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THE BANACH-ZARECKI THEOREM FOR FUNCTIONS WITH VALUES IN METRIC SPACES

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Abstract. Using an old result of Luzin about his property (N), we prove a general version of the Banach-Zarecki theorem (on absolute continuity and Luzin's property (N)).

We prove a general version of the Banach-Zarecki theorem (see the Theorem below) about absolute continuity and Luzin's property (N). The original version for real-valued functions of a real variable was proved by Banach and independently by Zarecki (cf. [N]). For functions of a real variable with values in reflexive Banach spaces, the result is contained in [F, Theorem 2.10.13] with a sketch of the proof which also works if X has the Radon-Nikodým property. We observe that the general case of a function of a real variable with values in a metric space follows by an old result of Luzin [L] (see text after (1)).

By λ we shall denote the Lebesgue measure on $\mathbb{R}$ and by $\mathcal{H}^1$ we denote the 1-dimensional Hausdorff measure.

Let (X, ρ) be a metric space and let $f : [0,1] \to X$ be a function. We say that f is *absolutely continuous* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for any

$$0 \leq a_1 < b_1 \leq a_2 < \cdots \leq a_n < b_n \leq 1$$

with $\sum_{i=1}^n (b_i - a_i) < \delta$ we have $\sum_{i=1}^n \rho(f(b_i), f(a_i)) < \varepsilon$. The symbol $\bigvee_c^d f$ stands for the variation of f on $[c,d] \subset [0,1]$. We say that f has (*Luzin's*) *property* (N) provided

$$\mathcal{H}^1(f(B)) = 0 \quad \text{whenever} \quad B \subset [0,1] \text{ with } \lambda(B) = 0. \tag{1}$$

Luzin [L, §47] proved that if $X = \mathbb{R}$ and f is continuous, then we obtain the same notion if we only use closed sets B in (1). (See [Fo] and [HPZZ] for proofs of more general results.)

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We shall also need the following simple lemma.

Lemma. *Let* (X, ρ) *be a metric space, let* $\{0,1\} \subset B \subset [0,1]$ *be closed, and let* $f : [0,1] \to X$ *be continuous. If* $\mathcal{H}^1(f(B)) = 0$*, then*

$$\bigvee_0^1 f = \sum_{i \in \mathcal{I}} \bigvee_{c_i}^{d_i} f,$$

where $I_i = (c_i, d_i)$ $(i \in \mathcal{I} \subset \mathbb{N})$ *are all (pairwise different) components of* $[0,1] \setminus B$.

Proof. We can embed the metric space X isometrically into a Banach space (see e.g. [BL, Lemma 1.1]). Put $\langle f \rangle := f([0,1])$ and, for each $A \subset [0,1]$ and $y \in X$, define

$$N(f|_A, y) = \mathrm{card}(\{x \in A : f(x) = y\}).$$

Using the vector version of the Banach indicatrix theorem ([F, Theorem 2.10.13]) and the obvious equality $N(f, y) = \sum_{i \in \mathcal{I}} N(f|_{I_i}, y)$ for $y \in \langle f \rangle \setminus f(B)$, we obtain

$$\begin{aligned} \bigvee_0^1 f &= \int_{\langle f \rangle} N(f,y)\, d\mathcal{H}^1 y = \int_{\langle f \rangle \setminus f(B)} N(f,y)\, d\mathcal{H}^1 y \\ &= \sum_{i \in \mathcal{I}} \int_{\langle f \rangle \setminus f(B)} N(f|_{I_i}, y)\, d\mathcal{H}^1 y = \sum_{i \in \mathcal{I}} \bigvee_{c_i}^{d_i} f. \end{aligned}$$

□

Now we can easily prove the general Banach-Zarecki theorem.

Theorem. *Let* (X, ρ) *be a metric space, and let* $f : [0,1] \to X$*. Then the following are equivalent:*

(i) f *is absolutely continuous;*
(ii) f *is continuous, has bounded variation and satisfies property* (N).

Proof. It is easy to see that (i) $\implies$ (ii). For a proof of property (N) we can just follow the standard "scalar" proof of [S, Theorem 6.1] with obvious modifications (namely writing $\mathrm{Osc}(H \cap I_n)$ instead of $M(H \cap I_n) - m(H \cap I_n)$ and $\mathrm{diam}(F(H \cap I_n))$ instead of $|F(H)|$).

Now suppose that (ii) holds. For $x \in [0,1]$ we define $v_f(x) = \bigvee_0^x f$. Since clearly $\rho(f(x), f(y)) \le |v_f(x) - v_f(y)|$, we easily see that it is sufficient to prove absolute continuity of v_f. To prove that, it's enough (since v_f is non-decreasing and continuous by [F, §2.5.16]) to establish that v_f has property (N) and apply the scalar version of the Banach-Zarecki Theorem (see e.g. [V, Theorem 3] or [F, 2.10.13]). By Luzin's theorem mentioned in the text following (1), it is enough to prove that $\lambda(v_f(B)) = 0$ for any closed $B \subset [0,1]$ with $\lambda(B) = 0$. Without any loss of generality, we can assume that $\{0,1\} \subset B$. Since f has property (N), we have $\mathcal{H}^1(f(B)) = 0$. Let $I_i = (c_i, d_i)$ $(i \in \mathcal{I} \subset \mathbb{N})$ be all (pairwise different) components of $[0,1] \setminus B$. The Lemma shows that

$$\begin{aligned} \lambda(v_f(\bigcup_{i \in \mathcal{I}} I_i)) &= \lambda(\bigcup_{i \in \mathcal{I}} (v_f(I_i))) \\ &= \sum_{i \in \mathcal{I}} (v_f(d_i) - v_f(c_i)) = \bigvee_0^1 f = \lambda(v_f([0,1])), \end{aligned}$$

as v_f is continuous non-decreasing and $v_f(d_i) - v_f(c_i) = \bigvee_{c_i}^{d_i} f$ for $i \in \mathcal{I}$. Observing that $v_f(B) \cap v_f(\bigcup_{i \in \mathcal{I}} I_i)$ is countable, we obtain

$$\lambda(v_f(B)) = \lambda(v_f([0,1]) \setminus v_f(\bigcup_i I_i)) = 0,$$

which completes the proof. □

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