PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 12, Pages 3505–3513
S 0002-9939(05)07911-6
Article electronically published on June 7, 2005

ON THE CONSTRUCTION OF A CLASS OF BIDIMENSIONAL NONSEPARABLE COMPACTLY SUPPORTED WAVELETS

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(Communicated by David R. Larson)

Abstract. Chui and Wang discussed the construction of one-dimensional compactly supported wavelets under a general framework, and constructed one-dimensional compactly supported spline wavelets. In this paper, under a mild condition, the construction of $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$-wavelets is obtained.

1. Introduction

Throughout this paper, M is always referred to as the matrix $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. The *Fourier transform* of f is defined by $\hat{f}(\xi) = \frac{1}{2\pi}\int_{R^2} dx\, f(x)e^{-ix\xi}$ for $f \in L^1(R^2)$. A real-valued measurable function f defined on R^2 is said to be symmetric (antisymmetric) about $\frac{x_0}{2} \in R^2$ if $f(\cdot) = f(x_0 - \cdot)$ $(f(\cdot) = -f(x_0 - \cdot))$ a.e. In recent years, bidimensional nonseparable wavelets have been extensively studied since they offer the hope of a more isotropic analysis ([1]–[9]).

A ladder of closed subspaces $\{V_j\}_{j\in Z}$ of $L^2(R^2)$ is called a *multiresolution analysis* related to M (MRA) if the following conditions hold:

(1) $V_j \subset V_{j-1}$ for $j \in Z$;
(2) $\bigcap_{j\in Z} V_j = \{0\}$, $\overline{\bigcup_{j\in Z} V_j} = L^2(R^2)$;
(3) $f(\cdot) \in V_j$ if and only if $f(M^j\cdot) \in V_0$ for $j \in Z$;
(4) there exists a function $\phi(\cdot)$ in V_0 such that the set $\{\phi(\cdot - k)\}_{k\in Z^2}$ is a Riesz basis for V_0.

Here $\phi(\cdot)$ is also called a *scaling function* of the MRA. Since $V_0 \subset V_{-1}$, $\phi(\cdot)$ has to satisfy some *M-refinement equation*

$$\phi(\cdot) = \sqrt{2}\sum_{n\in Z^2} h_n \phi(M\cdot - n), \tag{1.1}$$

where $\{h_n\}_{j\in Z^2}$ is called the mask, and

$$H_0(\cdot) = \frac{1}{\sqrt{2}}\sum_{n\in Z^2} h_n e^{-in\cdot} \tag{1.2}$$

is called the *symbol* of ϕ.

Received by the editors October 9, 2001 and, in revised form, July 2, 2004 and July 6, 2004.
2000 *Mathematics Subject Classification.* Primary 42C40.
Key words and phrases. Scaling function, wavelet, Riesz basis.
This work was partially supported by the Natural Science Foundation of China and the Natural Science Foundation of Beijing.

We denote by W_j the orthogonal complement of V_j in V_{j-1} for $j \in Z$. If we find a ψ such that $\{\psi(\cdot - k)\colon k \in Z^2\}$ is a Riesz basis for the orthogonal complement W_0 of V_0 in V_{-1}, then it is easy to check that $\{\psi_{j,k}(\cdot)\colon j \in Z, k \in Z^2\}$ is a Riesz basis for $L^2(R^2)$, where $f_{j,k}(\cdot) = 2^{-\frac{j}{2}} f(M^{-j} \cdot - k)$ for any function f defined on R^2 and $j \in Z$, $k \in Z^2$. In particular, when $\{\phi(\cdot - k)\colon k \in Z^2\}$ is an orthonormal basis for V_0, and

$$\psi(\cdot) = \sqrt{2} \sum_{n \in Z^2} (-1)^{n_1+n_2} \overline{h_{(1,0)^T - n}} \phi(M \cdot - j), \tag{1.3}$$

it is well known that $\{\psi(\cdot - k)\colon k \in Z^2\}$ is an orthonormal basis for W_0, and that $\{\psi_{j,k}(\cdot)\colon j \in Z, k \in Z^2\}$ is an orthonormal basis for $L^2(R^2)$.

In one dimension, it is well known that, except for the Haar wavelet, there exists no compactly supported, orthonormal, symmetric or antisymmetric, and real-valued wavelet ([10, Theorem 8.1.4]). But it is not the case if orthogonality is replaced by Riesz basis. Chui and Wang obtained an approach to the construction of a compactly supported Riesz basis and constructed compactly supported spline wavelets with symmetry or antisymmetry ([11]–[14]). Their approach depends on the determination of zeros of polynomials, which is not easy in higher dimensions. So it is a natural and interesting problem to construct a nonseparable compactly supported Riesz basis in higher dimensions.

In this paper, under the hypothesis that some bivariate polynomial has no zeros, we obtain a general construction of compactly supported M-wavelets, which inherits the symmetry of the corresponding scaling functions and satisfies the vanishing moment condition originating in the symbols of the scaling functions. Some examples are also given to illustrate the general theory, and a conjecture of an infinite family of examples is put forward. Our main results can be stated as follows.

Theorem 1.1. *Assume that ϕ is a scaling function of an MRA $\{V_j\}_{j \in Z}$ satisfying* (1.1)*, its symbol H_0 defined as in* (1.2) *is a Laurent polynomial, and W_0 is the orthogonal complement of V_0 in V_{-1}. Define*

$$g_n = (-1)^{n_1+n_2} \langle \phi_{-1,(1,0)^T - n}, \phi \rangle \tag{1.4}$$

for $n \in Z^2$, and

$$\psi(\cdot) = \sqrt{2} \sum_{n \in Z^2} g_n \phi(M \cdot - n).$$

Then

(1) *$\psi \in W_0$;*

(2) *$\psi(\cdot - n)\colon n \in Z^2\}$ is a Riesz basis for W_0 if and only if*

$$\sum_{n \in Z^2} \left(\sum_{l \in Z^2} (-1)^{l_1+l_2} h_l g_{Mn + (1,0)^T - l} \right) e^{-in\xi}$$

has no zeros in $[-\pi, \pi]^2$.

Remark 1.1. It is obvious that $\{g_n\}$ is finitely supported, and thus, ψ is compactly supported. To know whether $\sum_{n \in Z^2} (\sum_{l \in Z^2} (-1)^{l_1+l_2} h_l g_{Mn+(1,0)^T - l}) e^{-in\xi}$

has zeros in $[-\pi,\pi]^2$, it is enough to map the graph of

$$\left|\sum_{n\in Z^2}\left(\sum_{l\in Z^2}(-1)^{l_1+l_2}h_l g_{Mn+(1,0)^T-l}\right)e^{-in\xi}\right|^2$$

in $[-\pi,\pi^2]$, and it is easy with the help of Matlab.

Theorem 1.2. *Under the hypothesis of Theorem* 1.1*, suppose ϕ is real-valued, and $h_n\in R$ for $n\in Z^2$. Then*

1. *ψ is symmetric about $x=(\frac{1}{2},\frac{1}{2})^T$ $(x=(1,0)^T)$ when ϕ is symmetric about $x=0$ $(x=(\frac{1}{2},\frac{1}{2})^T)$;*
2. *ψ is antisymmetric about $x=(\frac{1}{2},1)^T$ $(x=(1,-\frac{1}{2})^T)$ when ϕ is symmetric about $x=(\frac{1}{2},0)^T$ $(x=(0,\frac{1}{2})^T)$.*

Remark 1.2. A compactly supported M-refinable function ϕ must be M^2-refinable (i.e. 2-refinable), and satisfy $\hat{\phi}(\xi)\neq 0$ for a.e. $\xi\in R^2$. Hence, it follows from [16, Proposition 2.4.2.9] that, if ϕ is real-valued and symmetric about $\frac{c}{2}$, then $c\in Z^2$. For any ϕ compactly supported, M-refinable, real-valued, and symmetric about some $c\in Z^2$, one may make a reasonable integer shift so that the shifted version is symmetric about $x=0$, or $x=(\frac{1}{2},\frac{1}{2})^T$, or $x=(\frac{1}{2},0)^T$, or $x=(0,\frac{1}{2})^T$, and preserve other properties. So the hypothesis on ϕ is reasonable.

Conjecture. Let $N\in\mathbf{N}$. Define $\phi_N(x_1,x_2)=\tilde{\phi}_N(x_1-x_2)\tilde{\phi}_N(x_2)$, where

$$\tilde{\phi}_{2N}(\cdot)=\overbrace{\chi_{[-\frac{1}{2},\frac{1}{2}]}*\chi_{[-\frac{1}{2},\frac{1}{2}]}\cdots*\chi_{[-\frac{1}{2},\frac{1}{2}]}(\cdot)}^{2N\ \text{functions}},$$
$$\tilde{\phi}_{2N-1}(\cdot)=\overbrace{\chi_{[-\frac{1}{2},\frac{1}{2}]}*\chi_{[-\frac{1}{2},\frac{1}{2}]}\cdots*\chi_{[-\frac{1}{2},\frac{1}{2}]}(\cdot-\tfrac{1}{2})}^{2N-1\ \text{functions}}$$

for $N\in\mathbf{N}$. We conjecture that ϕ_N satisfies the hypothesis of Theorem 1.1, and the corresponding ψ_N satisfies that $\{\psi_N(\cdot-n)\colon n\in Z^2\}$ is a Riesz basis for W_0.

2. PROOFS OF THE THEOREMS

Lemma 2.1. *Under the hypothesis of Theorem* 1.1*, suppose*

$$H_0(\xi)=\left[1-\frac{1}{2}\left(\sin^2\frac{\xi_1}{2}+\sin^2\frac{\xi_2}{2}\right)\right]^N\mathcal{L}(\xi)$$

for some positive integer N and some Laurent polynomial $\mathcal{L}$. Then

$$\int_{R^2}dx\,x^\alpha\psi(x)=0$$

for $|\alpha|\le N-1$, where $|\alpha|=\alpha_1+\alpha_2$ for $\alpha=(\alpha_1,\alpha_2)\in Z^2$, $\alpha_1,\ \alpha_2\ge 0$, $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}$ for $x\in R^2$.

Proof. Since $\psi\perp V_0$, we have

$$\begin{aligned}0&=\sum_{l\in Z^2}\hat{\psi}(\cdot+2\pi l)\overline{\hat{\phi}(\cdot+2\pi l)}\\&=H_1(M^{-1}\cdot)F(M^{-1}\cdot)+H_1(M^{-1}\cdot+(\pi,\pi)^T)F(M^{-1}\cdot+(\pi,\pi)^T),\end{aligned}$$

where $H_1(\cdot) = \frac{1}{\sqrt{2}} \sum_{n \in Z^2} g_n e^{-in\cdot}$, and $F(\cdot) = \overline{H_0(\cdot)} \sum_{l \in Z^2} |\hat{\phi}(\cdot + 2\pi l)|^2$. It follows that $H_1(\cdot) F(\cdot) = -H_1(\cdot + (\pi,\pi)^T) F(\cdot + (\pi,\pi)^T)$, and consequently,

$$\sum_{0 \le l \le \alpha} \binom{\alpha}{l} D^l H_1(0) D^{\alpha - l} F(0) = - \sum_{0 \le l \le \alpha} \binom{\alpha}{l} D^l H_1((\pi,\pi)^T) D^{\alpha - l} F((\pi,\pi)^T) \tag{2.1}$$

for $|\alpha| \le N-1$, where $0 \le l \le \alpha$ means that $0 \le l_1 \le \alpha_1$, $0 \le l_2 \le \alpha_2$, $\binom{\alpha}{l} = \binom{\alpha_1}{l_1}\binom{\alpha_2}{l_2}$, and $D^l f(x_1,x_2) = \frac{\partial^l f(x_1,x_2)}{\partial x_1^{l_1} \partial x_2^{l_2}}$ for $l = (l_1,l_2)$, $\alpha = (\alpha_1,\alpha_2) \in Z^2$, l_1, l_2, α_1, $\alpha_2 \ge 0$.

When $|\alpha| = 0$, $\alpha = (0,0)$. It follows from (2.1) that

$$H_1(0)F(0) = -H_1((\pi,\pi)^T) F((\pi,\pi)^T) = 0.$$

Since ϕ has stable integer shifts, which leads to that $|H_0(\cdot)|^2 + |H_0(\cdot + (\pi,\pi)^T|^2 \neq 0$, and $H_0((\pi,\pi)^T) = 0$, we have $F(0) \neq 0$. Hence,

$$H_1(0) = 0. \tag{2.2}$$

When $|\alpha| = 1$, $\alpha = (0,1)$ or $\alpha = (1,0)$. For $\alpha = (0,1)$, it follows from (2.1) and (2.2) that $D^{(0,1)} H_1(0) F(0) = 0$, which implies

$$D^{(0,1)} H_1(0) = 0. \tag{2.3}$$

Analogously,

$$D^{(1,0)} H_1(0) = 0. \tag{2.4}$$

Assuming $D^\alpha H_1(0) = 0$ for $|\alpha| \le s < N-1$, then, for any α with $|\alpha| = s+1 \le N-1$, it follows from (2.1) that

$$D^\alpha H_1(0) F(0) = - \sum_{0 \le l \le \alpha} \binom{\alpha}{l} D^l H_1((\pi,\pi)^T) D^{\alpha - 1} F((\pi,\pi)^T) = 0. \tag{2.5}$$

Therefore,

$$D^\alpha H_1(0) = 0 \text{ for } |\alpha| \le N-1. \tag{2.6}$$

Since $\hat{\psi}(\xi) = H_1(M^{-1}\xi)\hat{\phi}(M^{-1}\xi)$, define

$$\eta = (\eta_1, \eta_2)^T = \left(\frac{\xi_1 + \xi_2}{2}, \frac{\xi_1 - \xi_2}{2} \right)^T, H_2(\xi) = H_1(M^{-1}\xi), G(\xi) = \phi(M^{-1}\xi).$$

Then $\hat{\psi}(\xi) = H_1(\eta) G(\xi) = H_2(\xi) G(\xi)$. Hence,

$$D^\alpha \hat{\psi}(\xi) = \sum_{0 \le l \le \alpha} \binom{\alpha}{l} D^l H_2(\xi) D^{\alpha - l} G(\xi) \tag{2.7}$$

for $|\alpha| \le N-1$. It is easy to check that, for any l with $|l| \le N-1$, $D^l H_2(\xi)$ can be represented as a linear combination of $D^s H_1(\eta)$ with $|s| = |l|$. Since $\eta = 0$ for $\xi = 0$, it follows from (2.6) that $D^l H_2(0) = 0$ for $|l| \le N-1$. This together with (2.7) yields that

$$D^\alpha \hat{\psi}(0) = 0 \tag{2.8}$$

for $|\alpha| \le N-1$. Therefore, $\int_R dx\, x^\alpha \psi(x) = 0$ for $|\alpha| \le N-1$. This proof is completed. □

Analogously, we have

Lemma 2.2. *Under the hypothesis of Theorem* 1.1, *suppose*

$$H_0(\xi) = \left(\frac{1+e^{-i\xi_1}}{2}\right)^N \mathcal{L}(\xi)$$

for some positive integer N and some Laurent polynomial $\mathcal{L}$. Then

$$\int_{R^2} dx\, x_1^\alpha \psi(x) = 0$$

for $0 \leq \alpha \leq N-1$.

Remark 2.1. The hypothesis on H_0 in Lemma 2.1 and Lemma 2.2 is reasonable, which can be seen in [2].

Proof of Theorem 1.1. (1) We only need to prove that $\psi \perp V_0$ since $\psi \in V_{-1}$. It is obvious that

$$\langle \psi, \phi(\cdot - m)\rangle = \sum_{n\in Z^2} (-1)^{n_1+n_2} \langle \phi_{-1,(1,0)^T-n}, \phi\rangle \langle \phi_{-1,n-Mm}, \phi\rangle.$$

Putting $n - Mm = (1,0)^T - n'$, we obtain that

$$\langle \psi, \phi(\cdot - m)\rangle = -\langle \psi, \phi(\cdot - m)\rangle.$$

Hence, $\langle \psi, \phi(\cdot - m)\rangle = 0$, and (1) follows.

Now we divide the argument into three steps to prove (2).

(i) $\{\psi(\cdot - n)\colon n \in Z^2\}$ is a Riesz basis for W_0 if and only if $\{\phi(\cdot - n), \psi(\cdot - n)\colon n \in Z^2\}$ is a Riesz basis for V_{-1}.

The "only if" part is obvious. In the following, we prove the "if" part. Suppose $\{\phi(\cdot - n), \psi(\cdot - n)\colon n \in Z^2\}$ is a Riesz basis for V_{-1}. Then we only need to prove that $W_0 = \{\sum_{n\in Z^2} c_n \psi(\cdot - n)\colon c \in l^2(Z^2)\}$, which is reduced to

$$W_0 \subset \left\{ \sum_{n\in Z^2} c_n \psi(\cdot - n)\colon c \in l^2(Z^2)\right\}$$

since $\psi \in W_0$, and W_0 is invariant under integer shifts. Since $W_0 \subset V_{-1}$, for $f \in W_0$, we have $f(\cdot) = \sum_{n\in Z^2} c_n \psi(\cdot - n) + \sum_{n\in Z^2} d_n \phi(\cdot - n)$ for some $c, d \in l^2(Z^2)$. Define $\tilde{\phi}(\cdot)$ by

$$\hat{\tilde{\phi}}(\cdot) = \frac{\hat{\phi}(\cdot)}{4\pi^2 \sum_{k\in Z^2} |\hat{\phi}(\cdot + 2k\pi)|^2}.$$

It is easy to check that $\{\tilde{\phi}(\cdot - n)\colon n \in Z^2\}$ is the dual of $\{\phi(\cdot - n)\colon n \in Z^2\}$. It follows that $0 = \langle f, \tilde{\phi}(\cdot - m)\rangle = d_m$ for $m \in Z^2$, and thus

$$f(\cdot) = \sum_{n\in Z^2} c_n \psi(\cdot - n) \in \left\{ \sum_{n\in Z^2} c_n \psi(\cdot - n)\colon c \in l^2(Z^2)\right\}.$$

(ii) Define $\tilde{\psi}(\cdot) = \sqrt{2} \sum_{n\in Z^2} (-1)^{n_1+n_2} \overline{h_{(1,0)^T-n}} \phi(M\cdot - n)$. Then $\{\phi(\cdot - n), \tilde{\psi}(\cdot - n)\colon n \in Z^2\}$ is a Riesz basis for V_{-1}.

Define $\phi_0(\cdot) = \phi(M\cdot)$, $\phi_1(\cdot) = \phi(M\cdot - (1,0)^T)$. Since $\{\phi(\cdot - n)\colon n \in Z^2\}$ is a Riesz basis for V_0, $\{\phi(M\cdot - n)\colon n \in Z^2\}$ is a Riesz basis for V_{-1}. Hence,

$\{\phi_0(\cdot - n), \phi_1(\cdot - n) \colon n \in Z^2\}$ is a Riesz basis for V_{-1}. It is easy to check that

$$\begin{aligned}\phi(\cdot) &= \sqrt{2}\sum_{n\in Z^2} h_{Mn}\phi_0(\cdot - n) + \sqrt{2}\sum_{n\in Z^2} h_{Mn+(1,0)^T}\phi_1(\cdot - n),\\ \tilde{\psi}(\cdot) &= \sqrt{2}\sum_{n\in Z^2} \overline{h_{(1,0)^T-Mn}}\phi_0(\cdot - n) - \sqrt{2}\sum_{n\in Z^2} \overline{h_{-Mn}}\phi_1(\cdot - n).\end{aligned}$$

$$\begin{aligned}&\det\begin{pmatrix} \sqrt{2}\sum_{n\in Z^2} h_{Mn}e^{-in\xi} \sqrt{2}\sum_{n\in Z^n} h_{Mn+1(1,0)^T}e^{-in\xi} \\ \sqrt{2}\sum_{n\in Z^2}\overline{h_{(1,0)^T-Mn}}e^{-in\xi} - \sqrt{2}\sum_{n\in Z^2}\overline{h_{-Mn}}e^{-in\xi}\end{pmatrix}\\ &\quad= -2[|H_0(M^{-1}\xi)|^2 + |H_0(M^{-1}\xi + (\pi,\pi)^T)|^2]\\ &\quad\neq 0\end{aligned}$$

for $\xi \in R^2$, where the last inequality is due to the fact that $\{\phi(\cdot - n) \colon n \in Z^2\}$ is a Riesz basis for V_0. By [15, Theorem 4.3], $\{\phi(\cdot - n), \tilde{\psi}(\cdot - n) \colon n \in Z^2\}$ is a Riesz basis for V_{-1}.

(iii) $\{\phi(\cdot - n), \psi(\cdot - n) \colon n \in Z^2\}$ is a Riesz basis for V_{-1} if and only if

$$\sum_{n\in Z^2}\left(\sum_{l\in Z^2}(-1)^{l_1+l_2}h_l g_{Mn+(1,0)^T-l}\right)e^{-in\xi} \neq 0$$

for $\xi \in [-\pi,\pi]^2$.

Since $\{\phi(\cdot - n) \colon n \in Z^2\}$ is a Riesz basis for V_0,

$$\sum_{n\in Z^2}\left(\sum_{l\in Z^2} h_{l+Mn}\overline{h_l}\right)e^{-in\xi} = |H_0(M^{-1}\xi)|^2 + |H_0(M^{-1}\xi + (\pi,\pi)^T)|^2 \neq 0$$

for $\xi \in R^2$. Define

$$\begin{aligned}A(\xi) &= \frac{\sum_{n\in Z^2}\left(\sum_{l\in Z^2} g_{l+Mn}\overline{h_l}\right)e^{-in\xi}}{\sum_{n\in Z^2}\left(\sum_{l\in Z^2} h_{l+Mn}\overline{h_l}\right)e^{-in\xi}},\\ B(\xi) &= -\frac{\sum_{n\in Z^2}\left(\sum_{l\in Z^2}(-1)^{l_1+l_2}h_l g_{(1,0)^T-l+Mn}\right)e^{-in\xi}}{\sum_{n\in Z^2}\left(\sum_{l\in Z^2} h_{l+Mn}\overline{h_l}\right)e^{-in\xi}},\\ \widetilde{H}_1(\xi) &= -e^{-i\xi_1}\overline{H_0(\xi + (\pi,\pi)^T)}, \quad H_1(\xi) = \frac{1}{\sqrt{2}}\sum_{n\in Z^2} g_n e^{-in\xi}.\end{aligned}$$

It is easy to check that

$$H_1(M^{-1}\xi) = A(\xi)H_0(M^{-1}\xi) + B(\xi)\widetilde{H}_1(M^{-1}\xi).$$

Multiplying with $\hat{\phi}(M^{-1}\xi)$, we obtain that

$$\widehat{\psi}(\xi) = A(\xi)\hat{\phi}(\xi) + B(\xi)\hat{\tilde{\psi}}(\xi),$$

where $\tilde{\psi}$ is defined as in (ii). By (ii), it follows from [15, Theorem 4.3] that $\{\phi(\cdot - n), \psi(\cdot - n) \colon n \in Z^2\}$ is a Riesz basis for V_{-1} if and only if $B(\xi) \neq 0$ for $\xi \in R^2$, and equivalently,

$$\sum_{n\in Z^2}\left(\sum_{l\in Z^2}(-1)^{l}h_l g_{Mn+(1,0)^T-l}\right)e^{-in\xi} \neq 0$$

for $\xi \in [-\pi,\pi]^2$. Hence, (iii) holds.

(iii) together with (i) yields (2). The proof is completed. □

Proof of Theorem 1.2. We only prove (1) under the condition that ϕ is symmetric about $x = 0$, and the other parts can be proved analogously. Suppose ϕ is symmetric about $x = 0$. Then $\langle \phi_{-1,(1,0)^T-n}, \phi\rangle = \langle \phi_{-1,n-(1,0)^T}, \phi\rangle$ for $n \in Z^2$. It follows that $g_n = g_{(2,0)^T-n}$ for $n \in Z^2$, which is equivalent to $H_1(\xi) = e^{-i2\xi_1}H_1(-\xi)$, where $H_1(\cdot) = \frac{1}{\sqrt{2}}\sum_{n\in Z^2} g_n e^{-in}$. So we have $\hat{\psi}(-\xi) = e^{i(\xi_1+\xi_2)}\hat{\psi}(\xi)$, which implies that ψ is symmetric about $x = (\frac{1}{2}, \frac{1}{2})^T$. $\square$

Example 2.1. Let

$$\phi(x) = \begin{cases} (1-|x_1-x_2|)(1-|x_2|) & \text{for } |x_1-x_2| \le 1,\ |x_2| \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$V_j = \overline{\operatorname{span}\{\phi_{j,k} \colon k \in Z\}} \text{ for } j \in Z.$$

Then

(1) $\{V_j\}_{j\in Z^2}$ is an MRA related to M with ϕ being a corresponding scaling function.

(2) Let W_0 be the orthogonal complement of V_0 in V_{-1}. Define

$$\begin{aligned}\psi(\cdot) =: &-\frac{1}{72}\phi(M\cdot-(-2,-1)^T) + \frac{1}{12}\phi(M\cdot-(-1,-1)^T) - \frac{1}{36}\phi(M\cdot-(0,-1)^T) \\ &+ \frac{5}{36}\phi(M\cdot-(1,-1)^T) - \frac{1}{72}\phi(M\cdot-(2,-1)^T) - \frac{1}{18}\phi(M\cdot-(-1,0)^T) \\ &+ \frac{1}{3}\phi(M\cdot) - \frac{1}{3}\phi(M\cdot-(1,0)^T) + \frac{1}{3}\phi(M\cdot-(2,0)^T) \\ &- \frac{1}{18}\phi(M\cdot-(3,0)^T) - \frac{1}{72}\phi(M\cdot-(0,1)^T) + \frac{1}{12}\phi(M\cdot-(1,1)^T) \\ &- \frac{5}{36}\phi(M\cdot-(2,1)^T) + \frac{5}{36}\phi(M\cdot-(3,1)^T) - \frac{1}{72}\phi(M\cdot-(4,1)^T).\end{aligned}$$

Then $\{\psi(\cdot - n) \colon n \in Z^2\}$ is a Riesz basis for W_0, ψ is symmetric about $x = (\frac{1}{2}, \frac{1}{2})^T$, and

$$\int_{R^2} dx\, \psi(x) = \int_{R^2} dx\, x_1\psi(x) = 0.$$

Proof. (1) It is easy to check that $\hat{\phi}(\cdot) = H_0(M^{-1}\cdot)\hat{\phi}(M^{-1})$, where

$$H_0(\xi) = \frac{1}{2} + \frac{1}{4}e^{i\xi_1} + \frac{1}{4}e^{-i\xi_1} \text{ for } \xi \in R^2. \tag{2.9}$$

So ϕ is M-refinable, and thus $M^2 = 2I$-refinable. It follows that

$$\bigcup_{j\in Z} V_j = \bigcup_{j\in Z} V_{2j} \text{ and } \bigcap_{j\in Z} V_j = \bigcap_{j\in Z} V_{2j}. \tag{2.10}$$

Then applying [17, Corollary 4.14] and [17, Theorem 4.5], we have

$$\bigcap_{j\in Z} V_j = \{0\} \text{ and } \overline{\bigcup_{j\in Z} V_j} = L^2(R^2).$$

By simple computation, we obtain that

$$\begin{aligned}&\sum_{l\in Z^2}|\hat{\phi}(\xi+2\pi l)|^2\\&=\frac{1}{4\pi^2}\left[\sum_{k\in Z}\left(\frac{2}{\xi_1+2\pi k}\sin\frac{\xi_1+2\pi k}{2}\right)^4\right]\left[\sum_{k\in Z}\left(\frac{2}{\xi_1+\xi_2+2\pi k}\sin\frac{\xi_1+\xi_2+2\pi k}{2}\right)^4\right]\\&>0\end{aligned}$$

for $\xi \in R^2$. Since ϕ is compactly supported, $\sum_{l\in Z^2}|\hat{\phi}(\cdot+2\pi l)|^2$ is continuous. Hence, $A\leq\sum_{l\in Z^2}|\hat{\phi}(\cdot+2\pi l)|^2\leq B$ for some $0<A<B<\infty$, and thus $\{\phi(\cdot-n)\colon n\in Z^2\}$ is a Riesz basis for V_0. Note that ϕ is M-refinable. This together with (2.10) yields that $\{V_j\}_{j\in Z^2}$ is an MRA related to M.

(2) Let g_n be defined as in Theorem 1.1 for $n\in Z^2$. It is easy to check that

$$\begin{aligned}&\left|\sum_{n\in Z^2}\left(\sum_{l\in Z^2}(-1)^{l_1+l_2}h_l g_{Mn+(1,0)^T-l}\right)e^{-in\xi}\right|^2\\&\quad=\frac{1}{16}\left(1+\frac{1}{6}\cos\xi_1+\frac{1}{4}\cos\xi_2+\frac{7}{9}\cos(\xi_1+\xi_2)+\frac{1}{4}\cos(2\xi_1+\xi_2)\right)^2\\&\qquad+\frac{1}{16}\left(\frac{1}{18}\sin\xi_1+\frac{1}{18}\sin\xi_2+\frac{1}{18}\sin(\xi_1+\xi_2)\right)^2,\end{aligned}$$

which is nonzero in $[-\pi,\pi]^2$ by some estimation. Therefore, by Theorem 1.1, Lemma 2.2, and Theorem 1.2, $\{\psi(\cdot-n)\colon n\in Z^2\}$ is a Riesz basis for W_0, ψ is symmetric about $x=(\frac{1}{2},\frac{1}{2})^T$, and

$$\int_{R^2}dx\,\psi(x)=\int_{R^2}dx\,x_1\psi(x)=0.$$

The proof is completed. □

Example 2.2. Let

$$\phi(x)=:\begin{cases}(\frac{(x_1-x_2)^2}{2}+x_1-x_2+\frac{1}{2})(\frac{x_2^2}{2}+x_2+\frac{1}{2}) & -1\leq x_1-x_2\leq 0,\ -1\leq x_2\leq 0,\\(\frac{(x_1-x_2)^2}{2}+x_1-x_2+\frac{1}{2})(-x_2^2+x_2+\frac{1}{2}) & -1\leq x_1-x_2\leq 0,\ 0\leq x_2\leq 1,\\(\frac{(x_1-x_2)^2}{2}+x_1-x_2+\frac{1}{2})(\frac{x_2^2}{2}-2x_2+2) & -1\leq x_1-x_2\leq 0,\ 1\leq x_2\leq 2,\\(-(x_1-x_2)^2+x_1-x_2+\frac{1}{2})(\frac{x_2^2}{2}+x_2+\frac{1}{2}) & 0\leq x_1-x_2\leq 1,\ -1\leq x_2\leq 0,\\(-(x_1-x_2)^2+x_1-x_2+\frac{1}{2})(-x_2^2+x_2+\frac{1}{2}) & 0\leq x_1-x_2\leq 1,\ 0\leq x_2\leq 1,\\(-(x_1-x_2)^2+x_1-x_2+\frac{1}{2})(\frac{x_2^2}{2}-2x_2+2) & 0\leq x_1-x_2\leq 1,\ 1\leq x_2\leq 2,\\(\frac{(x_1-x_2)^2}{2}-2(x_1-x_2)+2)(\frac{x_2^2}{2}+x_2+\frac{1}{2}) & 1\leq x_1-x_2\leq 2,\ -1\leq x_2\leq 0,\\(\frac{(x_1-x_2)^2}{2}-2(x_1-x_2)+2)(-x_2^2+x_2+\frac{1}{2}) & 1\leq x_1-x_2\leq 2,\ 0\leq x_2\leq 1,\\(\frac{(x_1-x_2)^2}{2}-2(x_1-x_2)+2)(\frac{x_2^2}{2}-2x_2+2) & 1\leq x_1-x_2\leq 2,\ 1\leq x_2\leq 2,\\0 & \text{otherwise,}\end{cases}$$

$V_j=\overline{\operatorname{span}\{\phi_{j,k}\colon k\in Z\}}$ for $j\in Z$, W_0 be the orthogonal complementary subspace of V_0 in V_{-1}, and ψ be defined as in Theorem 1.1. Then, by similar arguments to those of Example 2.1, we can show that $\{V_j\}_{j\in Z^2}$ is an MRA related to M, that

$\{\psi(\cdot - n)\colon n \in Z^2\}$ is a Riesz basis for W_0, ψ is antisymmetric about $x = (1, \frac{1}{2})^T$, and

$$\int_R dx\psi(x) = \int_R dx\, x_1\psi(x) = \int_R dx\, x_1^2\psi(x) = 0.$$

Acknowledgement

The author would like to thank the referee for his encouragement and valuable suggestions.

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