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A SIMPLE PROOF OF A CURIOUS CONGRUENCE BY ZHAO

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Abstract. The author gives a simple proof of the following curious congruence for odd prime $p > 3$ which was established by Jianqiang Zhao:

$$\sum_{\substack{i+j+k=p \\ i,\ j,\ k>0}} \frac{1}{ijk} \equiv -2B_{p-3} (\bmod\ p).$$

1. Introduction

The well-known Wolstenholme's theorem [2, p. 89] is that for any odd prime $p > 3$,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1} \equiv 0 \ (\bmod\ p^2).$$

There is another important equivalent statement of Wolstenholme's theorem by using combinatorics. D. F. Bailey [1] generalized it to the following form.

Theorem 1.1 (Bailey [1, Theorem 4]). *Let n and r be non-negative integers and $p > 3$ an odd prime. Then*

$$\binom{np}{rp} \equiv \binom{n}{r} (\bmod\ p^3),$$

where we set $\binom{n}{r} = 0$ if $n < r$.

Lehmer [3] proved that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1} \equiv 0 \ (\bmod\ p^3)$ if and only if p divides the numerator of B_{p-1}. Here we define the Bernoulli numbers B_k and Bernoulli polynomials $B_k(x)$ by the Maclaurin series, respectively

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}, \qquad \frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

Zhihong Sun [4] proved the following congruence

Theorem 1.2 ([4, Theorem 5.1(a)]). *For any odd prime $p > 3$,*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1} \equiv -\frac{p^2}{3} B_{p-3} \ (\bmod\ p^3).$$

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In this note we prove the following congruence:

Theorem 1.3. *For any odd prime* $p > 3$,

$$\sum_{\substack{i+j+k=p \\ i,\ j,\ k>0}} \frac{1}{ijk} \equiv -2B_{p-3} (\text{mod } p).$$

Remark. Jianqiang Zhao [6] first gave the above interesting congruence and said that it would be very interesting to find a direct proof of this congruence without using partial sums of multiple zeta value series. This is the original motivation of our study.

2. Some lemmas

Lemma 1 (von Staudt-Clausen [5, p. 56])**.** *Let* m *be a positive integer. Then*

$$B_{2m} + \sum_{p-1|2m} \frac{1}{p} \in \mathbb{Z},$$

where the sum is over those primes p *such that* $p-1$ *divides* $2m$ *(in particular,* 2 *and* 3 *appear in the denominator of each Bernoulli number). Consequently,* pB_{2m} *is* p*-integral for all* m *and all* p.

Lemma 2 (Kummer's congruence [5, p. 64])**.** *Suppose* $m \equiv n \not\equiv 0 (\text{mod } p-1)$ *are positive even integers. Then*

$$\frac{B_m}{m} \equiv \frac{B_n}{n} (\text{mod } p).$$

Lemma 3. *Let* $p > 3$ *be an odd prime. Then*

$$\sum_{1 \le i < j \le p-1} \frac{1}{ij} \equiv -\frac{p}{3} B_{p-3} (\text{mod } p^2).$$

Proof. For m a positive integer,

$$\sum_{i=1}^{p-1} i^m = \frac{B_{m+1}(p) - B_{m+1}}{m+1} = \frac{1}{m+1} \sum_{r=1}^{m+1} \binom{m+1}{r} B_{m+1-r} p^r$$

$$= pB_m + \frac{p^2}{2} m B_{m-1} + \sum_{r=3}^{m+1} \binom{m}{r-1} pB_{m+1-r} \frac{p^{r-3}}{r} p^2.$$

By Lemma 1, pB_{m+1-r}, p^{r-3}/r are p-integral for $r \ge 3$. Hence

$$1^m + 2^m + 3^m + \cdots + (p-1)^m \equiv pB_m + \frac{p^2}{2} m B_{m-1} (\text{mod } p^2).$$

Using Euler's theorem and Lemma 1, we have

$$\sum_{i=1}^{p-1} \frac{1}{i^2} \equiv \sum_{i=1}^{p-1} i^{\phi(p^2)-2} \equiv pB_{\phi(p^2)-2} (\text{mod } p^2),$$

where $\phi(*)$ is the Euler totient function. Using Kummer congruence, we have

$$\frac{B_{\phi(p^2)-2}}{\phi(p^2)-2} \equiv \frac{B_{p-3}}{p-3} (\text{mod } p),$$

that is, $B_{\phi(p^2)-2} \equiv \frac{2}{3} B_{p-3} (\text{mod } p)$. So

$$\sum_{i=1}^{p-1} \frac{1}{i^2} \equiv \frac{2p}{3} B_{p-3} (\text{ mod } p^2).$$

Hence

$$\sum_{1 \leq i < j \leq p-1} \frac{1}{ij} = \frac{1}{2} \left(\sum_{i=1}^{p-1} \frac{1}{i} \cdot \sum_{j=1}^{p-1} \frac{1}{j} - \sum_{i=1}^{p-1} \frac{1}{i^2} \right) \equiv -\frac{p}{3} B_{p-3} (\text{mod } p^2).$$

The proof of Lemma 3 is complete.

3. The proof of Theorem 1.3

It is easy to see that

$$\sum_{\substack{i+j+k=p \\ i,\ j,\ k>0}} \frac{1}{ijk} = \frac{1}{p} \sum_{\substack{i+j+k=p \\ i,\ j,\ k>0}} \frac{i+j+k}{ijk} = \frac{3}{p} \sum_{\substack{i+j<p \\ i,\ j>0}} \frac{1}{ij}.$$

Write $m = i + j$ so that $1 \leq j < m \leq p - 1$. We see that

$$\frac{1}{ij} = \frac{i+j}{ijm} = \frac{1}{m} (\frac{1}{i} + \frac{1}{j}),$$

and hence

$$\sum_{\substack{i+j \leq p-1 \\ i,\ j>0}} \frac{1}{ij} = 2 \sum_{1 \leq j < m \leq p-1} \frac{1}{jm} \equiv -\frac{2p}{3} B_{p-3} (\text{mod } p^2).$$

The proof of Theorem 1.3 is complete.

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$$\sum_{\substack{i+j+k=p \\ i,\ j,\ k>0}} \binom{p}{i} \binom{p}{j} \binom{p}{k} = \binom{3p}{p} - 3 \binom{2p}{p} + 3.$$

References

[1] D. F. Baily, *Two p^3 variations of Lucas' theorem*, Jour. Number Theory **35** (1990), 208-215. MR1057323 (91f:11008)

[2] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1979. MR0568909 (81i:10002)

[3] E. Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Ann. Math. **39** (1938), 350-360. MR1503412

[4] Zhihong Sun, *Congruence concerning Bernoulli numbers and Bernoulli polynomials*, Discrete Applied Mathematics **105** (2000), 193-223. MR1780472 (2001m:11022)

[5] L. C. Washington, *Introduction to Cyclotomic Fields, 2nd ed.*, Springer-Verlag, New York, 1997. MR1421575 (97h:11130)

[6] Jianqiang Zhao, *Partial sums of multiple zeta value series I: generalizations of Wolstenholme's theorem*, xxx.lanl.gov/abs/math.NT/0301252, 19pages.

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