

CYCLIC VECTORS IN THE FOCK SPACE OVER THE COMPLEX PLANE

KOU HEI IZUCHI

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ABSTRACT. In this paper, we characterize the cyclic vectors in the Fock space over the complex plane. We prove that a function $f(z)$ is cyclic in the Fock space if and only if $f(z)$ is a nonvanishing function in $L_a^2(\mathbb{C})$.

1. INTRODUCTION

Let \mathbb{D} be the open unit disk of the complex plane \mathbb{C} . We denote the polynomial ring on \mathbb{C} by \mathcal{C} , and the space of all entire functions by $Hol(\mathbb{C})$. Let X be a Banach space of holomorphic functions on a domain Ω in \mathbb{C} . For a subset E of X , let \overline{E} be the norm closure of E in X . In this paper, a function f in X is said to be cyclic if $f\mathcal{C} \subset X$ and $\overline{f\mathcal{C}} = X$. In the classical Hardy space $H^2(\mathbb{D})$, it is well known that a function in $H^2(\mathbb{D})$ is cyclic if and only if it is $H^2(\mathbb{D})$ -outer; see [5]. Also in the Bergman space $L_a^2(\mathbb{D})$, it is known that f is cyclic if and only if f is $L_a^2(\mathbb{D})$ -outer; see [7].

The Fock space, or the so-called Segal-Bargmann space, is the space of all μ -square-integrable entire functions on \mathbb{C} , where

$$d\mu(z) = \exp\left(-\frac{|z|^2}{2}\right) dA(z)/2\pi$$

is the Gaussian measure on \mathbb{C} and dA is the ordinary Lebesgue measure. We denote this space by $L_a^2(\mathbb{C}) = L_a^2(\mathbb{C}, \mu)$. Then $L_a^2(\mathbb{C})$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-\frac{|z|^2}{2}} dA(z)$$

for $f, g \in L_a^2(\mathbb{C})$, the reproducing kernel functions are given by $K_\lambda(z) = e^{\frac{\lambda \bar{z}}{2}}$, $\lambda \in \mathbb{C}$, and the polynomial ring \mathcal{C} is dense in $L_a^2(\mathbb{C})$; see [2]. We denote by k_λ the normalized reproducing kernel at $\lambda \in \mathbb{C}$, that is, $k_\lambda = \frac{K_\lambda}{\|K_\lambda\|}$. It is significant to know which functions in $L_a^2(\mathbb{C})$ are cyclic. Obviously a cyclic vector f is an entire function without zeros in \mathbb{C} , so that we can write $f = e^h$ for some $h \in Hol(\mathbb{C})$. In this paper, we study for which $h \in Hol(\mathbb{C})$, e^h is cyclic in $L_a^2(\mathbb{C})$. In the Fock space $L_a^2(\mathbb{C})$, there is no multiplier of $L_a^2(\mathbb{C})$ except constant functions; see [6]. So the usual multiplication operator M_z is not defined on $L_a^2(\mathbb{C})$. But M_z is defined

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on the dense subspace \mathcal{C} of $L_a^2(\mathbb{C})$. In this meaning, we may say that the definition of the cyclicity in $L_a^2(\mathbb{C})$ can be considered as cyclic vectors for the densely defined operator M_z . The following is our main theorem.

Theorem 1.1. *Let $h \in Hol(\mathbb{C})$. Then the following are equivalent:*

- (i) $e^h \in L_a^2(\mathbb{C})$.
- (ii) $h = \alpha z^2 + \beta z + \gamma$ for $\alpha, \beta, \gamma \in \mathbb{C}$ with $|\alpha| < \frac{1}{4}$.
- (iii) e^h is cyclic in $L_a^2(\mathbb{C})$.

It is known that there are nonvanishing functions in $H^2(\mathbb{D})$ and $L_a^2(\mathbb{D})$ which are not cyclic in the respective spaces. But the situation is quite different in the Fock space.

Corollary 1.2. *Every nonvanishing function in $L_a^2(\mathbb{C})$ is cyclic.*

2. PROOF OF THE MAIN THEOREM

The proof of Theorem 1.1 consists of two steps.

Step 1. Let $h \in Hol(\mathbb{C})$. Then $e^h \in L_a^2(\mathbb{C})$ if and only if $h(z) = \alpha z^2 + \beta z + \gamma$ with $|\alpha| < \frac{1}{4}$. Moreover if $e^h \in L_a^2(\mathbb{C})$, then $e^h \mathcal{C} \subset L_a^2(\mathbb{C})$.

Proof. Suppose that $e^h \in L_a^2(\mathbb{C})$. First, we shall prove that $h \in \mathcal{C}$ and $\deg h \leq 2$, where $\deg h$ denotes the polynomial degree of h . In [3], Chen, Guo, and Hou proved that $\lim_{|\lambda| \rightarrow \infty} \langle f, k_\lambda \rangle = 0$ for every $f \in L_a^2(\mathbb{C})$, so that $\Re h(\lambda) < \frac{|\lambda|^2}{4}$ for every $\lambda \in \mathbb{C}$ with sufficiently large modulus. Then obviously e^h is an entire function of finite class $p = 1$, and the order ρ of e^h is

$$\begin{aligned} \rho &= \limsup_{r \rightarrow \infty} \frac{\log [\max_{|\lambda|=r} \Re h(z)]}{\log r} \\ &\leq \limsup_{r \rightarrow \infty} \frac{\log \frac{r^2}{4}}{\log r} \\ &= 2. \end{aligned}$$

By [1, Corollary 4.5.11], we get that $h \in \mathcal{C}$ and $\deg h \leq 2$.

Now, we show that $e^{\alpha z^2 + \beta z + \gamma} \notin L_a^2(\mathbb{C})$ for $\alpha, \beta, \gamma \in \mathbb{C}$ with $|\alpha| \geq \frac{1}{4}$. It is enough to prove that $e^{(az-b)^2} \notin L_a^2(\mathbb{C})$ for $a, b \in \mathbb{C}$ with $|a| \geq \frac{1}{2}$. Without loss of generality, we may assume that $a \geq \frac{1}{2}$. Write $b = r(\cos \theta + i \sin \theta)$, $r \geq 0$. For $t \in \mathbb{R}$, we have

$$\left| \left\langle e^{(az-b)^2}, k_t \right\rangle \right| = \frac{1}{\|K_t\|} \left| \left\langle e^{(az-b)^2}, K_t \right\rangle \right| = \left| e^{b^2} \right| e^{(a^2 - \frac{1}{4})t^2 - 2atr \cos \theta}.$$

If $a > \frac{1}{2}$, then we have $\lim_{t \rightarrow \infty} \left| \left\langle e^{(az-b)^2}, k_t \right\rangle \right| = \infty$. If $a = \frac{1}{2}$, then we have $\lim_{t \rightarrow \infty} \left| \left\langle e^{(az-b)^2}, k_t \right\rangle \right| = \infty$ provided $\cos \theta < 0$, $\lim_{t \rightarrow -\infty} \left| \left\langle e^{(az-b)^2}, k_t \right\rangle \right| = \infty$ provided $\cos \theta > 0$, and $\left| \left\langle e^{(az-b)^2}, k_t \right\rangle \right| = 1$ for every $t \in \mathbb{R}$ provided $\cos \theta = 0$. Since $\lim_{|\lambda| \rightarrow \infty} \langle f, k_\lambda \rangle = 0$ for every $f \in L_a^2(\mathbb{C})$, we get that $e^{(az-b)^2} \notin L_a^2(\mathbb{C})$.

Finally, fix α, β, γ with $|\alpha| < \frac{1}{4}$ and put $h(z) = \alpha z^2 + \beta z + \gamma$. Let ϵ be a number satisfying

$$(1) \quad 0 < \epsilon < \frac{1}{2} \left(\frac{1}{4} - |\alpha| \right).$$

For large $|z|$, $|p(z)| \leq e^{\epsilon|z|^2}$ and $|\beta z + \gamma| \leq \epsilon|z|^2$. Thus

$$\begin{aligned} \left| p(z)e^{h(z)} \right|^2 &= |p(z)|^2 e^{2\Re h(z)} \\ &\leq e^{2\epsilon|z|^2} e^{2(|\alpha|+\epsilon)|z|^2} \\ &= e^{2(2\epsilon+|\alpha|)|z|^2}. \end{aligned}$$

Then

$$\left| p(z)e^{h(z)} \right|^2 e^{-\frac{|z|^2}{2}} \leq e^{-\delta|z|^2}$$

for large $|z|$, where $\delta = \frac{1}{2} - 2(2\epsilon + |\alpha|)$. By (1), we have that $\delta > 0$. Therefore there exists a positive constant C such that

$$\int_{\mathbb{C}} \left| p(z)e^{h(z)} \right|^2 e^{-\frac{|z|^2}{2}} dA(z)/2\pi \leq C \int_{\mathbb{C}} e^{-\delta|z|^2} dA(z)/2\pi.$$

Since the last integral is finite, we get that $e^{h(z)}\mathcal{C} \subset L^2_a(\mathbb{C})$. This completes the proof. \square

If f is cyclic, then obviously f is a nonvanishing function. So it remains only to prove that $e^{h(z)}$ is cyclic in $L^2_a(\mathbb{C})$ for $h(z) = \alpha z^2 + \beta z + \gamma$ with $|\alpha| < \frac{1}{4}$.

Step 2. If $h(z) = \alpha z^2 + \beta z + \gamma$ with $|\alpha| < \frac{1}{4}$, then $e^{h(z)}$ is cyclic in $L^2_a(\mathbb{C})$.

Proof. By Step 1, $e^{h(z)}\mathcal{C} \subset L^2_a(\mathbb{C})$. Let N be a positive integer satisfying

$$(2) \quad \left(1 + \frac{1}{N}\right)|\alpha| < \frac{1}{4}.$$

Put $p_n(z) = \sum_{k=0}^n \frac{\{-\frac{1}{N}(\alpha z^2 + \beta z + \gamma)\}^k}{k!}$. Let l be a nonnegative integer. Since

$$|p_n(z)| \leq \sum_{k=0}^{\infty} \frac{\left\{\frac{1}{N}(|\alpha||z|^2 + |\beta||z| + |\gamma|)\right\}^k}{k!} = e^{\frac{1}{N}(|\alpha||z|^2 + |\beta||z| + |\gamma|)},$$

we have

$$\begin{aligned} &\left| z^l p_n(z) e^{\alpha z^2 + \beta z + \gamma} - z^l e^{-\frac{1}{N}(\alpha z^2 + \beta z + \gamma)} e^{\alpha z^2 + \beta z + \gamma} \right| \\ &\leq |z|^l \left(e^{(1+\frac{1}{N})(|\alpha||z|^2 + |\beta||z| + |\gamma|)} + e^{(1-\frac{1}{N})(|\alpha||z|^2 + |\beta||z| + |\gamma|)} \right) \\ &\leq 2|z|^l e^{(1+\frac{1}{N})(|\alpha||z|^2 + |\beta||z| + |\gamma|)}. \end{aligned}$$

Hence

$$\begin{aligned} &\left| z^l p_n(z) e^{\alpha z^2 + \beta z + \gamma} - z^l e^{-\frac{1}{N}(\alpha z^2 + \beta z + \gamma)} e^{\alpha z^2 + \beta z + \gamma} \right|^2 e^{-\frac{|z|^2}{2}} \\ &\leq 4|z|^{2l} e^{\{2(1+\frac{1}{N})|\alpha| - \frac{1}{2}\}|z|^2 + 2(1+\frac{1}{N})|\beta||z| + 2(1+\frac{1}{N})|\gamma|}. \end{aligned}$$

By (2), we have that $2(1 + \frac{1}{N})|\alpha| - \frac{1}{2} < 0$, so that the last function is integrable with respect to dA . Since $p_n(z) \rightarrow e^{-\frac{1}{N}(\alpha z^2 + \beta z + \gamma)}$ pointwise on \mathbb{C} as $n \rightarrow \infty$, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} &\int_{\mathbb{C}} \left| z^l p_n(z) e^{\alpha z^2 + \beta z + \gamma} - z^l e^{-\frac{1}{N}(\alpha z^2 + \beta z + \gamma)} e^{\alpha z^2 + \beta z + \gamma} \right|^2 e^{-\frac{|z|^2}{2}} dA(z)/2\pi \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore we get

$$e^{(1-\frac{1}{N})(\alpha z^2 + \beta z + \gamma)}\mathcal{C} \subset \overline{e^{\alpha z^2 + \beta z + \gamma}\mathcal{C}}.$$

Applying this method again for $e^{(1-\frac{1}{N})(\alpha z^2+\beta z+\gamma)}$, we have

$$\int_{\mathbb{C}} \left| z^l p_n(z) e^{(1-\frac{1}{N})(\alpha z^2+\beta z+\gamma)} - z^l e^{-\frac{1}{N}(\alpha z^2+\beta z+\gamma)} e^{(1-\frac{1}{N})(\alpha z^2+\beta z+\gamma)} \right|^2 e^{-\frac{|z|^2}{2}} dA(z) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

so that

$$e^{(1-\frac{2}{N})(\alpha z^2+\beta z+\gamma)} \mathcal{C} \subset \overline{e^{(1-\frac{1}{N})(\alpha z^2+\beta z+\gamma)} \mathcal{C}}.$$

Continuing this way, we get

$$\mathcal{C} \subset \overline{e^{\frac{1}{N}(\alpha z^2+\beta z+\gamma)} \mathcal{C}} \subset \dots \subset \overline{e^{(1-\frac{1}{N})(\alpha z^2+\beta z+\gamma)} \mathcal{C}} \subset \overline{e^{\alpha z^2+\beta z+\gamma} \mathcal{C}}.$$

Thus we get $\overline{e^{\alpha z^2+\beta z+\gamma} \mathcal{C}} = L_a^2(\mathbb{C})$. \square

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REFERENCES

1. C. Berenstein and R. Gay, *Complex Variables*, Springer-Verlag, New York, 1991. MR1107514 (92f:30001)
2. X. Chen and K. Guo, *Analytic Hilbert Modules*, Chapman & Hall/CRC, 2003. MR1988884 (2004d:47024)
3. X. Chen, K. Guo, and S. Hou, *Analytic Hilbert spaces over the complex plane*, J. Math. Anal. Appl., **268** (2002), 684–700. MR1896222 (2003f:46038)
4. X. Chen and S. Hou, *A Beurling-type theorem for the Fock space*, Proc. Amer. Math. Soc., **131** (2003), 2791–2795. MR1974336 (2004c:46095)
5. J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981. MR0628971 (83g:30037)
6. K. Guo and D. Zheng, *Invariant subspaces, quasi-invariant subspaces and Hankel operators*, J. Funct. Anal., **187** (2001), 308–342. MR1875150 (2003b:47050)
7. H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, Springer-Verlag, New York, 2000. MR1758653 (2001c:46043)
8. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962. MR0133008 (24:A2844)
9. W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, 1987. MR0924157 (88k:00002)

GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY, NIIGATA UNIVERSITY, NIIGATA 950-2181, JAPAN

E-mail address: f04n010j@mail.cc.niigata-u.ac.jp