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ON THE LOCATION OF THE DISCRETE SPECTRUM FOR COMPLEX JACOBI MATRICES

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ABSTRACT. We study spectrum inclusion regions for complex Jacobi matrices that are compact perturbations of the discrete Laplacian. The condition sufficient for the lack of a discrete spectrum for such matrices is given.

INTRODUCTION

Let

$$J = \begin{pmatrix} b_1 & c_1 & & & \\ a_1 & b_2 & c_2 & & \\ & a_2 & b_3 & c_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{1}$$

be an infinite Jacobi matrix with complex entries. We assume that $a_n c_n \neq 0$, $n \in \mathbb{N} := \{1, 2, ...\}$, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 1, \qquad \lim_{n\to\infty} b_n = 0;$$

that is, the operator J generated by matrix (1) in $\ell^2(\mathbb{N})$ is a compact perturbation of the discrete Laplacian

$$J_0: \quad a_n = c_n = 1, \quad b_n = 0.$$

The structure of the spectrum for such operators is well known: $\sigma(J) = [-2, 2] \cup \sigma_d(J)$, where $\sigma_d(J)$ is at most a denumerable set on the complex plane $\mathbb{C}$ with all accumulation points in $[-2, 2]$. We refer to this portion of the spectrum as the *discrete spectrum* of J. The goal of our note is to single out domains on $\mathbb{C}$ free from the discrete spectrum. In particular, a condition on the matrix entries which provides the lack of $\sigma_d(J)$ comes in quite naturally. In the case of selfadjoint operators (1) ($a_n = c_n > 0,\ b_n = \overline{b}_n$) the problem is well elaborated and goes back to M.S. Birman and J. Schwinger (see also [1]–[3] and the references therein, and [4]–[5] for complex Jacobi matrices). The method applied here is totally different and quite elementary. It is adopted from [6] and based on certain lower bounds for

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the Jost function in the unit disk. The point is made upon the spectrum inclusion regions rather than the bounds for the spectral radius.

Recurrence relations

We start out with the three-term recurrence relation associated with the matrix J:

$$a_{m-1}y_{m-1} + b_m y_m + c_m y_{m+1} = (z + z^{-1})y_m, \quad m \in \mathbb{N}, \quad z \in \overline{\mathbb{D}} := \{|z| \le 1\}, \tag{2}$$

$z \neq 0$ (we put $a_0 = c_0 = 1$). It is clear that the initial data $\{y_0, y_1\}$ enables one to restore the whole solution $\{y_m(z)\}_{m\ge 0}$ of (2); that is, the dimension of the space of solutions is 2. Sometimes it is beneficial to deal with a slightly modified relation. If we multiply (2) by $k(j) = \prod_{i=j}^{\infty} a_i$ (the product will always be assumed to converge) and put $x_m = k(m)y_m$, we come to

$$x_{m-1} + b_m x_m + a_m c_m x_{m+1} = (z + z^{-1})x_m, \quad m \in \mathbb{N}. \tag{3}$$

From recurrence relations to discrete integral equations

The key role in what follows is played by certain solutions of (2), (3) that have a specific behavior at infinity. We show that such solutions exist as long as the coefficients in (2) tend to their limits fast enough.

Denote by G the Green kernel

$$G(n, m; z) = \begin{cases} \frac{z^{m-n} - z^{n-m}}{z - z^{-1}}, & m > n, \\ 0, & m \le n, \end{cases} \qquad n, m \in \mathbb{Z}_+ := \{0, 1, ...\}, \ z \neq 0. \tag{4}$$

It is clear that

$$G(n, m, z) = U_{m-n-1}\left(\frac{z + z^{-1}}{2}\right),$$

where U_k is the Chebyshev polynomial of the second kind. The recurrence relations for G are straightforward:

$$G(n, m+1; z) + G(n, m-1; z) - (z + z^{-1})G(n, m; z) = \delta(n, m), \tag{5}$$

$$G(n-1, m; z) + G(n+1, m; z) - (z + z^{-1})G(n, m; z) = \delta(n, m), \tag{6}$$

where $\delta(n, m)$ is the Kronecker symbol.

We begin with the following conditional result.

Proposition 1. *Suppose that equation* (3) *has a solution v_n with asymptotic behavior at infinity*

$$\lim_{n\to\infty} v_n(z) z^{-n} = 1 \tag{7}$$

for some $z \in \mathbb{D}$. Then v_n satisfies a discrete integral equation

$$v_n(z) = z^n + \sum_{m=n+1}^{\infty} J(n, m; z)\, v_m(z), \quad n \in \mathbb{Z}_+, \tag{8}$$

with

$$J(n, m; z) = -b_m G(n, m; z) + (1 - a_{m-1}c_{m-1})\, G(n, m-1; z). \tag{9}$$

Proof. Let us multiply (5) by v_m, (3) by $G(n,m)$ and subtract the latter from the former:

$$G(n,m+1)v_m+G(n,m-1)v_m-G(n,m)v_{m-1}-b_mG(n,m)v_m-a_mc_mG(n,m)v_{m+1} = \delta(n,m)v_m.$$

Summing up over m from n to N gives

$$v_n = \sum_{m=n}^{N} \{-b_mG(n,m) + (1-a_{m-1}c_{m-1})\,G(n,m-1)\}\,v_m + G(n,N+1)v_N - a_Nc_NG(n,N)v_{N+1}.$$

For $|z|<1$ we have by (4) and (7),

$$\lim_{N\to\infty} (G(n,N+1)v_N - a_nc_nG(n,N)v_{N+1}) = z^n,$$

which along with $J(n,n)=0$ leads to (8), as needed. □

The converse statement is equally simple.

Proposition 2. *Each solution* $\{v_n(z)\}_{n\ge 0}$, $z\in\overline{\mathbb{D}}$, *of equation* (8) *satisfies the three-term recurrence relation* (3).

Proof. Write, for $n\ge 1$,

$$\begin{aligned} v_{n-1}+v_{n+1} &= z^{n-1}+z^{n+1}+\sum_{m=n}^{\infty} J(n-1,m)v_m + \sum_{m=n+2}^{\infty} J(n+1,m)v_m \\ &= (z+z^{-1})z^n + J(n-1,n)v_n + J(n-1,n+1)v_{n+1} \\ &\quad + \sum_{m=n+2}^{\infty}\{J(n-1,m)+J(n+1,m)\}v_m. \end{aligned}$$

By (4), (9) and (6),

$$J(n-1,n;z) = -b_n, \qquad J(n-1,n+1;z) = -(z+z^{-1})b_{n+1}+1-a_nc_n$$

and

$$J(n-1,m;z)+J(n+1,m;z) = (z+z^{-1})J(n,m;z).$$

Hence

$$\begin{aligned} v_{n-1}+v_{n+1}+b_nv_n-(1-a_nc_n)v_{n+1} &= (z+z^{-1})\left(z^n + \sum_{m=n+1}^{\infty} J(n,m)v_m\right) \\ &= (z+z^{-1})v_n, \end{aligned}$$

which is exactly (3). □

The Jost solution

To analyze equation (8) it seems reasonable to introduce new variables:

$$\tilde v_n(z) := v_nz^{-n}, \quad \tilde J(n,m;z) := J(n,m;z)z^{m-n},$$

so that

$$\tilde v_n(z) = 1+\sum_{m=n+1}^{\infty} \tilde J(n,m;z)\tilde v_m(z), \quad n\in\mathbb{Z}_+. \tag{10}$$

Now $\tilde{J}(n,m;\cdot)$ is a polynomial and since

$$|G(n,m,z)z^{m-n}| = \frac{|z^{2(m-n)}-1|}{|z-z^{-1}|} \le |z| \min\left\{|m-n|, \frac{2}{|z^2-1|}\right\},$$

the kernel $\tilde{J}$ is bounded by

$$|\tilde{J}(n,m;z)| \le |z| d_m \min\left\{|m-n|, \frac{2}{|z^2-1|}\right\}, \tag{11}$$
$$d_m := |b_m| + |1-a_{m-1}c_{m-1}|, \quad z \in \overline{\mathbb{D}}.$$

The main result concerning equation (8) is the following.

Theorem 1. (i) *Suppose that*

$$\sum_{m=1}^{\infty} d_m < \infty. \tag{12}$$

Then equation (8) *has a unique solution* v_n *such that* v_n *is analytic in* $\mathbb{D}$*, continuous on* $\mathbb{D}_1 := \overline{\mathbb{D}} \setminus \{\pm 1\}$ *and* [1]

(13)

$$|v_n - z^n| \le |z|^n \left\{\frac{2|z|}{|z^2-1|} \sum_{m=n+1}^{\infty} d_m\right\} \exp\left\{\frac{2|z|}{|z^2-1|} \sum_{m=n+1}^{\infty} d_m\right\}, \quad z\in\mathbb{D}_1,\ n\in\mathbb{Z}_+.$$

(ii) *Suppose that*

$$\sum_{m=1}^{\infty} m d_m < \infty. \tag{14}$$

Then v_n *is analytic in* $\mathbb{D}$*, continuous on* $\overline{\mathbb{D}}$ *and*

$$|v_n - z^n| \le |z|^n \left\{\sum_{m=n+1}^{\infty} m d_m\right\} \exp\left\{\sum_{m=n+1}^{\infty} m d_m\right\}, \quad z \in \mathbb{D}, \quad n \in \mathbb{Z}_+. \tag{15}$$

Proof. The method of successive approximations does the job. Write (10) as

$$f_n(z) = g_n(z) + \sum_{m=n+1}^{\infty} \tilde{J}(n,m;z) f_m(z) \tag{16}$$

with

$$f_m(z) := \tilde{v}_m(z) - 1, \quad g_n(z) := \sum_{m=n+1}^{\infty} \tilde{J}(n,m;z). \tag{17}$$

(i) Put $\sigma_0(n) := \sum_{m=n+1}^{\infty} d_m$, $\phi(z) := 2|z||z^2-1|^{-1}$ and apply (11) in the form

$$|\tilde{J}(n,m;z)| \le \phi(z)\, d_m, \quad z \in \mathbb{D}_1. \tag{18}$$

Then the series in (17) converges uniformly on compact subsets of $\mathbb{D}_1$ and so g_n is analytic in $\mathbb{D}$ and continuous on $\mathbb{D}_1$. Let us begin with $f_{n,1} = g_n$ and denote

$$f_{n,j+1}(z) := \sum_{m=n+1}^{\infty} \tilde{J}(n,m;z) f_{m,j}(z).$$

[1] Following the terminology of the selfadjoint case, we call this solution the *Jost solution*. The function v_0 is known as the *Jost function*.

We prove by induction starting with $j=1$ that

$$|f_{n,j}(z)| \le \frac{(\phi(z)\sigma_0(n))^j}{(j-1)!}. \tag{19}$$

It is obvious for $j=1$ by (18). Next, let (19) be true. Then

$$|f_{n,j+1}(z)| \le \phi(z) \sum_{m=n+1}^{\infty} d_m |f_{m,j}(z)| \le \frac{(\phi(z))^{j+1}}{(j-1)!} \sum_{m=n+1}^{\infty} d_m \sigma_0^j(m).$$

An elementary inequality $(a+b)^{j+1} - a^{j+1} \ge (j+1)ba^j$ gives

$$\sum_{m=n+1}^{\infty} d_m \sigma_0^j(m) \le \frac{1}{j} \sum_{m=n+1}^{\infty} \{\sigma_0^{j+1}(m-1) - \sigma_0^{j+1}(m)\} = \frac{\sigma_0^{j+1}(n)}{j},$$

which proves (19) for $f_{n,j+1}$. Thereby the series

$$f_n(z) = \sum_{j=1}^{\infty} f_{n,j}(z)$$

converges uniformly on compact subsets of $\mathbb{D}_1$ and solves (16), being analytic in $\mathbb{D}$ and continuous on $\mathbb{D}_1$. The estimate (13) follows from (19) and $\tilde{v}_n = v_n z^{-n}$.

Suppose that there are two solutions f_n and $\tilde{f}_n$ of (16). Take the difference and apply (18):

$$h_n \le \sum_{m=n+1}^{\infty} \phi(z) h_m d_m = q_n, \qquad h_n := |f_n(z) - \tilde{f}_n(z)|. \tag{20}$$

Clearly, $q_n \to 0$ as $n \to \infty$ and $q_k = 0$ for some k implies by (20) $h_n \equiv 0$. If $q_n > 0$, then

$$\frac{q_{n-1} - q_n}{q_n} = \frac{h_n \phi(z) d_n}{q_n} \le \phi(z) d_n, \quad q_k \le \prod_{j=k+1}^{M} (1 + \phi(z) d_j)\, q_M, \tag{21}$$

which leads to $q_k = 0$ and again $h_n \equiv 0$. So the uniqueness is proved.

(ii) The same sort of reasoning is applicable with

$$|\tilde{J}(n,m;z)| \le |z||m-n| d_m \le m d_m$$

and

$$|f_{n,j}(z)| \le \frac{\sigma_1^j(n)}{(j-1)!}, \quad \sigma_1(n) := \sum_{m=n+1}^{\infty} m d_m$$

instead of (18) and (19), respectively. □

Corollary 1. *Let t be the root of the equation*

$$te^t = 1, \quad t \approx 0.567. \tag{22}$$

Under assumption (12) *the Jost function v_0 does not vanish in the domain*

$$\Omega := \{z \in \mathbb{D}; \quad |z - z^{-1}| > 2t^{-1} \sum_{m=1}^{\infty} d_m\}. \tag{23}$$

Under assumption (14) *v_0 does not vanish in $\mathbb{D}$ as long as*

$$\sum_{m=1}^{\infty} m d_m < t. \tag{24}$$

Eigenvalues of the Jacobi matrix and zeros of the Jost function

Going back to the matrix J, let $\lambda \in \sigma_d(J)$ with an eigenvector $h = \{h_n\}_{n\geq 1}$. Clearly, $h_1 \neq 0$. As a result there is only one linearly independent eigenvector corresponding to any eigenvalue λ (an appropriate linear combination of two would give zero). Denote by

$$Z = Z(J) = \{z \in \mathbb{D} : \, v_0(z) = 0\}$$

the zero set of the Jost function v_0 in $\mathbb{D}$. As $\{v_n\}_{n\geq 0} \in \ell^2$, for each $z_0 \in Z$ the vector $\{v_n(z_0)\}_{n\geq 1}$ is the eigenvector of J with the eigenvalue $\lambda_0 = z_0 + z_0^{-1}$. Conversely, let $\{g_n\}$ and $\{h_n\}$ be two solutions of (2). Define their "Wronskian" by

$$W_n(g,h) = g_n h_{n+1} - g_{n+1} h_n, \quad n \geq 0.$$

From (2) it follows that $a_{n-1}W_{n-1} = c_n W_n$, and iterating gives

$$W_n = \frac{a_{n-1} \ldots a_0}{c_n c_{n-1} \ldots c_1} W_0. \tag{25}$$

Suppose that both solutions are from ℓ^2. Then W_n goes to zero as $n \to \infty$ and since both products in the RHS of (25) converge, $W_0 = 0$. The latter means that g and h are linearly dependent, and so $u_0(z_0) = 0$. We end up with

Proposition 3.

$$\sigma_d(J) = \{z + z^{-1} : \quad z \in Z(J)\}.$$

Our main result concerning the discrete spectrum of the Jacobi matrices (1) can be displayed as follows.

Theorem 2. *Under assumption* (12) *the domain*

$$G(J) = \{z + z^{-1} : \quad z \in \Omega\}$$

with Ω (23) *is free from the discrete spectrum* $\sigma_d(J)$. *The matrix* J *has no discrete spectrum at all as soon as* (24) *holds, where* t *is the solution of* (22).

Remark. Suppose that

$$c = \frac{2}{t} \sum_{m=1}^{\infty} d_m < 2.$$

Then $\sigma_d(J)$ is contained in the union of two symmetric rectangles

$$\sigma_d(J) \subset \left\{ w : \, \sqrt{4 - c^2} < |\mathrm{Re}\, w| < \sqrt{4 + c^2}, \quad |\mathrm{Im}\, w| < \frac{c^2}{4} \right\}.$$

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