

## INVARIANT SUBSPACES FOR A CLASS OF COMPLETE PICK KERNELS

MICHAEL T. JURY

(Communicated by Joseph A. Ball)

ABSTRACT. Motivated by the work of McCullough and Trent, we investigate the  $z$ -invariant subspaces of the Hilbert function spaces associated to the Szegő kernels on the open unit disk. In particular, we characterize those kernels for which the  $z$ -invariant subspaces are hyperinvariant, and (partially) those for which the so-called BLH subspaces are cyclic, obtaining counterexamples to two questions posed by McCullough and Trent.

### 1. INTRODUCTION

Fix a set  $\Omega$  and a point  $\omega \in \Omega$ . Let  $k(y, x)$  be a positive definite kernel on  $\Omega$ , normalized so that  $k(\cdot, \omega) \equiv 1$ . We say that  $k$  is a *complete Pick kernel* if there exists a positive semidefinite function  $b : \Omega \times \Omega \rightarrow \mathbb{C}$  with  $|b(x, y)| < 1$  such that

$$1 - \frac{1}{k(y, x)} = b(y, x)$$

for all  $x$  and  $y$  in  $\Omega$ . Since  $b(y, x)$  is positive semidefinite, there exists an index set  $\mathcal{B}$  and functions  $b_j : \Omega \rightarrow \mathbb{C}$ ,  $j \in \mathcal{B}$ , such that

$$b(y, x) = \sum_{j \in \mathcal{B}} b_j(y) \overline{b_j(x)}.$$

McCullough and Trent show in [10] that in this case each function  $b_j$  defines a multiplier  $M_{b_j}$  of the associated Hilbert function space  $H(k)$ , and they prove the following analogue of the Beurling-Lax-Halmos theorem:

**Theorem 1.** *Let  $k$  be a complete Pick kernel on  $\Omega$ , and for a Hilbert space  $\mathcal{E}$  let  $H_{\mathcal{E}}(k)$  denote the Hilbert space of  $\mathcal{E}$ -valued functions  $H(k) \otimes \mathcal{E}$ . Let  $\mathcal{M}$  be a closed subspace of  $H_{\mathcal{E}}(k)$ . Then the following are equivalent:*

- (i)  $\mathcal{M}$  is invariant for each  $M_{b_j}$ .
- (ii) There exists an auxiliary Hilbert space  $\mathcal{F}$  and an inner multiplier  $\Phi : \Omega \rightarrow \mathcal{L}(\mathcal{F}, \mathcal{E})$  such that

$$\mathcal{M} = \Phi H_{\mathcal{F}}(k).$$

- (iii)  $\mathcal{M}$  is invariant for every multiplier  $M_{\phi}$  of  $H(k)$ .

---

Received by the editors July 14, 2000 and, in revised form, July 16, 2004.

2000 *Mathematics Subject Classification.* Primary 47B32; Secondary 47A15, 47A16.

*Key words and phrases.* Pick kernel, invariant space, cyclic vector.

©2005 American Mathematical Society  
Reverts to public domain 28 years from publication

Now consider  $\Omega = \mathbb{D}$  (the open unit disk) and let  $s$  be the Szegő kernel

$$s(z, w) = \frac{1}{1 - z\bar{w}}.$$

The kernel  $s$  is a complete Pick kernel, and the space  $H(s)$  is the usual Hardy space  $H^2$ , the space of analytic functions on  $\mathbb{D}$  with square-summable power series. In this case it is possible to show that one may always choose  $\mathcal{F}$  to be a subspace of  $\mathcal{E}$ , recovering the usual Beurling-Lax-Halmos theorem. Additionally, in the Hardy space the multiplier  $\Phi(\zeta)$  is a coisometry almost everywhere on the unit circle. W. Arveson proved a special case of Theorem 1 in [4], and showed that in quite general circumstances  $\dim \mathcal{F}$  must be infinite even when  $\dim \mathcal{E} = 1$ ; nonetheless if  $\dim \mathcal{E}$  is finite and  $\mathcal{M}$  has finite codimension, we can choose  $\mathcal{F}$  to be finite dimensional. See [4] for details. Greene, Richter, and Sundberg [7] have shown that when  $\Omega = \mathbb{B}^d$  (the unit ball of  $\mathbb{C}^d$ ) and the kernel  $k$  satisfies some mild additional assumptions, the vector-valued multiplier  $\Phi(\zeta)$  is a coisometry almost everywhere on the boundary of  $\mathbb{B}^d$ , strengthening the analogy with the usual Beurling theorem.

In the case of the Szegő kernel, the function  $b$  is  $b(z, w) = z\bar{w}$ . Thus the theorem says in particular that every  $M_z$ -invariant subspace of  $H^2$  is hyperinvariant (*i.e.* invariant for every bounded operator on  $H^2$  that commutes with  $M_z$ —here, the multipliers are  $M_\phi$ ,  $\phi \in H^\infty$ ). McCullough and Trent ask in [10] if every  $M_z$ -invariant subspace of  $H(k)$  is hyperinvariant whenever  $k$  is a complete analytic Pick kernel on  $\mathbb{D}$  and multiplication by  $z$  is bounded on  $H(k)$ . We show that this is not true in general, and give necessary and sufficient conditions for this to hold when the function  $b$  has the form

$$b(z, w) = f(z)\overline{f(w)}$$

for a univalent analytic function  $f : \mathbb{D} \rightarrow \mathbb{D}$ .

McCullough and Trent also investigate the cyclicity of the subspaces described by Theorem 1 (hereafter called *BLH subspaces*). They show that if  $k$  is a complete Pick kernel on  $\mathbb{D}$ ,  $M_z$  is bounded above and below, and  $\mathcal{M}$  is a BLH subspace, then the space  $\mathcal{M} \ominus z\mathcal{M}$  is one dimensional (*i.e.*  $M_z$  has the *codimension one property*). When  $k$  is the Szegő kernel, a nonzero vector in  $\mathcal{M} \ominus z\mathcal{M}$  is cyclic for  $M_z$  (the unilateral shift) restricted to  $\mathcal{M}$ . They ask if  $M_z$  is cyclic on the BLH subspaces for more general  $k$ , at least when  $k$  is a *total* Pick kernel (defined later) and  $M_z$  is bounded above and below. Again, by appropriate choice of the function  $f$ , we provide a counterexample. However, a complete description of the kernels for which  $M_z$  is cyclic on the BLH subspaces, even for this special case, seems elusive.

Let  $\Omega$  be a set. We say a function  $k : \Omega \times \Omega \rightarrow \mathbb{C}$  is a *positive definite kernel on  $\Omega$*  if for each finite set  $\{x_1, \dots, x_n\} \subseteq \Omega$ , the matrix

$$(k(x_i, x_j))_{i,j=1}^n$$

is positive definite. For each  $x \in \Omega$ , define a function  $k(\cdot, x)$  on  $\Omega$  by  $k(\cdot, x)(y) = k(y, x)$ . Define an inner product on the linear span of these functions by

$$\left\langle \sum_i a_i k(\cdot, x_i), \sum_j b_j k(\cdot, x_j) \right\rangle = \sum_{i,j} a_i \bar{b}_j k(x_j, x_i).$$

Let  $H(k)$  denote the Hilbert space obtained by completing the linear span of the functions  $k(\cdot, x)$  with respect to the inner product  $\langle \cdot, \cdot \rangle$ . We may regard vectors  $f$  in  $H(k)$  as functions on  $\Omega$ , with  $f(x) = \langle f, k(\cdot, x) \rangle$ .

A function  $\phi$  in  $H(k)$  is called a *multiplier* of  $H(k)$  if  $\phi g \in H(k)$  for every  $g \in H(k)$ . We then define the operator  $M_\phi : H(k) \rightarrow H(k)$  by  $M_\phi g = \phi g, g \in H(k)$ ; boundedness of  $M_\phi$  follows from the closed graph theorem.  $Mult(H(k))$  will denote the algebra of multipliers  $\{M_\phi: \phi \text{ is a multiplier of } H(k)\}$ . The Pick problem is to determine, given points  $x_1, \dots, x_n$  in  $\Omega$  and complex numbers  $\lambda_1, \dots, \lambda_n$ , if there exists a multiplier  $\phi$  on  $H(k)$  with  $\|M_\phi\| \leq 1$  such that  $\phi(x_i) = \lambda_i$  for each  $i$ . We may also formulate a matrix-valued version of this problem: for a positive integer  $m$ , a multiplier on the Hilbert space  $H(k) \otimes \mathbb{C}^m$  is an  $m \times m$  matrix-valued function  $\Phi$  on  $\Omega$  such that

$$\Phi \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \in H(k) \otimes \mathbb{C}^m.$$

As in the scalar case, we define the operator  $M_\Phi$  of “multiplication by  $\Phi$ ”. Now, given points  $x_1, \dots, x_n$  and  $m \times m$  matrices  $\Lambda_1, \dots, \Lambda_n$ , we ask whether or not there exists a multiplier  $\Phi$  on  $H(k) \otimes \mathbb{C}^m$  with  $\|M_\Phi\| \leq 1$  such that  $\Phi(x_i) = \Lambda_i$ .

**Definition 1.** A kernel  $k$  on  $\Omega$  has the  $m \times m$  *Pick property* if, for any finite set of points  $x_1, \dots, x_n$  in  $\Omega$  and any choice of  $m \times m$  matrices  $\Lambda_1, \dots, \Lambda_n$ , the following are equivalent:

- (1) There exists a multiplier  $\Phi$  on  $H(k) \otimes \mathbb{C}^m$  such that  $\|M_\Phi\| \leq 1$  and  $\Phi(x_i) = \Lambda_i$  for each  $i$ .
- (2) The  $mn \times mn$  matrix  $(1 - \Lambda_j^* \Lambda_i)k(x_j, x_i)$  is positive semidefinite.

A kernel  $k$  has the *complete Pick property* if it has the  $m \times m$  Pick property for every  $m$ .

Fix a set  $\Omega$  and a point  $\omega \in \Omega$ . As before we assume  $k$  is normalized so that  $k(\cdot, \omega) \equiv 1$ .  $k$  has the complete Pick property if and only if there exists a positive semidefinite function  $b : \Omega \times \Omega \rightarrow \mathbb{C}$  with  $|b(x, y)| < 1$  such that

$$(1) \quad 1 - \frac{1}{k(y, x)} = b(y, x)$$

for all  $x$  and  $y$  in  $\Omega$ . (See [11], [12], [9]; see also [2].) Since  $b(y, x)$  is positive semidefinite, there is an index set  $\mathcal{B}$  and functions  $b_j : \Omega \rightarrow \mathbb{C}, j \in \mathcal{B}$ , such that

$$b(y, x) = \sum b_j(y) \overline{b_j(x)}.$$

Following [10], a complete Pick kernel  $k$  is called *total* if whenever  $\mathcal{M}$  and  $\mathcal{N}$  are BLH subspaces with  $\mathcal{M} \subseteq \mathcal{N} \subseteq H(k)$ , the kernel

$$\frac{P_{\mathcal{M}}k(z, w)}{P_{\mathcal{N}}k(z, w)}$$

is positive semidefinite. (Here  $P_{\mathcal{M}}$  denotes orthogonal projection onto  $\mathcal{M}$ .) For example, the Szegő kernel is a total Pick kernel, as is shown in the next section using Beurling’s theorem and the divisibility properties of inner functions.

## 2. MAIN RESULTS

From now on we restrict ourselves to the following situation:  $f$  will be a univalent analytic function from  $\mathbb{D}$  into itself, and we will assume for convenience that  $f(0) =$

0. Let  $G$  denote the image of  $\mathbb{D}$  under the conformal mapping  $f$ . Define a positive definite kernel  $k_f : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$  by

$$k_f(z, w) = \frac{1}{1 - f(z)\overline{f(w)}}.$$

Choosing the base point  $\omega = 0$  (so that  $k_f(\cdot, 0) \equiv 1$ ), (1) is easily seen to hold with  $b(z, w) = f(z)\overline{f(w)}$ . The kernels  $k_f$  are called *Szegő kernels*, because they may be regarded as restrictions of the Szegő kernel  $s$  to the subset  $G = f(\mathbb{D}) \subseteq \mathbb{D}$  (see [2]).

In this situation, we show that  $M_z$  is a multiplier of  $H(k_f)$  if and only if there exists a function  $\psi \in H^\infty(\mathbb{D})$  such that  $\psi|_G = f^{-1}$ , and that every  $M_z$ -invariant subspace of  $H(k_f)$  is hyperinvariant (*i.e.* left invariant by every bounded operator on  $H(k_f)$  that commutes with  $M_z$ ) if and only if this function  $\psi$  is a weak-star generator of  $H^\infty(\mathbb{D})$ . (A function  $\phi$  is called a *weak-star generator of  $H^\infty(\mathbb{D})$*  if the polynomials in  $\phi$  are weak-star dense in  $H^\infty(\mathbb{D})$ .) Furthermore, we show that  $M_z$  is cyclic when restricted to the BLH subspaces if and only if the analytic Toeplitz operator  $T_\psi$  is cyclic on  $H^2$ .

To prove these results, we make use of the following propositions which describe the multipliers of  $H(k_f)$ :

**Proposition 1.** *A bounded analytic function  $\phi$  on  $\mathbb{D}$  is a multiplier of  $H(k_f)$  if and only if there exists a function  $\psi \in H^\infty$  such that  $\phi \circ f^{-1} = \psi|_G$ .*

*Proof.* We exploit the fact that the kernel  $k_f$  and the Szegő kernel  $s$  have the Pick property. Suppose  $\phi$  is a multiplier of  $H(k_f)$  with  $\|M_\phi\| = M$ , and let  $z_1, \dots, z_n$  be points in  $\mathbb{D}$ . By computing the Gramian of  $M^2 \cdot I - M_\phi M_{\phi^*}$  with respect to the set of vectors  $\{k_f(\cdot, z_j)\}$ , we see that the matrix  $(A_{ij})_{i,j=1}^n$  with  $(i, j)$  entry given by

$$A_{ij} = (M^2 - \phi(z_i)\overline{\phi(z_j)})k_f(z_i, z_j)$$

is positive semidefinite. Let  $\zeta_i = f(z_i)$ . Then

$$\begin{aligned} A_{ij} &= (M^2 - ((\phi \circ f^{-1})(\zeta_i)\overline{(\phi \circ f^{-1})(\zeta_j)}))k_f(f^{-1}(\zeta_i), f^{-1}(\zeta_j)) \\ &= (M^2 - ((\phi \circ f^{-1})(\zeta_i)\overline{(\phi \circ f^{-1})(\zeta_j)}))\frac{1}{(1 - \zeta_i\overline{\zeta_j})} \\ &= (M^2 - ((\phi \circ f^{-1})(\zeta_i)\overline{(\phi \circ f^{-1})(\zeta_j)}))s(\zeta_i, \zeta_j). \end{aligned}$$

Thus, by the Pick property for  $s$ , there exists a function  $h \in H^\infty$ ,  $\|h\|_\infty \leq M$ , such that  $h(\zeta_i) = (\phi \circ f^{-1})(\zeta_i)$ . Now let  $\Lambda = \{\lambda_i : i \in \mathbb{N}\}$  be a set of uniqueness for analytic functions on  $G$ , and let  $h_N$  be the function obtained in the manner above with  $\zeta_i = \lambda_i, i = 1, \dots, N$ . Since  $\|h_N\|_\infty \leq M$  for each  $N$ , by the Banach-Alaoglu theorem there exists a subsequence  $(h_{N_j})_{j=1}^\infty$  of  $(h_N)_{N=1}^\infty$  which is weak-star convergent in  $H^\infty$  to a function  $\psi$ . We have  $\|\psi\|_\infty \leq M$ , and  $\psi(\lambda_i) = (\phi \circ f^{-1})(\lambda_i)$  for each  $\lambda_i \in \Lambda$ . Since  $\Lambda$  is a set of uniqueness for  $G$ ,  $\psi|_G = \phi \circ f^{-1}$ . The reverse implication is proved by similar reasoning, reversing the roles of  $k_f$  and  $s$  and using the fact that  $k_f$  is a Pick kernel.  $\square$

The preceding argument is standard in spaces possessing complete Pick kernels; a more general form of this proposition can be found in [5].

For  $\psi \in H^\infty(\mathbb{D})$ , the operator  $T_\psi : H^2 \rightarrow H^2$  defined by  $T_\psi f = \psi f$  is called the *analytic Toeplitz operator with symbol  $\psi$* . The map  $\psi \rightarrow T_\psi$  is an isometric

Banach algebra isomorphism and a weak-star homeomorphism (see Hoffman [8]). We now show that each multiplier  $M_\phi$  of  $H(k_f)$  is unitarily equivalent to an analytic Toeplitz operator.

**Proposition 2.** *Let  $\phi$  be a multiplier of  $H(k_f)$  and let  $\psi$  be the analytic continuation of  $\phi \circ f^{-1}$  to  $\mathbb{D}$ , as in Proposition 1. Then  $M_\phi \cong T_\psi$ .*

*Proof.* The linear span of the functions  $\{s(\cdot, \lambda) : \lambda \in G\}$  is dense in  $H^2$ , for if  $f(\lambda) = \langle f, s(\cdot, \lambda) \rangle_{H^2} = 0$  for all  $\lambda$  in  $G$ , then  $f \equiv 0$  on  $G$  and hence on  $\mathbb{D}$ , since  $G$  is open.

Define a map  $V$  from  $\bigvee\{s(\cdot, \lambda) : \lambda \in G\}$  to  $\bigvee\{k_f(\cdot, w) : w \in \mathbb{D}\}$  by setting  $Vs(\cdot, \lambda) = k_f(\cdot, f^{-1}(\lambda))$  and extending linearly. Since

$$\begin{aligned} \langle Vs(\cdot, \lambda), Vs(\cdot, \mu) \rangle_{H(k_f)} &= \langle k_f(\cdot, f^{-1}(\lambda)), k_f(\cdot, f^{-1}(\mu)) \rangle \\ &= k_f(f^{-1}(\mu), f^{-1}(\lambda)) \\ &= \frac{1}{1 - \mu\bar{\lambda}} \\ &= \langle s(\cdot, \lambda), s(\cdot, \mu) \rangle_{H^2}, \end{aligned}$$

$V$  is an isometry from a dense subset of  $H^2$  onto a dense subset of  $H(k_f)$ , so it extends to a unitary operator (which we also denote by  $V$ ) from  $H^2$  onto  $H(k_f)$ . It is now easy to verify that  $T_\psi = V^*M_\phi V$ .  $\square$

The proof shows that every function in  $H(k_f)$  has the form  $g \circ f$ , with  $g \in H^2$ , and  $\|g \circ f\|_{H(k_f)} = \|g\|_{H^2}$  (this is evident for kernel functions and follows in general by taking limits). the proposition also shows that every analytic Toeplitz operator gives rise to a multiplier of  $H(k_f)$ :

**Corollary 1.** *Let  $V$  be as in the proposition. For any  $\psi \in H^\infty$ ,  $\psi \circ f$  is a multiplier of  $H(k_f)$  and  $VT_\psi V^* = M_{\psi \circ f}$ .*

*Proof.* This is an immediate consequence of the preceding remark; alternatively we could observe that  $(\psi \circ f) \circ f^{-1} = \psi|_G$  and apply Propositions 1 and 2.  $\square$

From now on we assume that  $f$  is such that  $M_z$  is bounded on  $H(k_f)$ ; by Proposition 1 this means that there exists an  $H^\infty$  function  $\psi$  on  $\mathbb{D}$  such that  $\psi|_G = f^{-1}$ . The next proposition deals with the set of bounded operators that commute with  $M_z$  (here  $\{T\}'$  denotes the set of bounded operators that commute with the operator  $T$ ):

**Proposition 3.** *Suppose  $M_z$  is bounded on  $H(k_f)$  with  $M_z \cong T_\psi$ . If  $\psi$  is univalent as an analytic function on  $\mathbb{D}$ , then  $\{M_z\}' = \text{Mult}(H(k_f))$ .*

*Proof.* Since every multiplier commutes with  $M_z$ , it is enough to show that  $\{M_z\}' \subseteq \text{Mult}(H(k_f))$ ; by Proposition 2 and its corollary, this is equivalent to  $\{T_\psi\}' \subseteq \text{Mult}(H^2)$ . So suppose  $A$  is a bounded operator on  $H^2$  that commutes with  $T_\psi$ . Then  $A^*$  commutes with  $T_\psi^*$ , and hence with  $T_\psi^* - \overline{\psi(\lambda)}I$  for every  $\lambda \in \mathbb{D}$ . Thus  $\ker(T_\psi^* - \overline{\psi(\lambda)}I)$  is invariant for  $A^*$ . Since  $\psi$  was assumed univalent,  $\ker(T_\psi^* - \overline{\psi(\lambda)}I)$  is the one-dimensional space spanned by  $s(\cdot, \lambda)$ . Thus  $A^*s(\cdot, \lambda) = \overline{a(\lambda)}s(\cdot, \lambda)$  for some complex number  $\overline{a(\lambda)}$ . Now for every  $g \in H^2$  and every  $\lambda \in \mathbb{D}$ , we have

$$\langle Ag, s(\cdot, \lambda) \rangle = \langle g, A^*s(\cdot, \lambda) \rangle = a(\lambda)g(\lambda).$$

So  $ag = Ag$  is in  $H^2$  and  $A = T_a$ .  $\square$

The fact that  $\{T_\psi\}' = \text{Mult}(H^2)$  when  $\psi$  is univalent is actually a special case of a much more general theorem about the commutant of an analytic Toeplitz operator; see [17]. The proof also shows that in this situation,  $M_z^*$  is in the Cowen-Douglas class (see [6]).

Our first result will invoke the following theorem of D. Sarason [13]:

**Theorem 2.** *Let  $\psi \in H^\infty$ . Then  $T_\psi$  has the same invariant subspaces as  $T_z$  if and only if  $\psi$  is a weak-star generator of  $H^\infty$ .*

We can now describe those kernels  $k_f$  for which every  $M_z$ -invariant subspace is hyperinvariant:

**Theorem 3.** *With notations as above, the following are equivalent:*

- (1) *Every  $M_z$ -invariant subspace of  $H(k_f)$  is hyperinvariant.*
- (2) *Every  $M_z$ -invariant subspace of  $H(k_f)$  is  $M_f$ -invariant.*
- (3) *Every  $T_\psi$ -invariant subspace of  $H^2$  is  $T_z$ -invariant.*
- (4)  *$\psi$  is a weak-star generator of  $H^\infty$ .*

*Proof.* (1)  $\Rightarrow$  (2) is trivial. By Proposition 2,  $M_z \cong T_\psi$  and  $M_f \cong T_z$ ; the equivalence of (2) and (3) follows. (3)  $\Leftrightarrow$  (4) is just an application of Theorem 2 above, together with the fact that  $T_z$ -invariant subspaces are invariant for every analytic Toeplitz operator, the consequence of Beurling's theorem discussed earlier.

It remains to show (2)  $\Rightarrow$  (1). Since (2)  $\Leftrightarrow$  (4), the function  $\psi$  is a weak-star generator of  $H^\infty$  and hence is univalent in  $\mathbb{D}$  (see [14]), so by Proposition 3, the only bounded operators on  $H(k_f)$  that commute with  $M_z$  are the multipliers. By Theorem 1, each  $M_f$ -invariant subspace is invariant for every multiplier, and (1) follows.  $\square$

Turning now to cyclic vectors, we show that the kernels  $k_f$  are total Pick kernels, and describe the cyclicity of  $M_z$  on BLH subspaces in terms of the cyclicity of analytic Toeplitz operators.

**Proposition 4.** *The kernels  $k_f$  are total Pick kernels.*

*Proof.* We observe that  $\mathcal{M}$  is a BLH subspace of  $H(k_f)$  if and only if  $V^*\mathcal{M}$  is a BLH subspace (i.e. shift invariant subspace) of  $H^2$ , where  $V$  is the unitary map of Proposition 2. By Beurling's theorem,  $V^*\mathcal{M} = \phi H^2$ , where  $\phi$  is an inner function, so

$$\mathcal{M} = VV^*\mathcal{M} = V\phi H^2 = (\phi \circ f)H(k_f).$$

A straightforward calculation then shows that

$$P_{\mathcal{M}}k_f(z, w) = \phi(f(z))\overline{\phi(f(w))}k_f(z, w).$$

Recall that if  $\phi_1$  and  $\phi_2$  are inner functions and  $\phi_1 H^2 \subseteq \phi_2 H^2$ , then  $\phi_1/\phi_2$  is inner. Let  $\mathcal{M}$  and  $\mathcal{N}$  be BLH subspaces of  $H(k_f)$  with  $\mathcal{M} \subseteq \mathcal{N}$ , so there exist inner functions  $\phi_1$  and  $\phi_2$  with  $\phi_2$  dividing  $\phi_1$  so that  $\mathcal{M} = (\phi_1 \circ f)H(k_f)$  and  $\mathcal{N} = (\phi_2 \circ f)H(k_f)$ . Then

$$\begin{aligned} \frac{P_{\mathcal{M}}k_f(z, w)}{P_{\mathcal{N}}k_f(z, w)} &= \frac{\phi_1(f(z))\overline{\phi_1(f(w))}k_f(z, w)}{\phi_2(f(z))\overline{\phi_2(f(w))}k_f(z, w)} \\ &= \frac{\phi_1}{\phi_2}(f(z))\overline{\frac{\phi_1}{\phi_2}(f(w))}. \end{aligned}$$

The last expression is a positive semidefinite function on  $\mathbb{D} \times \mathbb{D}$ , so  $k_f$  is a total Pick kernel.  $\square$

**Theorem 4.** *Let  $k_f$  be a complete Pick kernel, and suppose  $M_z$  is bounded, so that  $M_z \cong T_\psi$ . Then the following are equivalent:*

- (1)  $M_z$  is cyclic on some BLH subspace  $\mathcal{M} \subseteq H(k_f)$ .
- (2)  $M_z$  is cyclic on every BLH subspace  $\mathcal{M} \subseteq H(k_f)$ .
- (3)  $T_\psi$  is cyclic on  $H^2$ .

*Proof.* In the proof of Proposition 4 we showed that the BLH subspaces of  $H(k_f)$  have the form  $(\phi \circ f)H(k_f)$ , where  $\phi$  is an inner function. From this it follows that a function  $g$  in a BLH subspace  $\mathcal{M} \subseteq H(k_f)$  is cyclic for  $M_z|_{\mathcal{M}}$  if and only if  $g/(\phi \circ f)$  is cyclic for  $M_z$  on  $H(k_f)$ , and from this follows the equivalence of assertions (1) and (2). Since  $M_z \cong T_\psi$ ,  $M_z$  is cyclic on the BLH subspaces if and only if  $T_\psi$  is cyclic on  $H^2$ .  $\square$

### 3. EXAMPLES AND REMARKS

Using Theorem 3 we can construct an analytic Pick kernel for which  $M_z$  is bounded but for which there exist  $M_z$ -invariant subspaces which are not hyperinvariant. To do this, we need only exhibit a univalent analytic function  $f : \mathbb{D} \rightarrow G$ , (with  $G \subseteq \mathbb{D}$ ) so that  $f^{-1}$  extends to a function  $\psi \in H^\infty$  which is *not* a weak-star generator of  $H^\infty$ .

Consider a bounded, simply-connected domain  $\Omega \subseteq \mathbb{C}$ , and let  $\psi$  be a Riemann map from  $\mathbb{D}$  onto  $\Omega$ . Sarason shows in [14] that  $\psi$  fails to be a weak-star generator of  $H^\infty$  if and only if  $\Omega$  has the following property:

- (\*) *There exists a domain  $B$  containing  $\Omega$  properly such that  $\sup_{z \in B} |f(z)| = \sup_{z \in \Omega} |f(z)|$  for all  $f$  bounded and analytic in  $B$ .*

For example, let  $B$  be an open disk, and fix a point  $z_0 \in B$  (not the center). Let  $\alpha = \text{dist}(z_0, \partial B)$  and let  $\overline{D}$  be the closed disk of radius  $\alpha$  centered at  $z_0$ . Then  $B \setminus \overline{D}$  has property (\*), by the maximum modulus principle.

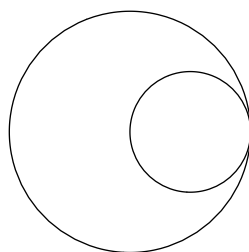


FIGURE 1. The domain  $B \setminus \overline{D}$

We now let  $\Omega$  be a domain satisfying (\*) with  $\mathbb{D} \subseteq \Omega$  and let  $\psi : \mathbb{D} \rightarrow \Omega$  be a Riemann map. Taking  $f$  to be the restriction of  $\psi^{-1}$  to  $\mathbb{D}$  does the job.

Regarding cyclic vectors, we can use Theorem 4 to exhibit a total Pick kernel for which multiplication by  $z$  is bounded above and below, but such that  $M_z$  is not cyclic on any of the BLH subspaces. By Theorem 4 it will suffice to exhibit an analytic Toeplitz operator which is bounded below but is not cyclic. For this a Riemann map  $\phi$  of  $\mathbb{D}$  onto a slit disk which contains  $\mathbb{D}$  suffices (see [16]); we then let

$f = \phi^{-1}|_{\mathbb{D}}$  as in the previous example. The cyclicity of analytic Toeplitz operators is a very subtle problem, which has not been solved completely; it is for this reason that we regard the “characterization” of Theorem 4 as incomplete. We remark, however, that  $T_\psi$  is cyclic when  $\psi$  is a weak-star generator of  $H^\infty$ , so combining Theorem 3 and the above remarks tells us that if every  $M_z$ -invariant subspace of  $H(k_f)$  is hyperinvariant (and hence a BLH subspace), then  $M_z$  is cyclic on each of these spaces.

In general, we expect the kernels  $k_f$  to be a good source of counterexamples for questions regarding multiplication by  $z$  on spaces possessing a complete Pick kernel: as seen in these two examples, one can choose  $f$  so that  $M_z$  is unitarily equivalent to an analytic Toeplitz operator with “bad” properties.

## REFERENCES

- [1] Agler, J. and McCarthy, J. E., *Pick interpolation and Hilbert function spaces*, American Mathematical Society, Providence, RI, 2002. MR1882259 (2003b:47001)
- [2] Agler, J. and McCarthy, J. E., *Complete Nevanlinna-Pick kernels*, Journal of Functional Analysis **175** (2000), 111-124. MR1774853 (2001h:47019)
- [3] Aleman, A., Richter, S., and Sundberg, C., *The majorization function and the index of invariant subspaces of the Bergman space*, Journal Analyse de Mathématique **86** (2002), 139-182. MR1894480 (2003g:30058)
- [4] Arveson, W., *The curvature invariant of a Hilbert module over  $\mathbb{C}[z_1, \dots, z_d]$* , J. Reine. Angew. Math. **522** (2000), 173-236. MR1758582 (2003a:47013)
- [5] Ball, J., Trent, T., and Vinnikov, V., *Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces*, in *Operator Theory and Analysis (Amsterdam 1997)*, OT122, 89-138, Birkhäuser, 2001. MR1846055 (2002f:47028)
- [6] Cowen, M. J. and Douglas, R. G., *Complex geometry and operator theory*, Acta Mathematica **141** (1978), 187-261. MR0501368 (80f:47012)
- [7] Greene, D., Richter, S., and Sundberg, C., *The structure of inner multipliers on spaces with complete Nevanlinna-Pick kernels*, Journal of Functional Analysis **194** (2002), 311-331. MR1934606 (2003h:46038)
- [8] Hoffman, K., *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, NJ, 1962. MR0133008 (24:A2844)
- [9] McCullough, S., *The local deBranges-Rovnyak construction and complete Nevanlinna-Pick kernels*, in *Algebraic Methods in Operator Theory*, 15-24, Birkhäuser, 1994. MR1284929 (95j:47016)
- [10] McCullough, S., and Trent, T., *Invariant subspaces and Nevanlinna-Pick kernels*, Journal of Functional Analysis **178** (2000), 226-249. MR1800795 (2002b:47006)
- [11] Quiggin, P., *For which reproducing kernel Hilbert spaces is Pick's theorem true?*, Integral Equations and Operator Theory **16** (1993), 244-266. MR1205001 (94a:47026)
- [12] ———, *Generalisations of Pick's theorem to reproducing kernel Hilbert spaces*, Ph.D. thesis, Lancaster University, 1994.
- [13] Sarason, D., *Invariant subspaces and unstarred operator algebras*, Pacific Journal of Mathematics **17** (1966), 511-517. MR0192365 (33:590)
- [14] ———, *Weak-star generators of  $H^\infty$* , Pacific Journal of Mathematics **17** (1966), 519-528. MR0211269 (35:2151)
- [15] ———, *Generalized interpolation in  $H^\infty$* , American Mathematical Society Transactions **27** (1967), 180-203. MR0208383 (34:8193)
- [16] Solomyak, B. M. and Volberg, A. L., *Multiplicity of analytic Toeplitz operators*, in *Toeplitz operators and spectral function theory*, 87-192, Birkhäuser, 1989. MR1030051 (91g:47021)
- [17] Thomson, J. E., *The commutant of a class of analytic Toeplitz operators II*, Indiana University Mathematics Journal **25** (1976), 793-800. MR0417843 (54:5891)

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, 150 N. UNIVERSITY ST., WEST LAFAYETTE, INDIANA 47907-2067

*E-mail address:* jury@math.purdue.edu