

BOUNDARY AND LENS RIGIDITY OF FINITE QUOTIENTS

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ABSTRACT. We consider compact Riemannian manifolds $(M, \partial M, g)$ with boundary ∂M and metric g on which a finite group Γ acts freely. We determine the extent to which certain rigidity properties of $(M, \partial M, g)$ descend to the quotient $(M/\Gamma, \partial/\Gamma, g)$. In particular, we show by example that if $(M, \partial M, g)$ is boundary rigid, then $(M/\Gamma, \partial/\Gamma, g)$ need not be. On the other hand, lens rigidity of $(M, \partial M, g)$ does pass to the quotient.

1. INTRODUCTION

In this paper we consider compact Riemannian manifolds $(M, \partial M, g)$ with boundary ∂M and metric g . On such an $(M, \partial M, g)$ the boundary distance function, $d_g : \partial M \times \partial M \rightarrow \mathbb{R}^+$, is the function which assigns to each pair p, q of boundary points the “chordal” distance between them (i.e. the infimum of the lengths of curves in M between p and q). The boundary rigidity question is to determine which $(M, \partial M, g)$ are determined by d_g . $(M, \partial M, g)$ is called boundary rigid if for any $(M_1, \partial M, g_1)$ (with the same boundary) such that $d_{g_1} = d_g$, there is a diffeomorphism $\phi : M_1 \rightarrow M$ which is the identity on ∂M such that $g_1 = \phi^*g$.

Not all $(M, \partial M, g)$ are boundary rigid since if there is an $x \in M$ such that no minimizing geodesic between boundary points passes through x (we will call such an x “inaccessible”), then we can change g arbitrarily in a small neighborhood of x without changing d_g (hence the metric near x is unaffected by d_g). Another easy example to consider is the hemisphere where any metric which is pointwise larger than the original metric (but the same at the boundary) fails to change d_g . There are also more interesting examples (e.g. [CK], [W]); see the survey [C1].

Thus it becomes important to consider special classes of $(M, \partial M, g)$. One such class is called *SGM* (strong geodesic minimizing). It is a modified version (see [C2]) of a class introduced by Gromov ([G]). One of the points of the definition is that it is in terms only of d_g . However, here we only give a “loose definition”. $(M, \partial M, g)$ is loosely *SGM* if every nongrazing geodesic segment in M is the unique minimizing path between its endpoints. Here by “nongrazing” we mean that the only possible intersection points with the boundary are the endpoints. The reader is referred to [C2] for the actual definition, however the above definition will be sufficient for this

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paper. We note here that if $(M, \partial M, g)$ is *SGM*, then it is “nontrapping” in the sense that every geodesic ray eventually hits the boundary.

There are a number of examples of boundary rigid manifolds (see [C1]) including all compact subdomains of \mathbb{R}^n (see [G]), H^n (or any negatively curved symmetric space; see [BCG1, BCG2]), and open hemispheres (see [M]). In this paper we will only use the fact ([C3]) that *SGM* subdomains of a flat torus are boundary rigid.

We consider the case where a finite group Γ acts freely by isometries on a boundary rigid $(M, \partial M, g)$, and ask whether the quotient $(M/\Gamma, \partial M/\Gamma, g)$ must also be boundary rigid. (Here and throughout the paper we will use the same notation for a metric and the induced metric on a quotient by isometries.)

First we give an easy example to give the idea of what can go wrong. Consider Z_n (n somewhat large, say 20) acting in the usual way by rotation on the unit disk D in the plane. Consider the annulus A obtained by removing a small open disk around the origin. A is boundary rigid (as a subset of the plane) but A/Γ is not *SGM* since there are lots of geodesics that do not minimize distances. (This of course does not mean that it is not boundary rigid.)

Thus when $(M, \partial M, g)$ is boundary rigid and *SGM* then $(M/\Gamma, \partial M/\Gamma, g)$ need not be *SGM*. So if we look at another metric g_1 on M/Γ with $d_{g_1} = d_g$ on $\partial M/\Gamma$ and lift it to M we will only know that the boundary distances on M agree for some pairs of boundary points. In fact

Example 1.1. There is a compact 3-dimensional boundary rigid manifold $(M, \partial M, g)$ which admits a free action by a finite group Γ of isometries such that $(M/\Gamma, \partial M/\Gamma, g)$ is not boundary rigid. In the example ∂M is connected and M/Γ is orientable.

We do this by constructing an example where there is an inaccessible x in the quotient $(M/\Gamma, \partial M/\Gamma, g)$. (There is a two-dimensional analogue, however, the three-dimensional case has the advantage that both M and M/Γ are orientable and both ∂M and $\partial M/\Gamma$ are connected.)

However, we now see (Corollary 1.3) that the fact that $(M/\Gamma, \partial M/\Gamma, g)$ need not be *SGM* is the only thing that can go wrong.

Let $U^+\partial M$ represent the space of unit vectors V whose base point is on the boundary and for which $\langle V, N \rangle \geq 0$, where N is the inward pointing unit normal vector. For $V \in U^+\partial M$ let $c_V : [0, L(V)] \rightarrow M$ (here $L(V)$ may take in values in $[0, \infty]$) be the unit speed geodesic segment with $c'_V(0) = V$ and $L(V)$ (if > 0) is the first number such that $c_V(L(V)) \in \partial M$. (If V is tangent to the boundary, then $L(V)$ could be 0 or not, while if c_V never hits the boundary again, then $L(V) = \infty$.)

The “scattering data” of $(M, \partial M, g)$ is the map which assigns to each $V \in U^+\partial M$ both $L(V)$ and $c'_V(L(V))$. That is, given the entry direction of a geodesic it gives the length and exit direction. If $(M, \partial M, g)$ and $(M_1, \partial M_1, g_1)$ are such that there is an isometry, $f : \partial M_1 \rightarrow \partial M$, between $g_1|_{\partial M_1}$ and $g|_{\partial M}$, then there is a natural identification of $U^+\partial M_1$ with $U^+\partial M$ (i.e. the normal projections to $T\partial M$ and $T\partial M_1$ are related by Df). If two such spaces have the same scattering data, then they behave the same as geodesic lenses. $(M, \partial M, g)$ is called lens rigid if for any such $(M_1, \partial M_1, g_1)$ with the same scattering data there is a diffeomorphism $f : M_1 \rightarrow M$ extending our given f on the boundary such that $f^*(g) = g_1$.

There are also a number of examples of $(M, \partial M, g)$ that are not lens rigid (see for example [CK]), but all known examples have the property that they are trapping (i.e. there is some geodesic ray in M that is defined for all positive values of t). A

natural assumption in considering lens rigidity is that $(M, \partial M, g)$ is nontrapping. By compactness of M that is equivalent to saying that there is a uniform upper bound on the length of geodesic segments.

Theorem 1.2. *If $(M, \partial M, g)$ is nontrapping and lens rigid and further admits a free action by a finite group Γ of isometries, then $(M/\Gamma, \partial M/\Gamma, g)$ is also nontrapping and lens rigid.*

The idea of the proof when g_1 is a different metric on the same space M/Γ is, of course, to lift the two metrics from M/Γ back to M and use the lens rigidity of M to get an isometry upstairs that we can project back down. However, when there are more than one boundary component, some interesting things can happen.

Consider the flat annulus with $\Gamma = Z_{20}$ acting on it as above (say the inner radius is r_0 —the outer was 1). Now consider a diffeomorphism f of the annulus that rotates the circle of radius r by an amount $R(r)$ in such a way that $R(r) = 0$ for r near r_0 and $R(r) = 2\pi/20$ for all r near 1. Now take $g_1 = f^*g_0$. It is easy to see that g_1 does not have the same scattering data as g_0 . We note that by our choice of f it commutes with rotations and hence with $\Gamma = Z_{20}$. Thus we get a metric g_1 on M/Γ which is easily seen to have the same scattering data as g_0 .

Thus if g_0 and g_1 have the same scattering data on M/Γ , they do not need to have the same data when lifted to M . Of course, by pushing f down to M/Γ , g_0 is still isometric to g_1 via an isometry that leaves the boundary fixed; however the isometry is just not the identity on π_1 . The proof will show that this happens in general.

Now we return to our original question. If both $(M, \partial M, g)$ and $(M/\Gamma, \partial M/\Gamma, g)$ are assumed to be *SGM*, then any $(M_1/\Gamma, \partial M/\Gamma, g_1)$ with $d_{g_1} = d_g$ will have the same scattering data as $(M/\Gamma, \partial M/\Gamma, g)$ (see [C2]) and hence, since *SGM* implies nontrapping, Theorem 1.2 yields:

Corollary 1.3. *If a boundary rigid manifold $(M, \partial M, g)$ admits a free action by a finite group Γ of isometries such that $(M/\Gamma, \partial M/\Gamma, g)$ is *SGM*, then $(M/\Gamma, \partial M/\Gamma, g)$ is also boundary rigid.*

The question addressed in this paper was posed by M. Porrati and R. Rabadan in [PR], where they discuss some possible applications in theoretical physics. The author thanks Gunther Uhlmann for passing the question on to him.

2. BOUNDARY RIGIDITY

In this section we construct Example 1.1.

Our space M will be an *SGM* subset of the flat cylinder $D^2 \times S^1$, where D^2 is a flat disc of radius 1 and S^1 has length $2n$, where n will be determined later. Since M is *SGM* and can be thought of as a subset of a flat 3-torus the result in [C2] shows that it is boundary rigid. $\Gamma = \mathbb{Z}_{2n}$ is the action on $D^2 \times S^1$ generated by inversion on D^2 and translation by one unit in the S^1 direction.

Fix a small ϵ and let P be a finite subset of the interior of D^2 that is ϵ -dense in D^2 and such that $P \cap -P = \emptyset$. Let $d = \min\{d(x, y) | x \in P \text{ and } y \in -P \text{ or } y \in \partial D\}$ (of course $d < \epsilon$). Let K^2 be a subset of D^2 obtained by removing the discs of radius $d/5$ centered at points of P .

$D^2 \times S^1$ is not *SGM* because there are geodesics in it that are not length minimizing. We will construct M by removing a $d/100$ tubular neighborhood, $N(d/100)$,

of the sets $K^2 \times \{2i\}$ and $-K^2 \times \{2i+1\}$ for $i = 1, \dots, n$. Note that Γ acts on M . Further note that there is an L (independent of n) such that any geodesic on M has length less than L . This is true because we were careful to make sure that the holes on successive copies of the removed K^2 and $-K^2$ do not line up. And hence there is a positive lower bound on the angle that any long geodesic must make with the vertical geodesics $\{x\} \times S^1$. We now choose $n > L$. Thus all geodesics in M minimize and so it is *SGM* and hence boundary rigid.

M/Γ is a subset of $D^2 \times [0, 1]/\sim$, where $(x, 0) \sim (-x, 1)$. There is one copy of $N(d/100)$ left in M/Γ , and it is centered about $t = 0$ (or $t = 1$). We note that if (x, t) and $(-x, 1 - t)$ lie on $\partial N(d/100)$, then there is a path along $\partial N(d/100)$ (going through one of the holes) of length less than $2\epsilon + d/50$ which by a choice of small ϵ we can assume is less than $\frac{1}{2}$.

Now consider the point x_0 in M/Γ which corresponds to the point $(0, 0) \times \frac{1}{2} \in D^2 \times S^1$. We claim that for every geodesic c in M/Γ between boundary points p and q , if c passes through x_0 , then c is not the minimizing path between p and q . Because of the symmetry about the point x_0 , if p can be represented as (x, t) (for some $x \in D^2$ and $t \in [0, 1]$), then q can be represented as $(-x, 1 - t)$. There are two cases to consider. If $x \in \partial D^2$, then the length of c is at least 2 (since the radius of D^2 is 1). But then c is not minimizing because there is a path along $\partial M/\Gamma$ from $p = (x, t)$ to $q = (-x, 1 - t)$ of length less than $1/2 + 2\epsilon + d/50 < 1$. To see this start with the “vertical” $\partial D^2 \times S^1/\Gamma$ from p to q which can be chosen to have length less than $1/2$. This path of course leaves $\partial M/\Gamma$ but can be modified with a path (going through a hole) as in the previous paragraph which has length less than $\frac{1}{2}$. If p (and hence q) lie on $\partial N(d/100)$, then the previous paragraph says that the distance in M/Γ between p and q is less than $\frac{1}{2}$. On the other hand any geodesic passing through x_0 has length at least 1. Thus again c cannot be minimizing. \square

Remark 2.1. One can construct a similar two-dimensional example by starting with a flat cylinder $[-1, 1] \times [0, 2n]/\sim$ with $(s, 0) \sim (s, 2n)$. We then make a similar construction using $[-1, 1]$ in place of D^2 . Our group Γ will be generated by the map that moves things vertically by one unit and flips across the central vertical line. So Γ is cyclic of order $2n$. This example works fine. It has the disadvantage that M/Γ is not orientable and ∂M has many components.

3. LENS RIGIDITY

Before proving the theorem we need to establish some facts about nontrapping spaces with the same scattering data.

We first show that when $(M, \partial M, g)$ is nontrapping, $(M_1, \partial M_1, g_1)$ is also nontrapping. The set of trapped vectors W (i.e. the geodesic ray $c_W(t)$ starting in the direction of W is defined for all $t > 0$) in M_1 is always closed since limiting rays will also be defined for all t . In our case, the compactness of M says there is an L such that all geodesic segments of M have length less than L , and, since the scattering data is the same, this is also true for all geodesic segments in M_1 if any point lies on the boundary. This makes it easy to see that the set of nontrapped vectors is closed, since any limit ray will still hit the boundary and will so be defined at most on an interval of length less than L . The connectedness of UM_1 along with the fact that there are some nontrapped vectors give the result.

We can now define a C^0 homeomorphism $F : UM_1 \rightarrow UM$ as follows: For $W \in UM_1$ by the nontrapping property there is a unique $V \in U^+ \partial M_1$ and a

unique $t \geq 0$ such that $W = c'_V(t)$. As in the case of *SGM* manifolds (see [C2]), we define $F(W) = c'_{f(V)}(t)$. It is not hard to see that F is continuous (it is in fact C^∞ except possibly at W 's such that the geodesic segment tangent to W is tangent to the boundary at both endpoints), invertible, $F(-W) = -F(W)$, and $F \circ g_1^t(W) = g^t \circ F(W)$ for all t in which either side (and hence both sides) is defined (by g^t we mean the geodesic flow in the metric g). It is also clear that if W has base point $x \in \partial M_1$, then the $F(W)$ has base point $f(x)$, where f is the isometry identifying our two boundaries.

As a homeomorphism, F induces an isomorphism $F_* : \pi_1(UM_1) \rightarrow \pi_1(UM)$. The projections $p : UM \rightarrow M$ and $p_1 : UM_1 \rightarrow M_1$ induce isomorphisms on π_1 if the dimension is greater than 2 so that we get an induced map $F_* : \pi_1(M_1) \rightarrow \pi_1(M)$. In two dimensions this is still the case since F will take the circle over a point on ∂M_1 to the circle over $f(x) \in \partial M$, and so takes the kernel of p_{1*} to the kernel of p_* .

Proof of Theorem 1.2. Let g be a lens rigid nontrapping metric on M and let Γ act freely by isometries. Let $(X_1, \partial X_1, g_1)$ be such that g and g_1 have the same scattering data with respect to an isometry $\tilde{f} : \partial X_1 \rightarrow \partial M/\Gamma$. Thus the above discussion gives us a C^0 time-preserving conjugacy $F : UX_1 \rightarrow UM/\Gamma$. This conjugacy will induce an isomorphism on π_1 and hence, since Γ is a normal subgroup of $\pi_1(M/\Gamma)$ and hence of $\pi_1(X_1)$, we have a finite cover M_1 of X_1 associated to Γ (so $X_1 = M_1/\Gamma$). We can lift F to a C^0 conjugacy (again called F) from UM_1 to UM that commutes with the action of Γ (i.e. $\gamma F = F\gamma$). Note that F induces an isometry, F_∂ , from ∂M_1 to ∂M (which again commutes with Γ). (We note that as in the example in the Introduction, even if we start with $X_1 = M/\Gamma$, then F_∂ need not be the identity.) Thus identifying the boundaries using F_∂ the conjugacy F tells us that the scattering data of $(M_1, \partial M_1, g_1)$ and $(M, \partial M, g)$ agree and hence, since g is lens rigid, there is a diffeomorphism $f : M_1 \rightarrow M$ agreeing with F_∂ on the boundaries, such that $f^*g = g_1$.

We are left to show that f descends to an isometry from $X_1 = M_1/\Gamma$ to M_0/Γ , i.e. to see that it takes orbits to orbits. We already know that it induces the appropriate map on the boundary (i.e. the identification we started with) because it agrees with F_∂ on the boundary. In fact we see that f commutes with Γ , since for any $\gamma \in \Gamma$ the isometry $f^{-1}\gamma^{-1}f\gamma$ restricted to the boundary is $F_\partial^{-1}\gamma^{-1}F_\partial\gamma = \text{Identity}$. But any isometry that is the identity on the boundary is the identity on the whole space. \square

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