

## SYMMETRIC STABLE PROCESSES IN PARABOLA-SHAPED REGIONS

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ABSTRACT. We identify the critical exponent of integrability of the first exit time of the rotation invariant stable Lévy process from a parabola-shaped region.

### 1. INTRODUCTION

For  $d = 2, 3, \dots$  and  $0 < \beta < 1$ , we define the *parabola-shaped region* in  $\mathbb{R}^d$  as

$$\mathcal{P}_\beta = \{x = (x_1, \tilde{x}) : x_1 > 0, \tilde{x} \in \mathbb{R}^{d-1}, |\tilde{x}| < x_1^\beta\}.$$

Let  $0 < \alpha < 2$ . By  $\{X_t\}$  we denote the isotropic  $\alpha$ -stable  $\mathbb{R}^d$ -valued Lévy process ([17]). The process is time-homogeneous, has right-continuous trajectories with left limits,  $\alpha$ -stable rotation invariant independent increments, and characteristic function

$$(1) \quad \mathbf{E}_x e^{i\xi(X_t-x)} = e^{-t|\xi|^\alpha}, \quad x \in \mathbb{R}^d, \quad \xi \in \mathbb{R}^d, \quad t \geq 0.$$

Here  $\mathbf{E}_x$  is the expectation with respect to the distribution  $\mathbf{P}_x$  of the process starting from  $x \in \mathbb{R}^d$ . For an open set  $U \subset \mathbb{R}^d$ , we define  $\tau_U = \inf\{t \geq 0; X_t \notin U\}$ , the first exit time of  $U$  ([17]). In the case of the parabola-shaped region  $\mathcal{P}_\beta$ , we simply write  $\tau_\beta$  for  $\tau_{\mathcal{P}_\beta}$ .

The main result of this note is the following result.

**Theorem 1.** *Let  $p \geq 0$ . Then  $E_x \tau_\beta^p < \infty$  for some (hence for all)  $x \in \mathcal{P}_\beta$  if and only if  $p < p_0$ , where*

$$p_0 = \frac{(d-1)(1-\beta) + \alpha}{\alpha\beta}.$$

Theorem 1 may be regarded as an addition to the research direction initiated in [3], where it was proved that for  $\beta = 1/2$ ,  $d = 2$  and  $\{X_t\}$  replaced by the *Brownian motion process*  $\{B_t\}$ ,  $\tau_\beta$  is subexponentially integrable. For  $\{B_t\}$ , the generalizations to all the considered domains  $\mathcal{P}_\beta$  (along with essential strengthening of the result of [3]) were subsequently obtained in [14], [15], [5], [11] and [2]. We should note that this direction of research was influenced to a large degree by the

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result of Burkholder [10] on the critical order of integrability of the exit times of  $\{B_t\}$  from cones. For more on the many generalizations of Burkholder's result we refer the reader to [4] and references therein.

Brownian motion is a limiting case of the isotropic  $\alpha$ -stable process and  $B_{2t}$  corresponds to  $\alpha = 2$  via the analogue of (1). Extension of some of the above-mentioned results pertaining to cones to the case  $0 < \alpha < 2$  (that is, for  $\{X_t\}$ ) were given in [12], [13], [16] and [1]; see also [7]. It should be emphasized that while there are many similarities between  $\{X_t\}$  and  $\{B_{2t}\}$ , there also exist essential differences in their respective properties and their proofs. For example the critical integrability of the exit time of  $\{X_t\}$  is less than 1 for every cone, however narrow it may be ([1]), while it is arbitrarily large for  $\{B_t\}$  in sufficiently narrow cones ([10]). A similar remark applies to regions  $\mathcal{P}_\beta$ . The critical exponent of integrability of  $\tau_\beta$  for  $\{X_t\}$  given in Theorem 1 is *qualitatively different* from that of  $\{B_t\}$  ([2]).

Finally, we have recently learned that Pedro J. Méndez-Hernández, in a paper in preparation, has obtained results similar to those presented. His results apply to more general regions defined by other "slowly" increasing functions.

The proof of Theorem 1 is based on estimates for harmonic measure of  $\{X_t\}$ , following the general idea used in [2] for  $\{B_t\}$ . The necessity of the condition  $p < p_0$  for finiteness of  $E_x \tau_\beta^p$  is proved in Lemma 2 and the sufficiency is proved in Lemma 6. The main technical result of the paper is Lemma 5, where we prove sharp estimates for the harmonic measure of cut-offs of the parabola-shaped region. At the end of the paper we discuss some remaining open problems.

## 2. PROOFS

We begin by reviewing the notation. By  $|\cdot|$  we denote the Euclidean norm in  $\mathbb{R}^d$ . For  $x \in \mathbb{R}^d$ ,  $r > 0$ , and a set  $A \subset \mathbb{R}^d$  we let  $B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$  and  $\text{dist}(A, x) = \inf\{|x - y| : y \in A\}$ .  $A^c$  is the complement of  $A$ . We always assume Borel measurability of the considered sets and functions. In equalities and inequalities, unless stated otherwise,  $c$  denotes some unspecified positive real number whose value depends *only* on  $d$ ,  $\alpha$  and  $\beta$ .

It is well known that  $(X_t, \mathbf{P}_x)$  is a strong Markov process with respect to the standard filtration ([17]).

For an open set  $U \subset \mathbb{R}^d$ , with exit time  $\tau_U$ , the  $\mathbf{P}_x$  distribution of  $X_{\tau_U}$ ,

$$\omega_U^x(A) := P_x\{X_{\tau_U} \in A, \tau_U < +\infty\}, \quad A \subset \mathbb{R}^d,$$

is a subprobability measure concentrated on  $U^c$  (probability measure if  $U$  is bounded), called the *harmonic measure*. When  $r > 0$ ,  $|x| < r$  and  $B = B(0, r) \subset \mathbb{R}^d$ , the corresponding harmonic measure has density function  $P_r(x, y) = d\omega_B^x/dy$  (the *Poisson kernel* for the ball). We have

$$(2) \quad P_r(x, y) = C(d, \alpha) \left[ \frac{r^2 - |x|^2}{|y|^2 - r^2} \right]^{\alpha/2} |y - x|^{-d} \quad \text{if } |y| > r,$$

where  $C(d, \alpha) = \Gamma(d/2)\pi^{-d/2-1} \sin(\pi\alpha/2)$ , and  $P_r(x, y) = 0$  otherwise ([6]).

The *scaling* property of  $X_t$  plays a role in this paper. Namely, for every  $r > 0$  and  $x \in \mathbb{R}^d$  the  $\mathbf{P}_x$  distribution of  $\{X_t\}$  is the same as the  $\mathbf{P}_{rx}$  distribution of  $\{r^{-1}X_{r^\alpha t}\}$  (see (1)). In particular, the  $\mathbf{P}_x$  distribution of  $\tau_U$  is the same as the  $\mathbf{P}_{rx}$  distribution of  $r^{-\alpha}\tau_{rU}$ . In short,  $\tau_{rU} = r^\alpha\tau_U$  in distribution.

Let  $\mathcal{P}'_\beta = \mathcal{P}_\beta \cap \{x_1 > 1, |\tilde{x}| < x_1^\beta/2\}$ . We claim that if  $x \in \mathcal{P}'_\beta$ , then  $B(x, |x|^\beta/5) \subset \mathcal{P}_\beta$ . Indeed, let  $x = (x_1, \tilde{x}) \in \mathcal{P}'_\beta$  and  $y = (y_1, \tilde{y}) \in B(x, |x|^\beta/5)$ . We have  $|\tilde{x}| < x_1^\beta/2 < x_1/2$ , hence

$$1 < x_1 \leq |x| = \sqrt{x_1^2 + |\tilde{x}|^2} < \sqrt{5/4}x_1 < 5x_1/4.$$

Then

$$y_1 \geq x_1 - |x|^\beta/5 > x_1 - |x|/5 > x_1 - (5/4)x_1/5 = 3x_1/4,$$

and so

$$|\tilde{y}| \leq |\tilde{x}| + |x|^\beta/5 < x_1^\beta/2 + (5x_1/4)^\beta/5 < 3x_1^\beta/4 < 3(4y_1/3)^\beta/4 < y_1^\beta,$$

which yields  $y \in \mathcal{P}_\beta$ , as claimed.

**Lemma 2.** *If  $p \geq p_0$ , then  $E_x \tau_{\mathcal{P}_\beta}^p = \infty$  for every  $x \in \mathcal{P}_\beta$ .*

*Proof.* Define  $\tau = \inf\{t \geq 0 : X_t \notin B(X_0, |X_0|^\beta/5)\}$ . For  $y \in \mathbb{R}^d$  and (nonnegative)  $p$  by scaling, we have  $\mathbf{E}_y \tau^p = \mathbf{E}_0 \tau_{B(0,1)}^p (|y|^\beta/5)^{p\alpha} = c|y|^{\alpha\beta p}$ . Let  $x \in \mathcal{P}_\beta$ ,  $r = \text{dist}(x, \mathcal{P}_\beta^c)$ ,  $B = B(x, r)$ , and let  $R$  be so large that  $B \subset \mathcal{P}_\beta \cap \{y_1 \leq R\}$ . By strong Markov property we have

$$\begin{aligned} \mathbf{E}_x \tau_{\mathcal{P}_\beta}^p &\geq \mathbf{E}_x \{X_{\tau_B} \in \mathcal{P}'_\beta; \mathbf{E}_{X_{\tau_B}} \tau_{\mathcal{P}_\beta}^p\} \geq \mathbf{E}_x \{X_{\tau_B} \in \mathcal{P}'_\beta; \mathbf{E}_{X_{\tau_B}} \tau^p\} \\ &= c \mathbf{E}_x \{X_{\tau_B} \in \mathcal{P}'_\beta; |X_{\tau_B}|^{\alpha\beta p}\} \geq c \int_{\mathcal{P}'_\beta \cap \{y_1 > R\}} P_r(0, y - x) |y|^{\alpha\beta p} dy \\ &\geq cr^\alpha C(d, \alpha) \int_R^\infty dy_1 \int_{|\tilde{y}| < y_1^\beta/2} |y - x|^{-d-\alpha} |y|^{\alpha\beta p} d\tilde{y} \\ &\geq \text{const.} \int_R^\infty y_1^{\beta(d-1) + \alpha\beta p - d - \alpha} dy_1. \end{aligned}$$

If  $p \geq p_0$ , then  $\beta(d-1) + \alpha\beta p - d - \alpha \geq \beta(d-1) + (d-1)(1-\beta) + \alpha - d - \alpha = -1$  and the last integral is positive infinity, proving the lemma.  $\square$

For  $0 \leq u < v \leq \infty$ , we let  $\mathcal{P}_\beta^{u,v} = \mathcal{P}_\beta \cap \{(x_1, \tilde{x}) : u \leq x_1 < v\}$ . Let  $0 < s < \infty$  and define  $\tau_s = \tau(s) = \tau_{\mathcal{P}_\beta^{0,s}}$ . Consider the cylinder

$$(3) \quad \mathcal{C} = \{x = (x_1, \tilde{x}) \in \mathbb{R}^d : -\infty < x_1 < s, |\tilde{x}| < s^\beta\}.$$

Clearly for  $A \subset \mathcal{P}_\beta$ ,

$$(4) \quad \mathbf{P}_x \{X_{\tau_s} \in A\} \leq \mathbf{P}_x \{X_{\tau_c} \in A\}, \quad x \in \mathbb{R}^d.$$

By Lemma 4.3 of [9] for  $A \subset \mathbb{R}^d \cap \{z_1 \geq s\}$  we have

$$(5) \quad \mathbf{P}_x \{X_{\tau_c} \in A\} \leq c \int_{z=(z_1, \tilde{z}) \in A} \frac{s^{\alpha\beta}}{|z-x|^{d+\alpha}} \left[ \frac{s^{\alpha\beta/2}}{(z_1-s)^{\alpha/2}} \vee 1 \right] dz, \quad x \in \mathcal{C}.$$

The following lemma will simplify the use of the estimate (5).

**Lemma 3.** *Let  $s \geq 1$  and  $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$  with  $x_1 \leq s/2$ . Let  $s \leq u < v \leq \infty$  and assume that either  $u \geq s + s^\beta$  or  $u = s$  and  $v \geq s + s^\beta$ . Then*

$$(6) \quad \mathbf{P}_x \{X_{\tau_s} \in \mathcal{P}_\beta^{u,v}\} \leq cs^{\alpha\beta} \int_u^v t^{-\alpha\beta p_0 - 1} dt.$$

*Proof.* Denote

$$f(y) = \frac{d\omega_{\mathcal{P}_\beta^{0,s}}^x}{dy}(y).$$

Let  $y = (y_1, \tilde{y}) \in \mathcal{P}_\beta^{s,\infty}$ . From (4) and (5) we can conclude that

$$f(y) \leq c \frac{s^{\alpha\beta}}{|y-x|^{d+\alpha}} \left[ \frac{s^{\alpha\beta/2}}{(y_1-s)^{\alpha/2}} \vee 1 \right].$$

Since  $s \geq 1$ , we have  $|\tilde{y}| \leq y_1$  and  $|y| \leq \sqrt{2}y_1$ . As  $|y-x| \geq y_1 - x_1 \geq y_1/2$ ,

$$f(y) \leq cs^{\alpha\beta} y_1^{-d-\alpha} \left[ \frac{s^{\alpha\beta/2}}{(y_1-s)^{\alpha/2}} \vee 1 \right].$$

If  $u \geq s + s^\beta$ , then

$$\begin{aligned} \mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta^{u,v}\} &\leq cs^{\alpha\beta} \int_u^v y_1^{-d-\alpha} \int_{|\tilde{y}| < y_1^\beta} d\tilde{y} dy_1 \\ &= cs^{\alpha\beta} \int_u^v t^{-d-\alpha+\beta(d-1)} dt = cs^{\alpha\beta} \int_u^v t^{-\alpha\beta p_0 - 1} dt. \end{aligned}$$

If  $u = s$  and  $v \geq s + s^\beta$ , we consider

$$I(s) = s^{\alpha\beta} \int_s^{s+s^\beta} t^{-\alpha\beta p_0 - 1} dt$$

and

$$II(s) = s^{\alpha\beta} \int_s^{s+s^\beta} t^{-\alpha\beta p_0 - 1} \frac{s^{\alpha\beta/2}}{(t-s)^{\alpha/2}} dt.$$

It is enough to verify that  $II(s) \leq cI(s)$ . Note that  $s + s^\beta \leq 2s$ , hence

$$I(s) \geq s^{\alpha\beta} (2s)^{-\alpha\beta p_0 - 1} s^\beta = cs^{-\alpha\beta p_0 + \alpha\beta + \beta - 1}.$$

Since

$$II(s) \leq s^{\alpha\beta} s^{-\alpha\beta p_0 - 1} s^{\alpha\beta/2} \int_0^{s^\beta} z^{-\alpha/2} dz = cs^{-\alpha\beta p_0 + \alpha\beta + \beta - 1},$$

we get  $II(s) \leq cI(s)$ . □

The following result is an immediate consequence of Lemma 3.

**Lemma 4.** *If  $s \geq 1$ ,  $x = (x_1, \tilde{x}) \in \mathbb{R}^d$  and  $x_1 \leq s/2$ , then*

$$(7) \quad \mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} \leq cs^{-\alpha\beta(p_0-1)}.$$

Estimate (7) is sharp if, for example,  $x = (s/2, \tilde{0})$ , where  $\tilde{0} = (0, \dots, 0) \in \mathbb{R}^{d-1}$ . Indeed, let  $B = B(x, cs^\beta) \subset \mathcal{P}_\beta^{0,s}$ , where  $c > 0$  is small enough. By (2) we have

$$\begin{aligned} \mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} &\geq \mathbf{P}_x\{X_{\tau_B} \in \mathcal{P}_\beta^{s,\infty}\} \\ &\geq c(s^\beta)^\alpha \int_s^\infty dz_1 \int_{\{\tilde{z} \in \mathbb{R}^{d-1} : |\tilde{z}| < z_1^\beta\}} |(z_1, \tilde{z}) - x|^{-d-\alpha} d\tilde{z} \\ (8) \quad &\geq cs^{\alpha\beta} \int_s^\infty t^{\beta(d-1)-d-\alpha} dt = cs^{-\alpha\beta(p_0-1)}. \end{aligned}$$

We can, however, improve the estimate when  $x$  is small relative to  $s$ .

**Lemma 5.** *If  $s \geq 1$ ,  $x = (x_1, \tilde{x}) \in \mathbb{R}^d$  and  $x_1 \leq s/2$ , then*

$$(9) \quad \mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} \leq C(x_1 \vee 1)^{\alpha\beta} s^{-\alpha\beta p_0}.$$

*Proof.* If  $s \leq S$ , where  $0 < S < \infty$  is fixed, then (9) trivially holds with  $C = S^{\alpha\beta p_0}$ . Hence from now on we assume  $s \geq S$ , where  $S = 2^{k_0}$  and  $k_0 \geq 2$  is such that

$$\frac{8^{\alpha\beta} c}{\alpha\beta(p_0 - 1)} (2^{k_0-1})^{-\alpha\beta(p_0-1)} < \frac{1}{2}.$$

The reason for this choice of  $k_0$  will be made clear later on. Here and in what follows  $c$  is the constant of Lemma 3. We will prove (9) by induction. If  $s/4 \leq x_1 \leq s/2$ , then by Lemma 4

$$\mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} \leq c_1 s^{\alpha\beta} s^{-\alpha\beta p_0} \leq 4^{\alpha\beta} c_1 x_1^{\alpha\beta} s^{-\alpha\beta p_0} = c_1 (x_1 \vee 1)^{\alpha\beta} s^{-\alpha\beta p_0}.$$

Assume that  $n$  is a natural number and (9) holds for all  $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$  such that  $s/2^{n+1} \leq x_1 \leq s/2$ .

Let  $s/2^n \geq 1$  and  $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$  be such that  $s/2^{n+2} \leq x_1 < s/2^{n+1}$ . Note that  $1 \leq s/2^n \leq 4x_1$  here. We have

$$\begin{aligned} \mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} &\leq \mathbf{P}_x\{X_{\tau(s/2^n)} \in \mathcal{P}_\beta^{s/2, \infty}\} \\ &\quad + \mathbf{E}_x\{X_{\tau(s/2^n)} \in \mathcal{P}_\beta^{s/2^n, s/2}; \mathbf{P}_{X_{\tau(s/2^n)}}\{X_{\tau_s} \in \mathcal{P}_\beta\}\} \\ &= I + II. \end{aligned}$$

By Lemma 3,

$$\begin{aligned} I &\leq c(s/2^n)^{\alpha\beta} \int_{s/2}^\infty t^{-\alpha\beta p_0-1} dt = c2^{\alpha\beta p_0} (s/2^n)^{\alpha\beta} s^{-\alpha\beta p_0} \\ &\leq c2^{\alpha\beta p_0+2\alpha} x_1^{\alpha\beta} s^{-\alpha\beta p_0} \leq c_2 (x_1 \vee 1)^{\alpha\beta} s^{-\alpha\beta p_0}. \end{aligned}$$

By Lemma 3 and induction

$$\begin{aligned} II &\leq 2^{\alpha\beta} c(s/2^n)^{\alpha\beta} \int_{s/2^n}^{s/2} t^{-\alpha\beta p_0-1} C t^{\alpha\beta} s^{-\alpha\beta p_0} dt \\ &\leq \frac{2^{\alpha\beta} c}{\alpha\beta(p_0 - 1)} (s/2^n)^{-\alpha\beta(p_0-1)} C (s/2^n)^{\alpha\beta} s^{-\alpha\beta p_0} \\ &\leq \frac{8^{\alpha\beta} c}{\alpha\beta(p_0 - 1)} (s/2^n)^{-\alpha\beta(p_0-1)} C (x_1 \vee 1)^{\alpha\beta} s^{-\alpha\beta p_0}. \end{aligned}$$

Thus

$$R = \frac{I + II}{(x_1 \vee 1)^{\alpha\beta} s^{-\alpha\beta p_0}} \leq c_2 + \frac{8^{\alpha\beta} c}{\alpha\beta(p_0 - 1)} (s/2^n)^{-\alpha\beta(p_0-1)} C.$$

For  $n$  such that  $s/2^n \geq 2^{k_0}$  we have  $R \leq c_2 + C/2$  and we can take  $C = 2(c_1 \vee c_2)$  in our inductive assumption to the effect that  $R \leq c_2 + c_1 \vee c_2 \leq C$ . Therefore, (9) holds for every  $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$ , satisfying  $s/2^{n+2} \leq x_1 \leq s/2$ .

By induction, (9) is true with  $C = 2(c_1 \vee c_2)$  for all  $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$ , satisfying  $2^{k_0-2} \leq x_1 \leq s/2$ .

If  $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$  and  $0 < x_1 \leq 2^{k_0-2}$ , then

$$\begin{aligned} \mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} &\leq \mathbf{P}_x\{X_{\tau(2^{k_0}/2)} \in \mathcal{P}_\beta^{s/2, \infty}\} \\ &+ \mathbf{E}_x\{X_{\tau(2^{k_0}/2)} \in \mathcal{P}_\beta^{2^{k_0}/2, s/2}; \mathbf{P}_{X_{\tau(2^{k_0}/2)}}\{X_{\tau_s} \in \mathcal{P}_\beta\}\} \\ &\leq c(2^{k_0}/2)^{\alpha\beta} \int_{s/2}^\infty t^{-\alpha\beta p_0-1} dt \\ &+ 2^{\alpha\beta} c(2^{k_0}/2)^{\alpha\beta} \int_{2^{k_0}/2}^{s/2} t^{-\alpha\beta p_0-1} 2(c_1 \vee c_2) t^{\alpha\beta} s^{-\alpha\beta p_0} dt \\ &\leq c_3 s^{-\alpha\beta p_0} \leq c_3(x_1 \vee 1)^{\alpha\beta} s^{-\alpha\beta p_0}. \end{aligned}$$

We used here our previous estimates. The proof is complete. □

Note that (9) is sharp if, for example,  $x = (x_1, \tilde{0})$  and  $1 \leq x_1 \leq s/2$ . This can be verified as in (8) to the effect that for such  $x$  the upper bound in Lemma 5 is of the same order as  $\mathbf{P}_x\{X_{\tau_B} \in \mathcal{P}_\beta^{s, \infty}\}$ , where  $B$  is the largest ball centered at  $x$  and such that  $B \subset \mathcal{P}_\beta$ .

Informally,  $\{X_t\}$  goes to  $\mathcal{P}_\beta^{s, \infty}$  “mostly” by a direct jump from  $B$ .

This informal rule seems to be related to a “thinness” of  $\mathcal{P}_\beta$  at infinity (or its inversion [8] at 0). Such a rule is false for cones ([1]).

**Lemma 6.** *If  $p < p_0$ , then  $E_x \tau_\beta^p < \infty$  for every  $x \in \mathcal{P}_\beta$ .*

*Proof.* Let  $1/\lambda_1$  be the first eigenvalue of the Green operator

$$G_B f(\tilde{x}) = \mathcal{E}_{\tilde{x}} \int_0^{\eta_B} f(Y_t) dt, \quad \tilde{x} \in \mathbb{R}^{d-1},$$

for our isotropic stable process  $Y_t$  in  $\mathbb{R}^{d-1}$ . Here,  $B$  is the unit ball in  $\mathbb{R}^{d-1}$ ,  $\eta_D$  is the first exit time of  $Y$  from  $D \subset \mathbb{R}^{d-1}$ , and  $\mathcal{P}, \mathcal{E}$ , are, respectively, the distribution and expectation corresponding to  $Y$ . For  $r > 0$ , by *scaling*,  $1/(r^{-\alpha} \lambda_1)$  is the eigenvalue for  $G_{rB}$  and  $\mathcal{P}_{\tilde{x}}\{\eta_{rB} > t\} \leq ce^{-tr^{-\alpha} \lambda_1}$ ,  $0 < t < \infty$ . Let  $\mathcal{C}$  be the cylinder as in (3). For  $t > 0$ , fixed  $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$  and  $s \geq 1 \vee 2x_1$ , we have by Lemma 5 that

$$\begin{aligned} \mathbf{P}_x\{\tau_\beta > t\} &= \mathbf{P}_x\{\tau_\beta > t, \tau_\beta = \tau_s\} + \mathbf{P}_x\{\tau_\beta > t, \tau_\beta > \tau_s\} \\ &\leq \mathbf{P}_x\{\tau_{\mathcal{C}} > t\} + \mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} \leq ce^{-ts^{-\alpha} \lambda_1} + cs^{-\alpha\beta p_0}. \end{aligned}$$

Let us take  $s = t^{(1-\epsilon)/(\alpha\beta)}$ , where  $0 < \epsilon \leq 1/2$ . We get

$$\mathbf{P}_x\{\tau_\beta > t\} \leq ce^{-t^\epsilon \lambda_1} + ct^{-(1-\epsilon)p_0} \leq ct^{-(1-\epsilon)p_0}$$

for large  $t$ . Thus, letting  $p = (1 - 2\epsilon)p_0$  we get

$$\mathbf{E}_x \tau_\beta^p = \int_0^\infty pt^{p-1} \mathbf{P}_x\{\tau_\beta > t\} dt \leq const. + const. \int_1^\infty t^{-\epsilon p_0-1} dt < \infty.$$

This finishes the proof. □

*Proof of Theorem 1.* The result follows from Lemma 2 and Lemma 6. □

We conclude with three remarks.

Because of scaling of  $\{X_t\}$ , Theorem 1 holds with the same  $p_0$  for the more general parabola-shaped regions of the form

$$\{x = (x_1, \tilde{x}) : x_1 > 0, \tilde{x} \in \mathbb{R}^{d-1}, |\tilde{x}| < ax_1^\beta\},$$

for any  $0 < a < \infty$ .

If  $\beta \downarrow 0$ , then  $\mathcal{P}_\beta$  approaches an infinite cylinder, for which the exit time has exponential moments (compare the proof of Lemma 6).

The second endpoint  $\beta \uparrow 1$  suggests studying the rate of convergence (to 1) of the critical exponent of integrability of the exit time from the right circular cone as the opening of the cone tends to 0. A partial result in this direction is given in [13]. [1] contains more information on our stable processes in cones and a hint on how to approach the problem.

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