

A NOTE ON ASYMPTOTICALLY FLAT METRICS ON \mathbb{R}^3 WHICH ARE SCALAR-FLAT AND ADMIT MINIMAL SPHERES

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ABSTRACT. We use constructions by Miao and Chruściel-Delay to produce asymptotically flat metrics on \mathbb{R}^3 which have zero scalar curvature and multiple stable minimal spheres. Such metrics are solutions of the time-symmetric vacuum constraint equations of general relativity, and in this context the horizons of black holes are stable minimal spheres. We also note that under pointwise sectional curvature bounds, asymptotically flat metrics of nonnegative scalar curvature and small mass do not admit minimal spheres, and hence are topologically \mathbb{R}^3 .

1. INTRODUCTION

In this note we combine a construction by Miao [Mi] with a construction by Chruściel and Delay [CD1] to produce asymptotically flat metrics on \mathbb{R}^3 which have zero scalar curvature and multiple minimal spheres (Thm. 3.2). Such metrics are solutions of the time-symmetric vacuum constraint equations of general relativity; in such context the (apparent) horizons of black holes are stable minimal spheres [HE]. We will thus use the term *horizon* to mean a stable minimal sphere.

A prototype example is given by the spatial Schwarzschild metric on punctured Euclidean space, which in conformally flat coordinates is given by

$$(1.1) \quad g^S(x) = \left(1 + \frac{m}{2r}\right)^4 \delta,$$

where $r = |x|$ and δ is the flat metric. The scalar curvature of g^S is zero, and the set $\Sigma = \{x : |x| = m/2\}$ is a stable minimal sphere of area $16\pi m^2$, which absolutely minimizes area in its homology class. The number m is the ADM mass; geometrically it measures deviation from the flat metric, and physically it corresponds to the mass an observer near infinity would attribute to the gravitational “force” he is experiencing (which is really just encoded in the geometry of spacetime). The Schwarzschild spacetime which evolves from this initial data contains a black hole with Σ as its horizon.

The Schwarzschild solution has nontrivial topology. It follows from work of Meeks, Simon and Yau [MSY] that in the presence of nontrivial topology there will exist a stable minimal sphere. As a consequence we remark below (Thm. 2.3) that the Penrose inequality implies that nontrivial topology (with curvature bounds)

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yields a minimal amount of mass. In any case, the natural question then is how to construct horizons when the topology is trivial. The first existence result was obtained by Beig and Ó Murchadha [BO] using critical sequences of metrics for the conformal Laplacian. We also mention recent work of Yan [Y] which gives metric criteria to guarantee the existence of minimal spheres. The work we use here is due to Miao [Mi]; he constructs examples by filling in the Schwarzschild metric to produce a metric on \mathbb{R}^3 with nonnegative scalar curvature, and then cleverly uses two types of scalar curvature deformation (one local and one conformal) to deform the metric to zero scalar curvature so that the horizon persists.

2. MASS AND TOPOLOGY

We recall some basic definitions and theorems on the mass. A smooth, complete Riemannian three-manifold (M, g) is called *asymptotically flat* (AF) if there is a compact set $K \subset M$ such that $M \setminus K$ is the union of finitely many components (*ends*) each diffeomorphic to the exterior of the closed unit ball \overline{B} in \mathbb{R}^3 , with decay conditions on the metric: if N denotes an end, and $\Phi : N \rightarrow \mathbb{R}^3 \setminus \overline{B}$ is an AF coordinate chart, we require the tensor $(\Phi_*g - \delta)(x)$ to decay suitably as $|x|$ tends to infinity. For example, scalar-flat metrics which are conformally flat near infinity with conformal factor u tending to 1 are AF; here we have $g = u^4\delta$ near infinity, with $\Delta u = 0$ from the vanishing of the scalar curvature. Using the spherical harmonic expansion of the conformal factor

$$(2.1) \quad u(x) = 1 + \frac{A}{|x|} + O(|x|^{-2})$$

we have decay conditions of the form $\partial^\alpha(g_{ij} - \delta_{ij})(x) = O(|x|^{-1-|\alpha|})$. We note that g^S is AF with two ends; the coordinates in Eq. (1.1) are AF as $r \rightarrow \infty$, while the radial coordinate change which sends $r \mapsto \frac{m^2}{4r}$ is an isometry which gives AF coordinates as $r \rightarrow 0$.

In this setting one can define the ADM mass m of an end [ADM] using AF coordinates $x = (x^1, x^2, x^3)$ as

$$(2.2) \quad m = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{|x|=r} \sum_{i,j} (g_{ij,i} - g_{ii,j}) \nu^j d\xi,$$

where $d\xi$ is the Euclidean surface measure and $\nu = x/r$ is the Euclidean normal. For a discussion of the decay conditions yielding existence of the limit, and independence of the limit on the AF chart, see [B]. It is a simple computation to show that if the metric g is conformally flat and scalar-flat near infinity, then the mass m appears in the expansion (2.1) as $A = \frac{m}{2}$; so m in the definition of the Schwarzschild metric g^S is the ADM mass of either end of g^S . In fact it is easy to compute

$$(2.3) \quad \begin{aligned} \int_{|x|=r} \sum_{i,j} (g_{ij,i}^S - g_{ii,j}^S) \nu^j d\xi &= \int_{|x|=r} -4\left(1 + \frac{m}{2r}\right)^3 \left(\frac{m}{2r^2}\right) \sum_{i,j} \left(\frac{x^i}{r} \delta_{ij} - \frac{x^j}{r}\right) \frac{x^j}{r} d\xi \\ &= 16\pi m \left(1 + \frac{m}{2r}\right)^3. \end{aligned}$$

A fundamental result about mass is the following Positive Mass Theorem [SY2], [S].

Theorem 2.1 (Positive Mass Theorem). *Let (M, g) be AF with $R(g) \geq 0$, and let N be an end. Then the ADM mass of N is nonnegative, and is zero if and only if (M, g) is isometric to (\mathbb{R}^3, δ) .*

A sharpening of the Positive Mass Theorem known as the Penrose Inequality has recently been attained by Bray [Br1] and Huisken-Ilmanen [HI] independently. We state the version from [Br1]. We call a minimal sphere *outermost* with respect to an end if it is not enclosed by any minimal surface in the exterior region containing this end (cf. [Br1], [HI]).

Theorem 2.2 (Penrose Inequality). *Let (M, g) be AF with $R(g) \geq 0$. Let m be the ADM mass of an end, and let A be the total surface area of the outermost minimal spheres with respect to this end. Then*

$$m \geq \sqrt{\frac{A}{16\pi}}.$$

Equality holds if and only if the metric is isometric to the Schwarzschild metric of mass $m = \sqrt{\frac{A}{16\pi}}$ outside the horizons.

The physical motivation is that the outermost minimal spheres represent horizons of black holes whose mass contribution is through the above area expression; the inequality gives a lower bound for the total mass in terms of the masses of the black holes. See [Br1], [Br2], [BC] and references therein for a more precise discussion of these ideas.

It is conjectured that under suitable curvature normalization, if the mass of an asymptotically flat initial data set is small, then it is diffeomorphic to \mathbb{R}^3 (cf. Yau's second problem set in [SY1], pp. 371-372). We remark how this follows with a strong (pointwise) normalization of the curvature tensor, using the Penrose Inequality.

Theorem 2.3. *Let (M, g) be an asymptotically flat three-manifold with nonnegative scalar curvature. Suppose that the sectional curvatures are bounded above by $C > 0$. Then if the ADM mass m satisfies $m\sqrt{C} < \frac{1}{2}$, there are no horizons, and moreover the manifold is diffeomorphic to \mathbb{R}^3 .*

Proof. If M were not diffeomorphic to \mathbb{R}^3 , then there would be a stable minimal sphere in M . A proof of this fact relies on some fundamental three-manifold topology and existence results of Meeks-Simon-Yau [MSY]. Also, using the mean-convex barrier spheres near infinity, one then shows that there is in any end an outermost horizon. See [HI] for a proof of these claims. Let Σ be an outermost horizon, and let $A(\Sigma)$ denote its area.

We note that the Gauss equation implies an upper bound on the curvature of the horizon Σ as follows. Let p be a point in Σ , and let $\{e_1, e_2\}$ be a basis of $T_p(M)$ in which the second fundamental form II is diagonal. Let $\kappa_1 = \text{II}(e_1, e_1)$, $\kappa_2 = \text{II}(e_2, e_2)$ be the principal curvatures of Σ at p . Since Σ is minimal, we have $\kappa_1 + \kappa_2 = 0$, and so $\kappa_1\kappa_2 \leq 0$. The Gauss equation then yields

$$\kappa \equiv R(e_1, e_2, e_1, e_2) = \overline{R}(e_1, e_2, e_1, e_2) + \kappa_1\kappa_2 \leq C,$$

where R denotes the curvature tensor of Σ , and \overline{R} the curvature tensor of M .

We now invoke the Gauss-Bonnet theorem, which together with the preceding inequality yields

$$4\pi = \int_{\Sigma} \kappa d\mu_{\Sigma} \leq C A(\Sigma).$$

Combining this with the Penrose Inequality $m \geq \sqrt{\frac{A(\Sigma)}{16\pi}}$, we obtain $m\sqrt{C} \geq \frac{1}{2}$. \square

Remark. The conjecture in Yau's problem set suggests a weaker normalization of the curvature tensor such as the L^2 -norm. See [BF] for estimates of the L^2 -norm in terms of the mass and isoperimetric constant. A related conjecture asserts that every "large" solution of the constraint equations on \mathbb{R}^3 contains an apparent horizon, again using some normalization of the metric. See [Y] for sufficient geometric conditions to guarantee horizons for AF metrics with zero scalar curvature (time-symmetric vacuum case). We also mention the theorem of Schoen-Yau [SY3] which guarantees the existence of an apparent horizon in the case of high matter density.

3. MULTIPLE HORIZONS

We now discuss the construction of metrics on \mathbb{R}^3 with multiple horizons. We begin with the following proposition from [CD1].

Proposition 3.1. *There exist metrics on $M = \mathbb{R}^3 \setminus \{0 = x_0, \pm x_1, \dots, \pm x_k\}$ with zero scalar curvature of the following form: there are radii r, r_i and masses m, m_i so that in the ball $M \cap B(\pm x_i, r_i)$, the metric is precisely Schwarzschild of mass m_i centered at $\pm x_i$; outside $B(0, r)$, the metric is Schwarzschild of mass m centered at the origin. The masses and radii are chosen so that the metric contains the minimal spheres $|x \mp x_i| = \frac{m_i}{2}$ of the inner Schwarzschild geometries, i.e. $r_i > \frac{m_i}{2}$ for $i = 0, \dots, k$.*

We recall the proof of this, both for completeness and to amplify some of the details. We let L_g be the linearization of the scalar curvature operator at a metric g . The formal L^2 -adjoint L_g^* is given by $L_g^*f = -(\Delta_g f)g + \text{Hess}_g(f) - f\text{Ric}(g)$. On a domain Ω , L_g^* has trivial kernel for generic metrics, in the sense that if there were nontrivial kernel, the scalar curvature would be constant on Ω ; see also [BCS] for stronger such results. In [C] we prove the following localized deformation result: if L_g^* has trivial kernel in a smooth bounded domain Ω , there is an $\epsilon > 0$ so that for any $S \in C_c^\infty(\Omega)$ with $\|S\|_{C^{0,\alpha}} < \epsilon$, there is a smooth metric $g + h$ with $R(g + h) = R(g) + S$, so that h is supported in $\overline{\Omega}$ and satisfies a bound $\|h\|_{C^{2,\alpha}} \leq C\|S\|_{C^{0,\alpha}}$. This ϵ is uniform for metrics near g in $C^{4,\alpha}(\overline{\Omega})$.

For the purpose of constructing data as in Prop. 3.1, one also considers the singular situation in which the operator L_g^* has nontrivial kernel. In particular at the flat metric we consider the operator $L_\delta^*f = -(\Delta f)\delta + \text{Hess}(f)$, whose kernel K is precisely the span of $\{1, x^1, x^2, x^3\}$. In this setting the deformation result stated above fails, but a projected version of it still holds [C].

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain, and let $\zeta \in C_c^\infty(\Omega)$ be a bump function. There is a neighborhood U of the flat metric in $C^{4,\alpha}(\overline{\Omega})$ and an $\epsilon > 0$ so that for smooth metrics g in U and functions $S \in C_c^\infty(\Omega)$ with $\|S\|_{C^{0,\alpha}} < \epsilon$, there is a smooth metric $g + h$ so that h is supported in $\overline{\Omega}$ and satisfies a bound $\|h\|_{C^{2,\alpha}} \leq C\|S\|_{C^{0,\alpha}}$, and with*

$$R(g + h) - (R(g) + S) \in \zeta K.$$

As a consequence of this proposition, for g near the flat metric, one can solve $R(g + h) \in \zeta K$ for h . If furthermore it can be shown that the L^2 -projection of $R(g + h)$ to K vanishes too, then we will have in fact solved $R(g + h) = 0$. We will recall the basic aspects of the proof of Prop. 3.2 in the Appendix, emphasizing what is needed for the proof of Prop. 3.1.

Proof of Proposition 3.1 (following [CD1]). We start by gluing Schwarzschild metrics of mass m_i centered at $\pm x_i$ inside $B(\pm x_i, 2r_i)$ to the Schwarzschild metric of mass m centered at the origin. The patching will occur inside a compactly contained subdomain of the region $\Omega = B(0, r) \setminus \bigcup_{i=0}^k \overline{B(\pm x_i, r_i)}$. The masses and radii are chosen so that the balls $\overline{B(\pm x_i, 2r_i)}$ are disjoint and contained in $B(0, r)$, the Schwarzschild piece inside $B(\pm x_i, r_i)$ contains one AF end and a neighborhood of the horizon ($m_i/2 < r_i$), and outside of $B(0, r_0)$ the outer Schwarzschild of mass m is an AF end past the horizon ($m/2 < r_0$). By letting the mass parameters be very small positive numbers we can arrange the glued metric on Ω to be close to the Euclidean metric. Furthermore we can arrange it so that the glued metric is symmetric under the parity map $x \mapsto -x$. Under this symmetry the glued metric g has parity-symmetric scalar curvature, and furthermore the operator L_g^* is parity-symmetric, i.e. if $\hat{h} = \alpha^* h$, then $L_g^* \hat{f} = \widehat{L_g^* f}$. As such, solving $R(g + h) \in \zeta K$ as in Prop. 3.2 can be done so that $g + h$ is again parity-symmetric. The idea is that in solving the linearized problem we minimize a convex functional involving L_g^* , which has a unique solution. In the parity-symmetric case, if h is a solution produced by minimization, then \hat{h} will also come from a minimizer, which will yield $h = \hat{h}$ (see the Appendix for details). Thus we automatically get $\int_{\Omega} x^i R(g + h) dx = 0$.

On Ω for small masses $0 < m, m_i < \delta$, we have (by an elementary computation involving the Christoffel symbols) $\|R(g)\|_{C^{0,\alpha}} = O(\delta)$, and in fact $R(g) = \sum_{i,j} (g_{ij,ij} - g_{ii,jj}) + O(\delta^2)$. Moreover the bound $\|h\|_{C^{2,\alpha}} \leq C \|R(g)\|_{C^{0,\alpha}} = O(\delta)$ implies $R(g + h) = R(g) + \sum_{i,j} (h_{ij,ij} - h_{ii,jj}) + O(\delta^2)$. By applying the divergence theorem in conjunction with Eq. (2.3), using the fact that h and its derivatives vanish on $\partial\Omega$, and working with masses $0 < m, m_i < \delta$, we then get

$$\int_{\Omega} R(g + h) dx = 16\pi \left(m - m_0 - \sum_{i=1}^k 2m_i \right) + O(\delta^2).$$

So by keeping the (positive) masses small and varying them, we see that some arrangements will yield zero (by continuity). □

We now apply the construction of Miao [Mi] to this data to prove the main theorem (Thm. 3.2). His construction utilizes the following local scalar curvature deformation theorem of Lohkamp [L].

Theorem 3.1. *Let (M, g) be any smooth Riemannian manifold with dimension at least three. Let U be any open subset, and let f be any smooth function on M which agrees with $R(g)$ outside U and with $f < R(g)$ on U . Let $\epsilon > 0$ and let U_ϵ be the ϵ -neighborhood of U with respect to g . Then there is a smooth metric g_ϵ on M which agrees with g outside U_ϵ , satisfies $\|g_\epsilon - g\|_{C^0(M)} < \epsilon$ and so that*

$$f - \epsilon \leq R(g_\epsilon) \leq f.$$

This localized scalar curvature deformation and its proof are very different from that of [C], which is a perturbation result. In Thm. 3.1, the scalar curvature may move (downward) a lot; amazingly, the metric can still be controlled in C^0 , which is a key observation for Miao's construction.

We are now in a position to state the main theorem.

Theorem 3.2. *There exist AF metrics on \mathbb{R}^3 with zero scalar curvature and multiple horizons.*

Proof (following [Mi]). Let g be a metric produced in Prop. 3.1. We cap each of the Schwarzschild necks with flat balls; the resulting nonsmooth metric has positive scalar curvature distributionally, so it can be deformed to a smooth metric with positive scalar curvature. To be explicit, working at $x_0 = 0$, we let $w(x) = 3$ for $|x| \leq m_0/4$ and $w(x) = (1 + \frac{m_0}{2r})$ for $|x| \geq m_0/4$. Then w is continuous and superharmonic. Using a standard spherically symmetric approximate identity (mollifiers ϕ_σ supported in $|x| \leq \sigma$ tending to the Dirac distribution at the origin as $\sigma \downarrow 0$) we have that $w_\sigma = w * \phi_\sigma$ is smooth, positive and superharmonic. Let $1/5 < a < 1/4 < b < 1/3$; for σ small we have $w_\sigma(x) = 3$ for $|x| \leq am_0$ and $w_\sigma(x) = (1 + \frac{m_0}{2r})$ for $|x| \geq bm_0$, by the mean value property of harmonic functions. By the conformal transformation of the scalar curvature $R(w_\sigma^4 \delta) = -8w_\sigma^{-5} \Delta w_\sigma$, we see that the metric $w_\sigma^4 \delta$ has nonnegative compactly supported scalar curvature, is flat for $|x| \leq am_0$ and is exactly Schwarzschild for $|x| \geq bm_0$; in particular, this metric contains a neighborhood of the horizon. We can perform this about each neck, and since the mollification leaves the outer Schwarzschild regions alone, we can patch these metrics into the original metric g . The resulting configuration is a smooth metric \tilde{g} on \mathbb{R}^3 , with scalar curvature supported in an open set U , and with U_ϵ strictly inside the Schwarzschild horizons for small ϵ (U_ϵ is contained in the union of annular regions A_i^\pm of the form $A_i^\pm = \{x : m_i/5 \leq |x \mp x_i| \leq m_i/3\}$).

We can apply the Lohkamp theorem to deform \tilde{g} only on U_ϵ , strictly inside the horizons for small ϵ , so that the scalar curvature in this region lies in $(-\epsilon, 0)$, with the deformed metric g_ϵ remaining C^0 -close to \tilde{g} . For ϵ small enough the scalar curvature is not too negative, so we can then solve [SY2] for a conformal factor u_ϵ so that $\tilde{g}_\epsilon = u_\epsilon^4 g_\epsilon$ is AF ($u_\epsilon \rightarrow 1$ near infinity) with zero scalar curvature, *i.e.* $\Delta_{g_\epsilon} u_\epsilon - \frac{1}{8} R(g_\epsilon) u_\epsilon = 0$. Indeed let $v_\epsilon = u_\epsilon - 1$; then we want to solve $\Delta_{g_\epsilon} v_\epsilon - \frac{1}{8} R(g_\epsilon) v_\epsilon = \frac{1}{8} R(g_\epsilon)$ with v_ϵ decaying near infinity. For $R(g_\epsilon)$ with a small negative part, the operator $\Delta_{g_\epsilon} - \frac{1}{8} R(g_\epsilon)$ is a small perturbation of an isomorphism in appropriate weighted spaces [B]. As pointed out by Miao, the conditions for solvability will be met independently of ϵ small by the C^0 -control on the metrics (and the fact that the asymptotics are fixed). One gets a bound $1 \leq u_\epsilon \leq 1 + o(1)$; indeed the lower bound comes from the maximum principle. The upper bound follows by getting integral estimates on v_ϵ and using elliptic theory as in [Mi], [SY2]: one multiplies the equation $\Delta_{g_\epsilon} v_\epsilon - \frac{1}{8} R(g_\epsilon) v_\epsilon = \frac{1}{8} R(g_\epsilon)$ by v_ϵ and integrates by parts (using the decay of v_ϵ) to bound $|\nabla v_\epsilon|$ in $L^2(d\mu_\epsilon)$, where $d\mu_\epsilon$ is the volume measure induced by g_ϵ ; an application of the Hölder inequality yields

$$\|\nabla v_\epsilon\|_{L^2(d\mu_\epsilon)}^2 \leq \|R(g_\epsilon)\|_{L^{3/2}(d\mu_\epsilon)} \|v_\epsilon\|_{L^6(d\mu_\epsilon)}^2 + \|R(g_\epsilon)\|_{L^{6/5}(d\mu_\epsilon)} \|v_\epsilon\|_{L^6(d\mu_\epsilon)}.$$

One now applies the Sobolev inequality $\|v_\epsilon\|_{L^6(d\mu_\epsilon)} \leq C \|\nabla v_\epsilon\|_{L^2(d\mu_\epsilon)}$, where the constant C can be taken to be independent of ϵ since g_ϵ is C^0 -close to \tilde{g} . Using this

along with the arithmetic-geometric mean inequality gives

$$\begin{aligned} \|v_\epsilon\|_{L^6(d\mu_\epsilon)}^2 &\leq C\|R(g_\epsilon)\|_{L^{3/2}(d\mu_\epsilon)}\|v_\epsilon\|_{L^6(d\mu_\epsilon)}^2 + C\|R(g_\epsilon)\|_{L^{6/5}(d\mu_\epsilon)}\|v_\epsilon\|_{L^6(d\mu_\epsilon)} \\ &\leq C\|R(g_\epsilon)\|_{L^{3/2}(d\mu_\epsilon)}\|v_\epsilon\|_{L^6(d\mu_\epsilon)}^2 + \frac{C^2}{2}\|R(g_\epsilon)\|_{L^{6/5}(d\mu_\epsilon)}^2 + \frac{1}{2}\|v_\epsilon\|_{L^6(d\mu_\epsilon)}^2. \end{aligned}$$

We can absorb the third term onto the left side, and for small ϵ we can absorb the first term, leaving an estimate

$$\|v_\epsilon\|_{L^6(d\mu_\epsilon)}^2 \leq C\|R(g_\epsilon)\|_{L^{6/5}(d\mu_\epsilon)}^2 = o(1)$$

since C is independent of ϵ . The elliptic estimates (DeGiorgi-Nash-Moser, *cf.* [GT] Thm. 8.17) then give the desired upper bound.

Since $g_\epsilon = g$ outside $V = \bigcup_{i=0}^k B(\pm x_i, m_i/3)$ for small ϵ , we have $R(g_\epsilon) = 0$ outside V and hence $\Delta_g v_\epsilon = 0$ here as well. By the sup-bound on v_ϵ and the interior Schauder estimates, we have that u_ϵ converges to 1 in $C^{2,\alpha}$ on compact subsets of $\mathbb{R}^3 \setminus \bar{V}$, and hence \bar{g}_ϵ converges to g in $C^{2,\alpha}$ on compact subsets here as well. We can use a minimization argument near each Schwarzschild neck to show that for small ϵ there must be a horizon of \bar{g}_ϵ nearby. Indeed g_ϵ contains a neighborhood of the Schwarzschild horizon, so that there are strictly mean-convex barrier spheres on both sides of the horizon. By the C^2 -convergence, then, these spheres will be a mean-convex barrier for \bar{g}_ϵ for small ϵ . Hence by area minimization within each of these barriers there must be stable minimal spheres near the original Schwarzschild horizons. (See [CM] for the argument of this fact using geometric measure theory.) \square

We note that it is another issue altogether whether this initial data for the Einstein equation represents a spacetime with multiple black holes. This requires an analysis both of the geometry of the initial data as well as of the spacetime evolution of this data. Such an analysis has been carried out in Chruściel and Mazzeo [CM], from which it follows, for example, that for small masses, the data produced by [CD1] in Prop. 3.1 has as its outermost minimal spheres with respect to the end $M \setminus \bar{B}(0, r)$ precisely the union of the Schwarzschild horizons. However, it is not clear whether the data constructed in Thm. 3.2 can be modified to satisfy the conditions described therein (which would require good control on the metric in the annuli where the necks were capped and where the Lohkamp deformation was then applied).

4. APPENDIX

In this section we discuss how in the proof of Prop. 3.1 we can solve for h so that $h = \hat{h}$. This entails a careful sketch of the proof of Prop. 3.2 applied to this case, and we note in particular that we are considering $S = -R(g) \in C_c^\infty(\Omega)$. We first start by choosing a parity-symmetric bump function $\zeta \in C_c^\infty(\Omega)$, and a parity-symmetric bump function $\rho \in C_c^\infty(\bar{\Omega})$ to serve as a weight function, so that ρ is exponentially decaying in the distance d to the boundary near the boundary of Ω ($\rho = e^{-1/d}$ near $\partial\Omega$). We let S_g be the $L^2(\Omega, d\mu_g)$ -orthogonal complement of ζK , and let Π_g denote the projection operator to S_g . We let $H_\rho^2(\Omega, d\mu_g)$ be the Hilbert space of functions with $|\nabla^k u| \in L^2(\Omega, \rho d\mu_g) =: L_\rho^2(d\mu_g)$ for $k = 0, 1, 2$. We let $\mathcal{P}_g = \Pi_g \circ L_g$; note that $\mathcal{R} := \mathcal{P}_\delta$ is the linearization of the operator $g \mapsto \Pi_\delta R(g)$.

We can now define the functional \mathcal{G} on $H^2_\rho(\Omega, d\mu_g)$, corresponding to some $f \in L^2_\rho(\Omega, d\mu_g)$: $\mathcal{G}(u) = \int_\Omega (\frac{1}{2}|\mathcal{P}_g^*u|_g^2 - fu)\rho \, d\mu_g$. Here \mathcal{P}_g^* denotes the $L^2(\Omega, d\mu_g)$ formal adjoint. We minimize over the subspace $\mathcal{V}_g = H^2_\rho(\Omega, d\mu_g) \cap S_g$; on this space it is simple to verify $\mathcal{P}_g^* = L_g^*$. Also on this space one has the weighted elliptic estimate (metric dependence of the norms is suppressed for convenience) $\|u\|_{H^2_\rho(\Omega)} \leq C\|L_g^*u\|_{L^2_\rho(\Omega)}$. The estimate follows by noting $tr_g L_g^*u = -2\Delta_g u - uR(g)$, which implies $Hess_g(u) = L_g^*u - \frac{1}{2}(tr_g L_g^*u)g + u(Ric(g) - \frac{1}{2}R(g)g)$. We thus get an estimate $\|u\|_{H^2(\Omega)} \leq C(\|L_g^*u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})$, and on \mathcal{V}_g one can get rid of the lower-order term in the estimate by the Rellich compactness theorem. The key point is that the estimate does not require any boundary terms (which is of course false for general elliptic operators; L_g^* happens to be overdetermined-elliptic). It is elementary to derive the weighted estimate from here [C]. This estimate gives a lower bound of \mathcal{G} on \mathcal{V}_g :

$$(4.1) \quad \mathcal{G}(u) \geq \frac{1}{2C}\|u\|_{H^2_\rho(\Omega)}^2 - \|f\|_{L^2_\rho(\Omega)}\|u\|_{H^2_\rho(\Omega)}.$$

Standard functional analysis gives a minimizer of \mathcal{G} on \mathcal{V}_g [C]. The key for the parity-symmetry in this case is that the convexity of the functional gives a *unique* minimizer. Indeed, if u_1 and u_2 minimize \mathcal{G} on \mathcal{V}_g , we let $0 < s < 1$, and let $\epsilon^2 = \frac{1-s}{s}$ in the arithmetic-geometric mean inequality $ab \leq \frac{1}{2}(\epsilon^2 a^2 + \frac{b^2}{\epsilon^2})$. An application of Cauchy-Schwarz gives

$$\begin{aligned} s(1-s)(L_g^*u_1) \cdot_g (L_g^*u_2) &\leq \frac{\epsilon^2 s^2}{2} |L_g^*u_1|_g^2 + \frac{(1-s)^2}{2\epsilon^2} |L_g^*u_2|_g^2 \\ &= \frac{s(1-s)}{2} (|L_g^*u_1|_g^2 + |L_g^*u_2|_g^2) \end{aligned}$$

so that

$$\begin{aligned} \mathcal{G}(u_1) = \mathcal{G}(u_2) &\leq \mathcal{G}(su_1 + (1-s)u_2) \\ &\leq \int_\Omega \left(\frac{(\epsilon^2 + 1)s^2}{2} |L_g^*u_1|_g^2 + \frac{(\epsilon^2 + 1)(1-s)^2}{2\epsilon^2} |L_g^*u_2|_g^2 \right) \rho \, d\mu_g \\ &\quad - \int_\Omega (su_1 + (1-s)u_2)f \rho \, d\mu_g \\ &= s\mathcal{G}(u_1) + (1-s)\mathcal{G}(u_2) = \mathcal{G}(u_1) = \mathcal{G}(u_2). \end{aligned}$$

Thus we have equality in the Cauchy-Schwarz inequality (and in the AM-GM inequality, which gives $\epsilon^2 a = b$), so that $L_g^*u_1 = L_g^*u_2$; thus, since \mathcal{V}_g intersects the kernel of L_g^* trivially (for g close to δ), we have $u_1 = u_2$.

By construction we have ρ, ζ, g , and hence $d\mu_g$ and $R(g)$, parity-symmetric on Ω . As such, if $u \in \mathcal{V}_g$, then so is \hat{u} , and as we pointed out above, $L_g^*\hat{u} = \widehat{L_g^*u}$. So if f is parity-symmetric, we also have $\mathcal{G}(u) = \mathcal{G}(\hat{u})$, and hence the minimizer u_0 must be parity-symmetric.

We sketch the remainder of the argument that the solution $g + h$ produced in Prop. 3.2 is parity-symmetric. If we assume that $f\rho \in S_g$, then the Euler-Lagrange equation satisfied by u_0 is $\mathcal{P}_g\rho\mathcal{P}_g^*u_0 = f\rho$; to see this, one decomposes any $\eta \in C_c^\infty(\Omega)$ as $\eta = \eta_1 + \eta_2$ with $\eta_1 \in S_g$ and $\eta_2 \in \zeta K$, and then computes $\frac{d}{dt}|_{t=0}\mathcal{G}(u + t\eta_i)$ (which vanishes in either case). Elementary linear algebra shows

the following: for $\phi \in S_\delta \cap L^2_{\rho^{-1}}(\Omega)$, we let $f\rho$ be the $L^2(\Omega, d\mu_g)$ projection of ϕ to S_g ; the minimizer u_0 for \mathcal{G} corresponding to f as above satisfies the following equation:

$$(4.2) \quad \mathcal{R}\rho L_g^* u_0 = \phi.$$

In particular, in the proof of Prop. 3.1, $R(g)$ is supported away from the boundary, so we can let $\phi = -\Pi_\delta R(g)$; if g , and hence $R(g)$, is parity-symmetric, so are ϕ and f . We let $h_0 = \rho L_g^* u_0$; using Eq. (4.1) along with $\mathcal{G}(u_0) \leq 0$ gives a bound $\|h_0\|_{L^2_{\rho^{-1}}(\Omega)} \leq C\|R(g)\|_{L^2_{\rho^{-1}}(\Omega)}$, and we also get a pointwise bound $\|h_0\|_{C^{2,\alpha}(\Omega)} \leq C\|R(g)\|_{C^{0,\alpha}(\Omega)}$ by the Schauder estimates and the decay of h_0 (imposed from the decay of ρ). So for $R(g)$ small, then, h_0 is small, and $g_1 = g + h_0$ is a metric. By the Taylor formula, we see $\Pi_\delta R(g_1) = O(\|h_0\|_{C^{2,\alpha}(\Omega)}^2)$, so we have corrected the scalar curvature to be zero in the S_δ -direction, to first order. Moreover the scalar curvature $R(g_1)$ decays at the boundary (inherited from the decay of h_0), so that we can now let $\phi = -\Pi_\delta R(g_1)$ and solve the analogue $\mathcal{R}\rho L_g^* u_1 = \phi$ of Eq. (4.2) as above for u_1 . Since the metric g_1 is also parity-symmetric, iterating this procedure will yield a parity-symmetric sequence $g_m = g + \sum_{k=0}^{m-1} \rho L_g^* u_m$ (we note that the iteration is not simply Newton's method, but it is a Picard-type iteration, and the convergence is more delicate [C]). Using Schauder estimates, one obtains enough pointwise control to assert that g_m are all metrics (for $R(g)$ sufficiently small), and moreover since ρ is exponentially decaying at the boundary, we will have that $h_m = \rho L_g^* u_m$ smoothly extends by zero outside Ω . Finally, the estimates also show that $h = \sum_{m=0}^{\infty} h_m$ converges in $C^{2,\alpha}$, and $\sum_{m=0}^{\infty} u_m$ converges in $C^{4,\alpha}$ on compact subsets of Ω , so that $h = \rho L_g^* u$ on Ω . By the pointwise convergence, then, we have $h = \hat{h}$ is inherited from the individual terms in the series. Furthermore, u satisfies a quasilinear elliptic equation of the form $R(g + \rho L_g^* u) \in \zeta K$. By applying the interior Schauder estimates to this quasilinear equation, we see that h is smooth on Ω , and again that the decay of ρ causes h to decay (along with all derivatives) at $\partial\Omega$. For details, see [C]; also *cf.* [CS], and [CD2].

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