ [3]. By developing a new method, we are able to fill this gap and prove statement (b) for all values of p .

Notation. We define $t^2 = t \cdot t$, for $t \in \mathbb{R}^d$, and $xy = x \cdot y$ is the scalar product on \mathbb{R}^d . When using the duality $\langle f, g \rangle$ for f in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ and $g \in \mathcal{S}'(\mathbb{R}^d)$ a tempered distribution, the bracket is always understood to extend the inner product $\langle f, g \rangle = \int f(t) \overline{g(t)} dt$ on $L^2(\mathbb{R}^d)$.

Throughout the paper, we shall use the notation $A \lesssim B$ to indicate $A \leq cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ if $A \leq cB$ and $B \leq kA$, for suitable $c, k > 0$.

2. SOME TIME-FREQUENCY ANALYSIS

We collect a few facts about the STFT and Gabor frames needed in the sequel. First some properties of the STFT.

Lemma 1. *Write $z = (z_1, z_2) \in \mathbb{R}^{2d}$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$. For $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ and $f, g \in \mathcal{S}'(\mathbb{R}^d)$ we have*

$$\begin{aligned} T_{(z_1, z_2)}(\overline{V_{\varphi_1} f} V_{\varphi_2} g)(x, \omega) &= \overline{V_{\varphi_1}(M_{z_2} T_{z_1} f)}(x, \omega) V_{\varphi_2}(M_{z_2} T_{z_1} g)(x, \omega), \\ M_{(\zeta_1, \zeta_2)}(\overline{V_{\varphi_1} f} V_{\varphi_2} g)(x, \omega) &= \overline{V_{\varphi_1} f(x, \omega)} V_{(M_{\zeta_1} T_{-\zeta_2} \varphi_2)}(M_{\zeta_1} T_{-\zeta_2} g)(x, \omega), \\ M_{\zeta} T_z(\overline{V_{\varphi_1} f} V_{\varphi_2} g) &= \overline{V_{\varphi_1}(M_{z_2} T_{z_1} f)} V_{(M_{\zeta_1} T_{-\zeta_2} \varphi_2)}(M_{\zeta_1} T_{-\zeta_2} M_{z_2} T_{z_1} g). \end{aligned}$$

These formulas are obtained by a minimal modification of formulas (25), (26), and (27) in [3].

The STFT of the d -dimensional Gaussian $\varphi(x) := e^{-\pi x^2}$ is given by [10, Lemma 1.5.2] as

$$(7) \quad V_\varphi \varphi(x, \omega) = 2^{-d/2} e^{-\pi i x \omega} e^{-\frac{\pi}{2}(x^2 + \omega^2)}.$$

Gaussians are particularly suited as windows to measure modulation space norms because of the following discrete version of the modulation space norm.

Theorem 2. *If $\Phi(x, \omega) = e^{-\pi(x^2 + \omega^2)}$ and $\alpha\beta < 1$, then $F \in M^{p,q}(\mathbb{R}^{2d})$ if and only if*

$$(8) \quad \|\langle F, M_{\beta n} T_{\alpha k} \Phi \rangle_{n,k \in \mathbb{Z}^{2d}}\|_{\ell^{p,q}(\mathbb{Z}^{4d})} := \left(\sum_{n \in \mathbb{Z}^{2d}} \left(\sum_{k \in \mathbb{Z}^{2d}} |\langle F, M_{\beta n} T_{\alpha k} \Phi \rangle|^p \right)^{q/p} \right)^{1/q}$$

is finite. Moreover, $\|F\|_{M^{p,q}} \asymp \|\langle F, M_{\beta n} T_{\alpha k} \Phi \rangle_{n,k \in \mathbb{Z}^{2d}}\|_{\ell^{p,q}(\mathbb{Z}^{4d})}$.

Remark. Theorem 2 contains more than meets the eye and is, in fact, a summary of some fundamental results of Gabor analysis. If $p = q = 2$, then the norm equivalence $\|F\|_2 \asymp \|\langle F, M_{\beta n} T_{\alpha k} \Phi \rangle_{n,k \in \mathbb{Z}^{2d}}\|_2$ expresses that the collection $\{M_{\beta n} T_{\alpha k} \Phi : k, n \in \mathbb{Z}^{2d}\}$ is a *Gabor frame* for $L^2(\mathbb{R}^{2d})$. This is the case if and only if $\alpha\beta < 1$ by a deep result of Seip and Wallstén [15, 16]. They proved it in dimension $d = 1$, but by a tensor product argument the result carries over to higher dimensions. By [7] or [10, Corollary 12.2.6] the Gabor frame $\{M_{\beta n} T_{\alpha k} \Phi : k, n \in \mathbb{Z}^{2d}\}$ then extends to a Banach frame for the modulation spaces $M^{p,q}(\mathbb{R}^{2d})$ (and also weighted modulation spaces). In particular, the norm equivalence (8) holds.

The norm equivalence (8) is the main tool in the proof of Theorem 1.

3. NECESSARY SCHATTEN CLASS CONDITIONS FOR LOCALIZATION OPERATORS

We can now prove part (b) of Theorem 1. Recall that a bounded (in fact, compact) operator L on a Hilbert space \mathcal{H} belongs to the Schatten class $S_p(\mathcal{H})$ if the sequence $\{\lambda_n : n \in \mathbb{N}\}$ of eigenvalues of $(L^*L)^{1/2}$, the so-called singular values, belongs to ℓ^p . The norm on S_p is given by $\|L\|_{S_p} = \|\lambda\|_p$ (see the standard references [18, 17]). Equivalently, we may express the norm on $S_p(\mathcal{H})$ by

$$\|L\|_{S_p}^p = \sup \sum_j \|L e_j\|_{\mathcal{H}}^p,$$

where the supremum is taken over all orthonormal bases $\{e_j : j \in \mathcal{J}\}$ of \mathcal{H} . As a straightforward consequence we obtain the inequality

$$\left(\sum_{j \in \mathcal{J}} |\langle L e_j, e_j \rangle|^p \right)^{1/p} \leq \|L\|_{S_p}$$

for every orthonormal basis $\{e_j : j \in \mathcal{J}\}$.

This last inequality extends easily to frames as was already observed in [14].

Lemma 3. *Let $\{b_j\}_{j \in \mathcal{J}}$ be a frame for \mathcal{H} . If $L \in S_p(\mathcal{H})$, for $1 \leq p \leq \infty$, then*

$$(9) \quad \left(\sum_{j \in \mathcal{J}} |\langle L b_j, b_j \rangle|^p \right)^{1/p} \lesssim \|L\|_{S_p}.$$

Proof. We prove the claim for $p = 1$ and $p = \infty$ and then use interpolation.

According to the spectral theorem for compact (non-self-adjoint) operators [13, Thm. VI.17] for $L \in S_p(\mathcal{H})$ there exist two orthonormal sets $\{e_k : k \in \mathbb{N}\}$ and $\{f_k : k \in \mathbb{N}\}$, such that

$$Lf = \sum_{k=1}^{\infty} \lambda_k \langle f, e_k \rangle f_k,$$

where the λ_k 's are the singular values of L .

Case $p = 1$. Since $\{b_j\}$ is a frame, we have $\sum_j |\langle f, b_j \rangle|^2 \leq D \|f\|_{\mathcal{H}}^2$ for all $f \in \mathcal{H}$, and we can estimate as follows:

$$\begin{aligned} \sum_{j \in \mathcal{J}} |\langle Lb_j, b_j \rangle| &= \sum_{j \in \mathcal{J}} \left| \sum_{k \in \mathbb{N}} \lambda_k \langle b_j, e_k \rangle \langle f_k, b_j \rangle \right| \\ &\leq \sum_{k \in \mathbb{N}} \lambda_k \sum_{j \in \mathcal{J}} |\langle b_j, e_k \rangle| |\langle f_k, b_j \rangle| \\ &\leq \sum_{k \in \mathbb{N}} \lambda_k \left(\sum_{j \in \mathcal{J}} |\langle b_j, e_k \rangle|^2 \right)^{1/2} \left(\sum_{j \in \mathcal{J}} |\langle b_j, f_k \rangle|^2 \right)^{1/2} \\ &\leq D \sum_{k \in \mathbb{N}} \lambda_k \|e_k\| \|f_k\| \\ &= D \sum_{k \in \mathbb{N}} \lambda_k = D \|L\|_{S_1}, \end{aligned}$$

Case $p = \infty$. If L is compact or bounded on \mathcal{H} , then

$$\sup_{j \in \mathcal{J}} |\langle Lb_j, b_j \rangle| \leq \sup_{j \in \mathcal{J}} \|L\|_{op} \|b_j\|_{L^2}^2 \leq D \|L\|_{op}.$$

Using complex interpolation between S_1 and the bounded (or compact) operators on \mathcal{H} , the estimate follows for all $p \in [1, \infty]$. \square

Corollary 2. *Let $\{b_j\}_{j \in \mathcal{J}}$ be a frame for \mathcal{H} . If T is a bounded operator on \mathcal{H} and $L \in S_p(\mathcal{H})$, $1 \leq p \leq \infty$, then we have*

$$(10) \quad \left(\sum_{j \in \mathcal{J}} |\langle Lb_j, Tb_j \rangle|^p \right)^{1/p} \lesssim \|L\|_{S_p} \|T\|_{op}.$$

Proof. We have $\langle Lb_j, Tb_j \rangle = \langle T^*Lb_j, b_j \rangle$. Since S_p is an operator ideal, we have $\|T^*L\|_{S_p} \leq \|T^*\|_{op} \|L\|_{S_p} = \|T\|_{op} \|L\|_{S_p}$, and so we may apply Lemma 3 to T^*L . \square

Remark. For the above estimates we have only used that $\{b_j\}$ is a Bessel sequence.

Proof of Theorem 1. The idea is to compute the $M^{p,\infty}$ -norm of the symbol a of the operator $A_a^{\varphi_1, \varphi_2}$ by using a Gabor frame. We choose the Gaussian window $\Phi(x, \omega) = 2^{-d} e^{-\pi(x^2 + \omega^2)}$ and $0 < \alpha, \beta < 1$. Then the set $\{M_{\beta n} T_{\alpha k} \Phi\}_{k, n \in \mathbb{Z}^{2d}}$ is a Gabor frame, and we can estimate the $M^{p,\infty}(\mathbb{R}^{2d})$ -norm of a by (8) of Theorem 2 in the form

$$(11) \quad \|a\|_{M^{p,\infty}(\mathbb{R}^{2d})} \asymp \|\langle a, M_{\beta n} T_{\alpha k} \Phi \rangle_{n, k \in \mathbb{Z}^{2d}}\|_{\ell^{p,\infty}(\mathbb{Z}^{4d})}.$$

Therefore, we focus our attention on the Gabor coefficients in (11). Using (7), we can write Φ as

$$(12) \quad \Phi(x, \omega) = 2^{-d} e^{-\pi(x^2 + \omega^2)} = V_\varphi \varphi(x, \omega) \overline{V_\varphi \varphi(x, \omega)}.$$

Next, with $k = (k_1, k_2), n = (n_1, n_2) \in \mathbb{Z}^{2d}$ and Lemma 1, we write the time-frequency shifts of Φ as the product of two STFTs on \mathbb{R}^d :

$$\begin{aligned} M_{\beta n} T_{\alpha k} \Phi(x, \omega) &= M_{(\beta n_1, \beta n_2)} T_{(\alpha k_1, \alpha k_2)} (V_\varphi \varphi \overline{V_\varphi \varphi})(x, \omega) \\ &= V_{(M_{\beta n_1} T_{-\beta n_2} \varphi)} (M_{\beta n_1} T_{-\beta n_2} M_{\alpha k_2} T_{\alpha k_1} \varphi)(x, \omega) \overline{V_\varphi (M_{\alpha k_2} T_{\alpha k_1} \varphi)(x, \omega)}. \end{aligned}$$

Now we substitute this formula into the weak definition of localization operators in (4) and we obtain the following decisive relation between the Gabor coefficients of the symbol a and the localization operator $A_a^{\varphi_1, \varphi_2}$:

$$(13) \quad \langle a, M_{\beta n} T_{\alpha k} \Phi \rangle = \langle A_a^{\varphi, M_{\beta n_1} T_{-\beta n_2} \varphi} (M_{\alpha k_2} T_{\alpha k_1} \varphi), M_{\beta n_1} T_{-\beta n_2} M_{\alpha k_2} T_{\alpha k_1} \varphi \rangle.$$

Consequently, the $M^{p, \infty}$ -norm of a can be recast as

$$\begin{aligned} \|a\|_{M^{p, \infty}} &\asymp \|\langle a, M_{\beta n} T_{\alpha k} \Phi \rangle_{n, k \in \mathbb{Z}^{2d}}\|_{\ell^{p, \infty}(\mathbb{Z}^{4d})} \\ &= \sup_{n \in \mathbb{Z}^{2d}} \left(\sum_{k \in \mathbb{Z}^{2d}} |\langle a, M_{\beta n} T_{\alpha k} \Phi \rangle|^p \right)^{1/p} \\ &= \sup_{n_1, n_2 \in \mathbb{Z}^d} \left(\sum_{k_1, k_2 \in \mathbb{Z}^d} |\langle A_a^{\varphi, M_{\beta n_1} T_{-\beta n_2} \varphi} (M_{\alpha k_2} T_{\alpha k_1} \varphi), M_{\beta n_1} T_{-\beta n_2} M_{\alpha k_2} T_{\alpha k_1} \varphi \rangle|^p \right)^{1/p}, \end{aligned}$$

where $n = (n_1, n_2), k = (k_1, k_2) \in \mathbb{Z}^{2d}$. Since $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the occurring localization operators $A_a^{\varphi, M_{\beta n_1} T_{-\beta n_2} \varphi}$ are all in $S_p(L^2(\mathbb{R}^d))$ and by hypothesis they satisfy the uniform estimate

$$(14) \quad \sup_{(n_1, n_2) \in \mathbb{Z}^{2d}} \|A_a^{\varphi, (M_{\beta n_1} T_{-\beta n_2} \varphi)}\|_{S_p} \lesssim \sup_{(n_1, n_2) \in \mathbb{Z}^{2d}} \|\varphi\|_{M^1} \|M_{\beta n_1} T_{-\beta n_2} \varphi\|_{M^1} = \|\varphi\|_{M^1}^2,$$

where we have used the fact that time-frequency shifts are isometries on M^1 [10, Theorem 11.3.5].

Furthermore, for fixed $n = (n_1, n_2) \in \mathbb{Z}^{2d}$, the right-hand side of (13) has exactly the form discussed in Corollary 2. Since $\alpha < 1$, the set $\{M_{\alpha k_2} T_{\alpha k_1} \varphi : k_1, k_2 \in \mathbb{Z}^d\}$ is a (Gabor) frame for $L^2(\mathbb{R}^d)$. Choosing this frame and $T = M_{\beta n_1} T_{-\beta n_2}$ in Corollary 2, we obtain the desired estimate

$$\begin{aligned} \|a\|_{M^{p, \infty}} &\lesssim \sup_{n_1, n_2} \|\langle A_a^{\varphi, M_{\beta n_1} T_{-\beta n_2} \varphi} (M_{\alpha k_2} T_{\alpha k_1} \varphi), M_{\beta n_1} T_{-\beta n_2} M_{\alpha k_2} T_{\alpha k_1} \varphi \rangle_{k_1, k_2}\|_{\ell^p} \\ &\lesssim \sup_{n_1, n_2} \|A_a^{\varphi, M_{\beta n_1} T_{-\beta n_2} \varphi}\|_{S_p} \lesssim \|\varphi\|_{M^1}^2. \end{aligned}$$

Thus we have shown that $a \in M^{p, \infty}(\mathbb{R}^{2d})$, and the proof is finished. □

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