THE BANACH-ZARECKI THEOREM FOR FUNCTIONS WITH VALUES IN METRIC SPACES

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ABSTRACT. Using an old result of Luzin about his property \((N)\), we prove a general version of the Banach-Zarecki theorem (on absolute continuity and Luzin’s property \((N)\)).

We prove a general version of the Banach-Zarecki theorem (see the Theorem below) about absolute continuity and Luzin’s property \((N)\). The original version for real-valued functions of a real variable was proved by Banach and independently by Zarecki (cf. [N]). For functions of a real variable with values in reflexive Banach spaces, the result is contained in [F, Theorem 2.10.13] with a sketch of the proof which also works if \(X\) has the Radon-Nikodým property. We observe that the general case of a function of a real variable with values in a metric space follows by an old result of Luzin [L] (see text after (1)).

By \(\lambda\) we shall denote the Lebesgue measure on \(\mathbb{R}\) and by \(\mathcal{H}^1\) we denote the 1-dimensional Hausdorff measure.

Let \((X, \rho)\) be a metric space and let \(f : [0, 1] \to X\) be a function. We say that \(f\) is absolutely continuous if for each \(\varepsilon > 0\) there is a \(\delta > 0\) such that for any

\[
0 \leq a_1 < b_1 \leq a_2 < \cdots \leq a_n < b_n \leq 1
\]

with \(\sum_{i=1}^n (b_i - a_i) < \delta\) we have \(\sum_{i=1}^n \rho(f(b_i), f(a_i)) < \varepsilon\). The symbol \(\mathcal{V}_\varepsilon^d f\) stands for the variation of \(f\) on \([c, d] \subset [0, 1]\). We say that \(f\) has (Luzin’s) property \((N)\) provided

\[
\mathcal{H}^1(f(B)) = 0 \quad \text{whenever} \quad B \subset [0, 1] \text{ with } \lambda(B) = 0.
\]

Luzin [L, §47] proved that if \(X = \mathbb{R}\) and \(f\) is continuous, then we obtain the same notion if we only use closed sets \(B\) in \([1]\). (See [F6] and [HPZZ] for proofs of more general results.)
We shall also need the following simple lemma.

**Lemma.** Let \((X, \rho)\) be a metric space, let \(\{0, 1\} \subset B \subset [0,1] \) be closed, and let \(f : [0, 1] \to X\) be continuous. If \(\mathcal{H}^1(f(B)) = 0\), then

\[
\int_0^1 f = \sum_{i \in \mathcal{I}} \int_{c_i}^{d_i} f,
\]

where \(I_i = (c_i, d_i)\) \((i \in \mathcal{I} \subset \mathbb{N})\) are all (pairwise different) components of \([0, 1] \setminus B\).

**Proof.** We can embed the metric space \(X\) isometrically into a Banach space (see e.g. [BL, Lemma 1.1]). Put \( \langle f \rangle := f([0,1]) \) and, for each \(A \subset [0,1]\) and \(y \in X\), define

\[
N(f|A, y) = \text{card}\{x \in A : f(x) = y\}.
\]

Using the vector version of the Banach indicatrix theorem ([F, Theorem 2.10.13]) and the obvious equality \(N(f, y) = \sum_{i \in \mathcal{I}} N(f|I_i, y)\) for \(y \in \langle f \rangle \setminus f(B)\), we obtain

\[
\int_0^1 f = \int_{\langle f \rangle} N(f, y) \, d\mathcal{H}^1 y = \int_{\langle f \rangle \setminus f(B)} N(f, y) \, d\mathcal{H}^1 y = 1 \sum_{i \in \mathcal{I}} \int_{I_i \setminus f(B)} N(f|I_i, y) \, d\mathcal{H}^1 y = \sum_{i \in \mathcal{I}} \int_{I_i} f \, d\mathcal{H}^1 y = \sum_{i \in \mathcal{I}} d_i,
\]

\(\square\)

Now we can easily prove the general Banach-Zarecki theorem.

**Theorem.** Let \((X, \rho)\) be a metric space, and let \(f : [0, 1] \to X\). Then the following are equivalent:

(i) \(f\) is absolutely continuous;

(ii) \(f\) is continuous, has bounded variation and satisfies property \((N)\).

**Proof.** It is easy to see that (i) \(\implies\) (ii). For a proof of property \((N)\) we can just follow the standard “scalar” proof of [S, Theorem 6.1] with obvious modifications (namely writing \(\text{Osc}(H \cap I_n)\) instead of \(M(H \cap I_n) - m(H \cap I_n)\) and \(\text{diam}(F(H \cap I_n))\) instead of \(|F(H)|\)).

Now suppose that (ii) holds. For \(x \in [0,1]\) we define \(v_f(x) = \int_0^x f\). Since clearly \(\rho(f(x), f(y)) \leq |v_f(x) - v_f(y)|\), we easily see that it is sufficient to prove absolute continuity of \(v_f\). To prove that, it’s enough (since \(v_f\) is non-decreasing and continuous by [F, §2.5.16]) to establish that \(v_f\) has property \((N)\) and apply the scalar version of the Banach-Zarecki Theorem (see e.g. [V, Theorem 3] or [F, 2.10.13]). By Luzin’s theorem mentioned in the text following (1), it is enough to prove that \(\lambda(v_f(B)) = 0\) for any closed \(B \subset [0,1]\) with \(\lambda(B) = 0\). Without any loss of generality, we can assume that \(\{0, 1\} \subset B\). Since \(f\) has property \((N)\), we have \(\mathcal{H}^1(f(B)) = 0\). Let \(I_i = (c_i, d_i)\) \((i \in \mathcal{I} \subset \mathbb{N})\) be all (pairwise different) components of \([0, 1] \setminus B\). The Lemma shows that

\[
\lambda(v_f\bigcup_{i \in \mathcal{I}} I_i) = \lambda\bigcup_{i \in \mathcal{I}} (v_f(I_i)) = \sum_{i \in \mathcal{I}} (v_f(d_i) - v_f(c_i)) = \int_0^1 f = \lambda(v_f([0,1])),
\]

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as \( v_f \) is continuous non-decreasing and \( v_f(d_i) - v_f(c_i) = \bigvee_{I_i} f \) for \( i \in \mathcal{I} \). Observing that \( v_f(B) \cap v_f(\bigcup_{i \in \mathcal{I}} I_i) \) is countable, we obtain

\[
\lambda(v_f(B)) = \lambda(v_f([0,1]) \setminus v_f(\bigcup_i I_i)) = 0,
\]

which completes the proof. \( \square \)

**References**


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