

## A VAN DER CORPUT LEMMA FOR THE $p$ -ADIC NUMBERS

KEITH M. ROGERS

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ABSTRACT. We prove a version of van der Corput's lemma for polynomials over the  $p$ -adic numbers.

### 1. INTRODUCTION

The following lemma goes back to J.G. van der Corput [4]. It has many applications in number theory and harmonic analysis. In particular, it is key to the study of oscillatory integrals (see [8]). We note that only partial van der Corput-type lemmas are known in dimensions greater than one (see [3]). As a consequence the theory of oscillatory integrals in higher dimensions is relatively open.

**Lemma 1.** *Suppose that  $f : (a, b) \rightarrow \mathbb{R}$  is  $n$  times differentiable, where  $n \geq 2$ , and  $|f^{(n)}(x)| \geq \lambda > 0$  on  $(a, b)$ . Then*

$$\left| \int_a^b e^{if(x)} dx \right| \leq C_n \frac{n}{\lambda^{1/n}},$$

where  $C_n \leq 2^{5/3}$  for all  $n \geq 2$  and  $C_n \rightarrow 4/e$  as  $n \rightarrow \infty$ .

It can be shown by considering  $f(x) = x^n$  that the linear growth in  $n$  is optimal, and this more precise formulation is due to G.I. Arhipov, A.A. Karacuba and V.N. Čubarikov [1]. Consideration of the Chebyshev polynomials shows that the constant is asymptotically sharp [6]. The following corollary can be easily obtained using Stirling's formula.

**Corollary 2.** *Let  $f(x) = a_0 + a_1x + \cdots + a_nx^n$  be a real polynomial of degree  $n \geq 1$ . Then*

$$\left| \int_a^b e^{if(x)} dx \right| \leq \frac{2^{5/3}e}{|a_n|^{1/n}} < \frac{9}{|a_n|^{1/n}}$$

for all  $a, b \in \mathbb{R}$ .

We will prove a  $p$ -adic version of this corollary, opening the way for the study of oscillatory integrals on the  $p$ -adics. This problem was first considered by J. Wright [10], who proved lemmas for polynomials of degree two and monomials of degree three.

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2. INTRODUCTION TO THE  $p$ -ADIC NUMBERS

For a more complete introduction to the  $p$ -adic numbers, see [5] or [9]. Here we will outline what we need.

Fix a prime number  $p$ . Any nonzero rational number  $x$  can be uniquely expressed in the form  $p^k m/n$ , where  $m$  and  $n$  have no common divisors and neither is divisible by  $p$ . We then define the  $p$ -adic norm on the rational numbers by  $|x| = p^{-k}$  when  $x \neq 0$ , and  $|0| = 0$ . We obtain the  $p$ -adic numbers by completing  $\mathbb{Q}$  with respect to this norm. It is not difficult to show that the norm satisfies the following properties:

$$\begin{aligned} |xy| &= |x||y|, \\ |x+y| &\leq \max\{|x|, |y|\}. \end{aligned}$$

It follows from the second property that when  $|x| \neq |y|$ ,

$$|x+y| = \max\{|x|, |y|\}.$$

Similarly, it can be shown that

$$\{y : |y - x_1| \leq p^r\} = \{y : |y - x_2| \leq p^r\}$$

when  $|x_1 - x_2| \leq p^r$ , so every point within a ball can be considered to be its centre.

A nonzero  $p$ -adic number  $x$  with  $|x| = p^{-k}$  may be written in the form

$$x = \sum_{j=k}^{\infty} x_j p^j,$$

where  $0 \leq x_j \leq p-1$  and  $x_k \neq 0$ . This will be called the standard  $p$ -adic expansion, and the arithmetic is done formally with carrying. Define  $\chi : \mathbb{Q}_p \rightarrow \mathbb{C}$  by

$$\chi(x) = \begin{cases} \prod_{j=k}^{-1} e^{2\pi i x_j / p^j}, & |x| > 1, \\ 1, & |x| \leq 1. \end{cases}$$

The characters of  $\mathbb{Q}_p$  are all of the form  $\chi_\epsilon : \mathbb{Q}_p \rightarrow \mathbb{C}$ ;

$$\chi_\epsilon(x) = \chi(\epsilon x),$$

where  $\epsilon \in \mathbb{Q}_p$ . Finally  $\mathbb{Q}_p$  is a locally compact commutative group, so there is a Haar measure that necessarily satisfies  $d(ax) = |a|dx$ , where  $dx$  denotes an element of this measure. We normalise the measure so that  $\{x \in \mathbb{Q}_p : |x| \leq p^r\}$  has measure  $p^r$ .

The usual arguments can be employed to obtain the standard Fourier results.

3.  $p$ -ADIC VAN DER CORPUT LEMMAS

The main thrust of the paper is to prove the following lemma for  $p$ -adic polynomials. The Euclidean arguments will not be helpful as there is no order on the  $p$ -adic numbers.

**Lemma 3.** *Suppose that  $a_1, \dots, a_n \in \mathbb{Q}_p$ . Then*

$$\left| \int_{|x| \leq 1} \chi(a_1 x + \dots + a_n x^n) dx \right| \leq \frac{p^m}{|ma_m|^{1/m}},$$

where  $m = \max\{l : |la_l| \geq |ja_j| \text{ for all } j \neq l\}$ .

Before proving Lemma 3 we note some easy corollaries.

**Corollary 4.** *Suppose that  $a_1, \dots, a_n \in \mathbb{Q}_p$ . Then*

$$\left| \int_{|x| \leq 1} \chi(a_1x + \dots + a_nx^n) dx \right| \leq \frac{2p^n}{\lambda^{1/n}},$$

where  $\lambda = \max_{1 \leq j \leq n} |a_j|$ .

*Proof.* Suppose that  $|a_k| = \max_{1 \leq j \leq n} |a_j| = \lambda$ . By Lemma 3, we have

$$|I| = \left| \int_{|x| \leq 1} \chi(a_1x + \dots + a_nx^n) dx \right| \leq \frac{p^m}{|ma_m|^{1/m}},$$

where  $m = \max\{l : |la_l| \geq |ja_j| \text{ for all } j \neq l\}$ . Now as  $|ma_m| \geq |ka_k|$ ,

$$|I| \leq \frac{p^m}{|ka_k|^{1/m}} \leq \frac{k^{1/m}p^m}{|a_k|^{1/m}} \leq \frac{n^{1/m}p^m}{\lambda^{1/n}}.$$

Finally, it is easy to calculate that  $n^{1/m}p^m \leq 2p^n$  for all  $m = 1, \dots, n$ , so that

$$|I| \leq \frac{2p^n}{\lambda^{1/n}},$$

as desired. □

For an application of this result see [7], where maximal averages along curves defined by polynomials are  $L^q(\mathbb{Q}_p^n)$  bounded, where  $1 < q < \infty$ .

We will need some notation. Let

$$f(y) = a_0 + a_1y + \dots + a_ny^n,$$

and for all  $j = 1, \dots, n - 1$ , define  $b_j$  by

$$(1) \quad b_j(y) = \frac{f^{(j)}(y)}{j!} = a_j + \binom{j+1}{j} a_{j+1}y + \dots + \binom{n}{j} a_ny^{n-j}.$$

Then,

$$\begin{aligned} & a_0 + a_1(x+y) + \dots + a_n(x+y)^n \\ &= f(y) + f'(y)x + \frac{f^{(2)}(y)}{2!}x^2 + \dots + \frac{f^{(n)}(y)}{n!}x^n \\ &= a_0 + \dots + a_ny^n + b_1(y)x + \dots + b_{n-1}(y)x^{n-1} + a_nx^n. \end{aligned}$$

The next corollary is the main result of the paper. It holds uniformly over all balls, and is the  $p$ -adic equivalent of Corollary 2 in Section 1.

**Corollary 5.** *Suppose that  $x_0, a_0, \dots, a_n \in \mathbb{Q}_p$ , where  $n \geq 1$ , and suppose that  $r \in \mathbb{Z}$ . Then*

$$\left| \int_{|x-x_0| \leq p^r} \chi(a_0 + a_1x + \dots + a_nx^n) dx \right| \leq \frac{2p^n}{|a_n|^{1/n}}.$$

*Proof.* Let  $y = p^r(x - x_0)$ , so that

$$\begin{aligned} I &= \int_{|x-x_0| \leq p^r} \chi(a_0 + a_1x + \dots + a_nx^n) dx \\ &= \int_{|y| \leq 1} \chi\left(a_0 + a_1\left(\frac{y}{p^r} + x_0\right) + \dots + a_n\left(\frac{y}{p^r} + x_0\right)^n\right) dx \\ &= p^r \int_{|y| \leq 1} \chi\left(a_0 + \dots + a_ny^n + \frac{b_1(x_0)y}{p^r} + \dots + \frac{b_{n-1}(x_0)y^{n-1}}{p^{(n-1)r}} + \frac{a_ny^n}{p^{nr}}\right) dy \\ &= p^r I_1, \end{aligned}$$

say, where the  $b_j$  are defined as in (1). We note that

$$\begin{aligned} |I_1| &= \left| \chi(a_0 + \dots + a_ny^n) \int_{|y| \leq 1} \chi\left(\frac{b_1(x_0)y}{p^r} + \dots + \frac{b_{n-1}(x_0)y^{n-1}}{p^{(n-1)r}} + \frac{a_ny^n}{p^{nr}}\right) dy \right| \\ &= \left| \int_{|y| \leq 1} \chi\left(\frac{b_1(x_0)y}{p^r} + \dots + \frac{b_{n-1}(x_0)y^{n-1}}{p^{(n-1)r}} + \frac{a_ny^n}{p^{nr}}\right) dy \right| \\ &= |I_2|, \end{aligned}$$

say. Thus,

$$|I| = p^r |I_2| \leq p^r \frac{2p^n}{|p^{-nr}a_n|^{1/n}} = \frac{2p^n}{|a_n|^{1/n}},$$

by Corollary 4. □

#### 4. PRELIMINARY LEMMAS

Our starting point is the following well-known lemma. For a proof see [9].

**Lemma 6.** *Suppose that  $a \in \mathbb{Q}_p$  and  $|a| > 1$ . Then*

$$\int_{|x| \leq 1} \chi(ax) dx = 0.$$

**Lemma 7.** *Suppose that  $|ma_m| > |ja_j|$  for all  $j > m$ , and  $|y| \leq 1$ . Then*

$$|mb_m(y)| = |ma_m| > |jb_j(y)|$$

for all  $j > m$ , where the  $b_j$  are defined as in (1).

*Proof.* Suppose that  $|ma_m| > |ja_j|$  for all  $j > m$ . Then

$$|ma_m| > \left| \binom{j-1}{m-1} \right| |ja_j|,$$

so that

$$|a_m| > \left| \binom{j}{m} a_j \right|$$

for all  $j > m$ . Thus

$$|mb_m(y)| = |m| \left| a_m + \binom{m+1}{m} a_{m+1}y + \dots + \binom{n}{m} a_ny^{n-m} \right| = |ma_m|$$

for all  $|y| \leq 1$ . Similarly, if  $k > j > m$ , then

$$|ma_m| > \left| \binom{k-1}{j-1} \right| |ka_k|,$$

so that

$$|ma_m| > \left| j \binom{k}{j} a_k \right|.$$

Putting these together,

$$|mb_m(y)| = |ma_m| > \left| ja_j + j \binom{j+1}{j} a_{j+1}y + \cdots + j \binom{n}{j} a_n y^{n-j} \right| = |jb_j(y)|$$

for all  $|y| \leq 1$ . □

**Lemma 8.** *Suppose that  $|a_1| > p$  and  $|a_1| > |ja_j|$  for  $j > 1$ . Then*

$$\int_{|x| \leq 1} \chi(a_1x + \cdots + a_nx^n) dx = 0.$$

*Proof.* Let  $|a_1| = p^{k+1}$ , where  $k \geq 1$ . We split the integral into  $p^k$  pieces:

$$\begin{aligned} I &= \int_{|x| \leq 1} \chi(a_1x + \cdots + a_nx^n) dx \\ &= \sum_{y=0}^{p^k-1} \int_{|h| \leq p^{-k}} \chi(a_1(y+h) + \cdots + a_n(y+h)^n) dh. \end{aligned}$$

Now

$$\begin{aligned} &a_1(y+h) + \cdots + a_n(y+h)^n \\ &= a_1y + \cdots + a_ny^n + b_1(y)h + \cdots + b_{n-1}(y)h^{n-1} + a_nh^n, \end{aligned}$$

where  $b_j$  is defined as in (1). Thus

$$I = \sum_{y=0}^{p^k-1} \chi(a_1y + \cdots + a_ny^n) I_1(y),$$

where

$$\begin{aligned} I_1(y) &= \int_{|h| \leq p^{-k}} \chi(b_1(y)h + \cdots + b_{n-1}(y)h^{n-1} + a_nh^n) dh \\ &= \frac{1}{p^k} \int_{|x| \leq 1} \chi(b_1(y)p^kx + \cdots + b_{n-1}(y)p^{(n-1)k}x^{n-1} + a_np^{nk}x^n) dx. \end{aligned}$$

When  $|y| \leq 1$ ,

$$|b_1(y)| = |a_1| > |jb_j(y)|$$

for all  $j > 1$ , by Lemma 7. Hence

$$|b_1(y)p^k| = \frac{|a_1|}{p^k} = p$$

and

$$|jb_j(y)p^{jk}|p^{jk} < |b_1(y)p^k|p^k = p^{k+1}.$$

So if  $j > 1$ , then

$$|b_j(y)p^{jk}| \leq \frac{1}{|j|p^{(j-1)k}} \leq \frac{j}{p^{(j-1)k}} \leq \frac{2}{2^{(2-1)1}} = 1.$$

Thus, by Lemma 6,

$$\begin{aligned} I_1(y) &= \frac{1}{p^k} \int_{|x| \leq 1} \chi(b_1(y)p^k x) \chi(b_2(y)p^{2k} x^2 + \dots + a_n p^{nk} x^n) dx \\ &= \frac{1}{p^k} \int_{|x| \leq 1} \chi(b_1(y)p^k x) dx = 0 \end{aligned}$$

for all  $|y| \leq 1$ , and we are done. □

**Lemma 9.** *Suppose that  $|ma_m| > p^2$  and  $|ma_m| > |ja_j|$  for all  $j \neq m$ . Then*

$$\int_{|x| \leq 1} \chi(a_1 x + \dots + a_n x^n) dx = \frac{1}{p} \int_{|x| \leq 1} \chi(a_1 p x + \dots + a_n p^n x^n) dx.$$

*Proof.* We split the integral into  $p$  pieces:

$$I = \int_{|x| \leq 1} \chi(a_1 x + \dots + a_n x^n) dx = \sum_{y=0}^{p-1} \chi(a_1 y + \dots + a_n y^n) I_1(y),$$

where

$$\begin{aligned} I_1(y) &= \int_{|h| \leq 1/p} \chi(b_1(y)h + \dots + b_{n-1}(y)h^{n-1} + a_n h^n) dh \\ &= \frac{1}{p} \int_{|x| \leq 1} \chi(b_1(y)px + \dots + b_{n-1}(y)p^{n-1}x^{n-1} + a_n p^n x^n) dx, \end{aligned}$$

and  $b_j$  is given by (1).

We aim to apply Lemma 8. When  $|y| = 1$ ,

$$|b_1(y)p| = |a_1 + 2a_2 y + \dots + na_n y^{n-1}|/p = |ma_m|/p > p.$$

Now if  $k > j \geq 2$ , then

$$|ma_m| \geq \left| \binom{k-1}{j-1} \right| |ka_k| = \left| j \binom{k}{j} a_k \right|$$

so that

$$|ma_m| \geq \left| ja_j + j \binom{j+1}{j} a_{j+1} y + \dots + j \binom{n}{j} a_n y^{n-j} \right| = |jb_j(y)|.$$

Hence if  $j \geq 2$ , then

$$|jb_j(y)p^j| \leq \frac{|ma_m|}{p^j} = \frac{|b_1(y)p|}{p^{j-1}} < |b_1(y)p|.$$

Thus, by Lemma 8, we have  $I_1(y) = 0$  for all  $y$  such that  $|y| = 1$ , so that  $I = I_1(0)$ . □

### 5. PROOF OF LEMMA 3

We use double induction on

$$m = \max\{l : |la_l| \geq |ja_j| \text{ for all } j \neq l\}$$

and

$$r = \max_{1 \leq j \leq n} \log_p |ja_j|.$$

First we note that

$$|I| = \left| \int_{|x| \leq 1} \chi(a_1x + \cdots + a_nx^n) dx \right| \leq \int_{|x| \leq 1} |\chi(a_1x + \cdots + a_nx^n)| dx = 1.$$

Suppose that  $m = 1$ . When  $r \leq 1$ , we have

$$\frac{p^m}{|ma_m|^{1/m}} = \frac{p}{|a_1|} \geq \frac{p}{p} = 1,$$

so that

$$|I| \leq 1 \leq \frac{p^m}{|ma_m|^{1/m}}$$

and we are done. When  $r > 1$ , we have  $|a_1| > p$ , and as  $m = 1$ , we have  $|a_1| > |ja_j|$  for all  $j > 1$ . Thus, when  $m = 1$  and  $r > 1$ , we obtain the result by Lemma 8.

Now suppose that  $m > 1$  and  $r \leq 2$ . Again we are done, as

$$\frac{p^m}{|ma_m|^{1/m}} \geq \frac{p^2}{p^{2/2}} \geq 1 \geq |I|.$$

So when  $m = 1$  or  $r \leq 2$ , we have the result.

Suppose the result holds when  $m \leq k - 1$  and  $r \leq s - 1$ , and suppose that  $m = k$  and  $r = s$ . When  $|y| \leq 1$ , we have

$$\{x : |x| \leq 1\} = \{x : |x - y| \leq 1\}$$

so that by a change of variables,

$$\begin{aligned} |I| &= \left| \int_{|x-y| \leq 1} \chi(a_1x + \cdots + a_nx^n) dx \right| \\ &= \left| \int_{|h| \leq 1} \chi(a_1(h+y) + \cdots + a_n(h+y)^n) dh \right| \\ &= \left| \int_{|h| \leq 1} \chi(b_1(y)h + \cdots + b_{n-1}(y)h^{n-1} + a_nh^n) dh \right|, \end{aligned}$$

for all  $|y| \leq 1$ , where the  $b_j$  are given by (1).

As  $m = k$ , we have  $|ka_k| > |ja_j|$  for all  $j > k$ . Thus when  $|y| \leq 1$ , we have  $|kb_k(y)| > |jb_j(y)|$  for all  $j > k$ , by Lemma 7. We choose  $y = y_1$ , so that

$$\max_{1 \leq j < k} |jb_j(y_1)| = \min_{|y| \leq 1} \max_{1 \leq j < k} |jb_j(y)|.$$

Either  $\max_{1 \leq j < k} |jb_j(y_1)| < |kb_k(y_1)|$  or  $\max_{1 \leq j < k} |jb_j(y_1)| \geq |kb_k(y_1)|$ .

When  $\max_{1 \leq j < k} |jb_j(y_1)| < |kb_k(y_1)|$ , we have

$$|kb_k(y_1)| > |jb_j(y_1)|$$

for all  $j \neq k$ , so we can apply Lemma 9 to obtain

$$|I| = \frac{1}{p} \left| \int_{|h| \leq 1} \chi(b_1(y_1)ph + \cdots + b_{n-1}(y_1)p^{n-1}h^{n-1} + a_np^nh^n) dh \right|.$$

Now as  $\max_{1 \leq j \leq n} |jb_j(y_1)p^j| \leq p^{s-1}$ , we have

$$r = \max_{1 \leq j \leq n} \log_p |jb_j(y_1)p^j| \leq s - 1.$$

Since  $|kb_k(y_1)| > |jb_j(y_1)|$  for all  $j > k$ ,

$$m = \max\{l : |lb_l(y_1)p^l| \geq |jb_j(y_1)p^j| \text{ for all } j \neq l\} = k_1 \leq k.$$

Hence

$$|I| \leq \frac{1}{p} \frac{p^{k_1}}{|k_1 b_{k_1}(y_1) p^{k_1}|^{1/k_1}} = \frac{p^{k_1}}{|k_1 b_{k_1}(y_1)|^{1/k_1}} \leq \frac{p^k}{|kb_k(y_1)|^{1/k}},$$

by induction. Thus

$$|I| \leq \frac{p^k}{|ka_k|^{1/k}},$$

as  $|kb_k(y_1)| = |ka_k|$  by Lemma 7.

When  $\max_{1 \leq j < k} |jb_j(y_1)| \geq |kb_k(y_1)|$ ,

$$\max_{1 \leq j < k} |jb_j(y)| \geq |kb_k(y)|,$$

for all  $|y| \leq 1$ , as

$$\max_{1 \leq j < k} |jb_j(y)| \geq \min_{|y| \leq 1} \max_{1 \leq j < k} |jb_j(y)| = \max_{1 \leq j < k} |jb_j(y_1)|,$$

by definition, and

$$|kb_k(y)| = |kb_k(y_1)|$$

by Lemma 7. We split the integral into  $p$  pieces, so that

$$I = \int_{|x| \leq 1} \chi(a_1 x + \dots + a_n x^n) dx = \sum_{y=0}^{p-1} \chi(a_1 y + \dots + a_n y^n) I_1(y),$$

where

$$\begin{aligned} I_1(y) &= \int_{|h| \leq 1/p} \chi(b_1(y)h + \dots + b_{n-1}(y)h^{n-1} + a_n h^n) dh \\ &= \frac{1}{p} \int_{|x| \leq 1} \chi(b_1(y)px + \dots + b_{n-1}(y)p^{n-1}x^{n-1} + a_n p^n x^n) dx, \end{aligned}$$

and  $b_j$  is given by (1). Now by Lemma 7, we have

$$|kb_k(y)| = |ka_k| > |lb_l(y)|$$

for all  $l > k$ , so that

$$\max_{1 \leq j < k} |jb_j(y)| \geq |kb_k(y_1)| = |ka_k| \geq |lb_l(y)|$$

for all  $l \geq k$ . Thus for  $y = 0, \dots, p - 1$ , there exists  $k_1 < k$  (where  $k_1$  depends on  $y$ ) such that

$$|k_1 b_{k_1}(y)| \geq |jb_j(y)|$$

and

$$|k_1 b_{k_1}(y) p^{k_1}| > |jb_j(y) p^j|$$

for all  $j > k_1$ . Hence for each  $y = 0, \dots, p - 1$  and  $I_1(y)$  defined by

$$I_1(y) = \frac{1}{p} \int_{|x| \leq 1} \chi(b_1(y)px + \dots + b_{n-1}(y)p^{n-1}x^{n-1} + a_n p^n x^n) dx,$$

we have

$$m = \max\{l : |lb_l(y)p^l| \geq |jb_j(y)p^j| \text{ for all } j \neq l\} = k_1 < k.$$

Thus

$$|I_1(y)| \leq \frac{1}{p} \frac{p^{k_1}}{|k_1 b_{k_1}(y) p^{k_1}|^{1/k_1}} = \frac{p^{k_1}}{|k_1 b_{k_1}(y)|^{1/k_1}},$$

by induction. Finally,

$$\frac{p^{k_1}}{|k_1 b_{k_1}(y)|^{1/k_1}} \leq \frac{p^{k-1}}{|k b_k(y)|^{1/k}} = \frac{p^{k-1}}{|k a_k|^{1/k}},$$

by Lemma 7, so that

$$|I| \leq \sum_{y=0}^{p-1} |I_1(y)| \leq p \frac{p^{k-1}}{|k a_k|^{1/k}} = \frac{p^k}{|k a_k|^{1/k}},$$

and we are done. □

### 6. SUBLEVEL SET ESTIMATES

Finally we obtain the corresponding sublevel set estimate. In the one-dimensional real case, the van der Corput lemma can be obtained from the sublevel set estimate. It is not clear, however, if this is possible in the  $p$ -adic setting.

**Corollary 10.** *Suppose that  $a_0, \dots, a_n \in \mathbb{Q}_p$ , where  $n \geq 1$ . Then*

$$|\{x \in \mathbb{Q}_p : |a_0 + a_1 x + \dots + a_n x^n| \leq 1\}| \leq \frac{C_n}{|a_n|^{1/n}}.$$

*Proof.* Let  $f(x) = a_0 + a_1 x + \dots + a_n x^n$  and define  $\phi$  to be the characteristic function of the unit ball centered at the origin, so that

$$\phi(x) = \begin{cases} 1 & \text{when } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 6,

$$\int_{\mathbb{Q}_p} \phi(\xi) \chi(x\xi) d\xi = \int_{|\xi| \leq 1} \chi(x\xi) d\xi = \phi(x),$$

so that

$$\int_{|\xi| \leq 1} \chi(f(x)\xi) d\xi = \phi(f(x)).$$

Thus, by Fubini's theorem,

$$\int_{|x| \leq p^r} \phi(f(x)) dx = \int_{|\xi| \leq 1} \int_{|x| \leq p^r} \chi(f(x)\xi) dx d\xi.$$

Now by Corollary 5, the inner integral is bounded above by  $2p^n/|\xi a_n|^{1/n}$ , so that

$$|\{x \in \mathbb{Q}_p : |x| \leq p^r, |f(x)| \leq 1\}| = \int_{|x| \leq p^r} \phi(f(x)) dx \leq \frac{2p^n}{|a_n|^{1/n}} \int_{|\xi| \leq 1} \frac{1}{|\xi|^{1/n}} d\xi.$$

The final integral is finite (see [9] for the calculation), so that

$$|\{x \in \mathbb{Q}_p : |x| \leq p^r, |f(x)| \leq 1\}| \leq C_n/|a_n|^{1/n}.$$

Finally we let  $r$  tend to infinity, and we are done. □

A similar result was recently proven by E. Breuillard and T. Gelander [2]. They use different techniques to show that if  $f \in \mathbb{Q}_p[x]$  is monic, then  $|\{x \in \mathbb{Q}_p : |f(x)| \leq 1\}| \leq C$ , where  $C$  is an absolute constant.

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SCHOOL OF MATHEMATICS, UNIVERSITY OF NEW SOUTH WALES, SYDNEY, NSW 2052, AUSTRALIA

*E-mail address:* K.M.Rogers.99@cantab.net

*Current address:* Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain