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EXTENSION OF SIMONS' INEQUALITY

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Abstract. We prove the following extended version of Simons' inequality and present its applications. Let S be a set and T be a subset of S. Let C be a subset of a Hausdorff topological vector space which is invariant under infinite convex combinations. Let $f : C \times S \longrightarrow \mathbb{R}$ be a bounded function such that the functions $f(\,\cdot\,, t) : C \longrightarrow \mathbb{R}$ are convex for all $t \in T$ and $f(\lambda x, s) = \lambda f(x, s)$ whenever $\lambda > 0$, x, $\lambda x \in C$ and $s \in S$. Let (x_n) be a sequence in C. Assume that, for every $x \in C_1 = \left\{\sum_{n=1}^{\infty} \lambda_n x_n : \quad \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n = 1\right\}$, there exists $t \in T$ satisfying $f(x, t) = \sup_{s \in S} f(x, s)$. Then

$$\inf_{x \in C_1} \sup_{s \in S} f(x, s) \leq \sup_{t \in T} \limsup_{n} f(x_n, t).$$

If $-C_1 \subset C$, then the set C_1 in the above inequality can be replaced by $\operatorname{conv}\{x_1, x_2, \ldots\}$.

1. Introduction and the main result

The remarkable inequality of Simons [S1, Lemma 2] (see also e.g. [HHZ, p. 49]) has important applications to real analysis and the geometry of Banach spaces (see e.g. [AG], [FG], [FHHMPZ], [G1], [G2], [GZ], [HHZ], [O2], [O3], [S1], [S2]).

Theorem 1 (Simons' inequality)**.** *Let S be a set and let T be a subset of S. Let (x_n) be a bounded sequence in $l_\infty(S)$, the Banach space of bounded real functions on S. Assume that, for every*

$$x \in C_1 = \left\{ \sum_{n=1}^{\infty} \lambda_n x_n : \quad \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n = 1 \right\},$$

there exists $t \in T$ satisfying

$$x(t) = \sup_{s \in S} x(s). \tag{1}$$

Then the following inequality holds:

$$\inf_{x \in C_0} \sup_{s \in S} x(s) \leq \sup_{t \in T} \limsup_{n} x_n(t), \tag{2}$$

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where C_0 denotes the convex hull of the sequence (x_n), that is,

$$C_0 = \operatorname{conv}\{x_1, x_2, \dots\} = \left\{ \sum_{n=1}^{m} \lambda_n x_n : \quad m \in \mathbb{N}, \lambda_n \geq 0, \sum_{n=1}^{m} \lambda_n = 1 \right\}.$$

Recently, Deville and Finet [DF] extended Theorem 1 by taking instead of $l_\infty(S)$ a topological vector space and by introducing a function $f : C_1 \times S \longrightarrow \mathbb{R}$ in conditions (1) and (2).

Theorem 2 (see [DF]). *Let S be a set and let C be a subset of a Hausdorff topological vector space which is invariant under infinite convex combinations. Let $f : C \times S \longrightarrow \mathbb{R}$ be a bounded function such that the functions $f(\,\cdot\,, s) : C \longrightarrow \mathbb{R}$, $s \in S$, are convex and there exists a neighbourhood $\mathcal{U}$ of 0 such that*

$$| f(x, s) - f(y, s) | \leq p_{\mathcal{U}}(x - y), \quad x, y \in C, s \in S,$$

where $p_{\mathcal{U}}$ is the Minkowski functional of $\mathcal{U}$. Assume that, for every $x \in C$, there exists $t \in S$ satisfying

$$f(x, t) = \sup_{s \in S} f(x, s).$$

Then, for every sequence (x_n) in C, the following inequality holds:

$$\inf_{x \in C} \sup_{s \in S} f(x, s) \leq \sup_{s \in S} \limsup_{n} f(x_n, s).$$

In the present article, inspired by Theorem 2, we shall prove the following extension of Theorem 1, where the continuity assumption on $f(\,\cdot\,, s)$, $s \in S$, in Theorem 2 is replaced by a purely algebraic one. Moreover, like in Theorem 1, the subset T is not necessarily equal to S (cf. Theorem 2), a fact that has been crucial in applications of Simons' inequality (see e.g. [AG], [FG], [G1], [G2], [GZ], [O3], [S1]).

Theorem 3. *Let S be a set and let T be a subset of S. Let C be a subset of a Hausdorff topological vector space which is invariant under infinite convex combinations. Let $f : C \times S \longrightarrow \mathbb{R}$ be a bounded function such that*

$$\text{the functions } f(\,\cdot\,, t) : C \longrightarrow \mathbb{R}, t \in T, \text{ are convex} \tag{3}$$

and

$$f(\lambda x, s) = \lambda f(x, s) \text{ whenever } \lambda > 0,\ x, \lambda x \in C \text{ and } s \in S. \tag{4}$$

Let (x_n) be a sequence in C. Assume that, for every

$$x \in C_1 = \left\{ \sum_{n=1}^{\infty} \lambda_n x_n : \quad \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n = 1 \right\},$$

there exists $t \in T$ satisfying

$$f(x, t) = \sup_{s \in S} f(x, s).$$

Then the following inequality holds:

$$\inf_{x \in C_1} \sup_{s \in S} f(x, s) \leq \sup_{t \in T} \limsup_{n} f(x_n, t). \tag{5}$$

If, moreover, $-C_1 \subset C$, then the set C_1 in (5) can be replaced by $\operatorname{conv}\{x_1, x_2, \dots\}$.

Remark 1. Let (x_n) be a bounded sequence in a Banach space and let C be the closed absolutely convex hull of (x_n). It is straightforward to verify that C is invariant under infinite convex combinations. Therefore Theorem 1 immediately follows from Theorem 3 if one sets

$$f(x,\, s) = x(s), \quad x \in C,\, s \in S.$$

The beautiful technical proof of Theorem 2 due to Deville and Finet [DF] uses ideas inherent in the original proof of Theorem 1 (see [S1] or [HHZ, pp. 49–51] or [G2]). In [O1] (see [FHHMPZ, pp. 80–81]), the second-named author found a simple direct proof of Theorem 1 that, unfortunately, does not work for Theorem 2. In contrast, Theorem 3 can and will be proved (in Section 2) relying on ideas of the proof of Theorem 1 in [O1].

The main application on a pointwise convergent sequence of vector-valued functions deduced from Theorem 2 in [DF] (see Corollary 2 in Section 3 below) is also immediate from our Theorem 3, thus providing an easier proof for this result.

2. PROOF OF THEOREM 3

The proof of Theorem 3 will be given as a consequence of some elementary lemmas. Let us recall that S is a set, T is a subset of S, and C is a subset of a Hausdorff topological vector space. The set C is assumed to be invariant under infinite convex combinations, meaning that

$$\sum_{n=1}^{\infty} \lambda_n\, x_n \ \in C$$

for every sequence (x_n) in C and for every sequence of non-negative numbers (λ_n) such that $\sum_{n=1}^{\infty} \lambda_n = 1$. In particular, C is a convex set. We also have a bounded function $f : C \times S \longrightarrow \mathbb{R}$ satisfying conditions (3) and (4).

We now fix some more notation. Let

$$\boldsymbol{\mathcal{C}} = \bigcup_{\alpha \geq 1} \alpha C.$$

We extend f from $C \times S$ to $\boldsymbol{\mathcal{C}} \times S$ defining

$$f(\alpha x,\, s) = \alpha f(x,\, s), \quad \alpha \geq 1,\, x \in C,\, s \in S.$$

Clearly, the function

$$f : \boldsymbol{\mathcal{C}} \times S \longrightarrow \mathbb{R}$$

is well-defined (by (4)), bounded on every subset $\alpha C \times S$, where $\alpha \geq 1$, and

$$f(\lambda z,\, s) = \lambda f(z,\, s), \quad \lambda \geq 1,\, z \in \boldsymbol{\mathcal{C}},\, s \in S. \tag{6}$$

Lemma 1. *If* $z_1,\, z_2 \in \boldsymbol{\mathcal{C}}$*, then* $z_1 + z_2 \in \boldsymbol{\mathcal{C}}$ *and*

$$f(z_1 + z_2,\, t) \leq f(z_1,\, t) + f(z_2,\, t), \quad t \in T,$$

meaning that the functions $f(\,\cdot\,,\, t) : \boldsymbol{\mathcal{C}} \longrightarrow \mathbb{R}$*,* $t \in T$*, are subadditive.*

Proof. If $z_1,\, z_2 \in \boldsymbol{\mathcal{C}}$, then $z_1 = \alpha x$ and $z_2 = \beta y$ with $\alpha,\, \beta \geq 1$ and $x,\, y \in C$. Hence

$$\gamma := \alpha + \beta \geq 1,\ \lambda := \frac{\alpha}{\gamma} \in (0,\, 1),\ \text{and}\ \beta = \gamma(1 - \lambda).$$

Thus $\lambda x + (1 - \lambda) y \in C$, and so

$$z_1 + z_2 = \gamma(\lambda x + (1 - \lambda) y) \in \boldsymbol{\mathcal{C}}.$$

Since the functions $f(\,\cdot\,,\,t)$, $t\in T$, are convex on C,

$$\begin{aligned} f(z_1+z_2,\,t) &= f(\gamma(\lambda x+(1-\lambda)y),\,t)\\ &= \gamma f(\lambda x+(1-\lambda)y,\,t)\\ &\le \gamma\lambda f(x,\,t)+\gamma(1-\lambda)f(y,\,t)\\ &= \alpha f(x,\,t)+\beta f(y,\,t)\\ &= f(z_1,\,t)+f(z_2,\,t). \end{aligned}$$

□

Let $(x_n)=(x_n)_{n=1}^\infty$ be a sequence in C. Below, we use the notation

$$C_k=\left\{\sum_{n=k}^\infty \lambda_n\,x_n:\quad \lambda_n\ge 0,\,\sum_{n=k}^\infty\lambda_n=1\right\},\qquad k\in\mathbb{N}.$$

Clearly $C\supset C_1\supset C_2\supset\ldots$.

Lemma 2. *If $y\in C_k$, $k\in\mathbb{N}$, and $t\in T$, then*

$$f(y,\,t)\le\sup_{n\ge k}f(x_n,\,t).$$

Proof. Let

$$y=\sum_{n=k}^\infty\lambda_n\,x_n,\,\lambda_n\ge 0,\,\sum_{n=k}^\infty\lambda_n=1.$$

If there exists an $l\in\mathbb{N}$ such that $\sum_{n=k}^{l}\lambda_n=1$, then the claim is immediate from the convexity of $f(\,\cdot\,,t):C\longrightarrow\mathbb{R}$ (assumption (3)).

Let us assume that $\sum_{n=k}^{l}\lambda_n<1$ for all $l\in\mathbb{N}$. For $\epsilon\in(0,\,1/2)$, choose $l\in\mathbb{N}$ so that

$$\lambda:=\sum_{n=k}^{l}\lambda_n>1-\epsilon.$$

Since $y\in C\subset\boldsymbol{\mathcal{C}}$ and $1/(1-\lambda)>1$, we have, by (6),

$$\frac{1}{1-\lambda}f(y,\,t)=f\left(\frac{y}{1-\lambda},\,t\right)=f(z_1+z_2,\,t),$$

where

$$z_1=\frac{\lambda}{1-\lambda}\sum_{n=k}^{l}\frac{\lambda_n}{\lambda}\,x_n\in\frac{\lambda}{1-\lambda}C\subset\boldsymbol{\mathcal{C}}$$

(notice that $\lambda/(1-\lambda)>(1-\epsilon)/\epsilon>1$) and

$$z_2=\sum_{n=l+1}^\infty\frac{\lambda_n}{1-\lambda}\,x_n\in C\subset\boldsymbol{\mathcal{C}}.$$

Using Lemma 1, (6), and (3), and denoting

$$M=\sup_{x\in C}\mid f(x,\,t)\mid,$$

we get that

$$f(z_1+z_2,\,t)\le\frac{\lambda}{1-\lambda}\sup_{n\ge k}f(x_n,\,t)+M.$$

Hence

$$\begin{aligned} f(y,\,t) &\leq \lambda \sup_{n\geq k} f(x_n,\,t) + (1-\lambda)\,M \\ &\leq \max\left\{\sup_{n\geq k} f(x_n,\,t);\ (1-\epsilon)\sup_{n\geq k} f(x_n,\,t)\right\} + \epsilon\,M, \end{aligned}$$

and the claim follows since $\epsilon \in \big(0,\,1/2\big)$ is arbitrary. □

Lemma 3. *Let $k \in \mathbb{N}$. If $\mu_m \geq 0$, $\sum_{m=k}^{\infty} \mu_m = 1$, and $z_m \in C_m$, then*

$$y := \sum_{m=k}^{\infty} \mu_m\, z_m \in C_k.$$

Proof. Notice that $y \in C$ since $z_m \in C_m \subset C$. Let

$$z_m = \sum_{n=m}^{\infty} \lambda_{nm}\, x_n, \quad \lambda_{nm} \geq 0,\ \sum_{n=m}^{\infty} \lambda_{nm} = 1.$$

Denote

$$a_{nm} = \mu_m \lambda_{nm}, \quad m \geq k,\, n \geq m;$$

then

$$y = \sum_{m=k}^{\infty} \sum_{n=m}^{\infty} a_{nm}\, x_n.$$

Let $(b_i)_{i=1}^{\infty}$ be any rearrangement of the elements of the matrix (a_{nm}) as a sequence. Since

$$\sum_{m=k}^{\infty} \sum_{n=m}^{\infty} a_{nm} = 1,$$

we also have

$$\sum_{i=1}^{\infty} b_i = 1.$$

In particular,

$$\sum_{n=k}^{\infty} \sum_{m=k}^{n} a_{nm} = 1. \tag{7}$$

Let us now denote

$$u_{nm} = a_{nm} x_n, \quad m \geq k,\, n \geq m,$$

and let $(v_i)_{i=1}^{\infty}$ be any rearrangement of the elements of the matrix (u_{nm}) as a sequence. Since $v_i = b_i\, x_{n(i)}$ with $\sum_{i=1}^{\infty} b_i = 1$, $b_i \geq 0$, and $x_{n(i)} \in C$, the series $\sum_{i=1}^{\infty} v_i$ converges in C. This implies that the series $\sum_{i=1}^{\infty} v_{\pi(i)}$ converges for every permutation π of the natural numbers. It is well known and can be easily verified that then the sum of $\sum_{i=1}^{\infty} v_{\pi(i)}$ does not depend on the permutation π. (In fact, if two rearrangements of the series gave different sums x and y, then one could combine these two rearrangements to obtain a rearrangement of the series such that a subsequence of its partial sums would converge to x and another to y.)

Denote this uniquely determined sum by x. In particular, by (7), we have that

$$x = \sum_{n=k}^{\infty} \sum_{m=k}^{n} a_{nm} x_n \in C_k.$$

Therefore it suffices to prove that $y = x$.

Recall that

$$y = \sum_{m=k}^{\infty} \sum_{n=m}^{\infty} u_{nm}.$$

Let $\mathcal{U}$ be any closed neighbourhood of 0 and let $\mathcal{V}$ be a neighbourhood of 0 satisfying $\mathcal{V} - \mathcal{V} \subset \mathcal{U}$. Choose $M_1 \geq k$ so that

$$\sum_{m=k}^{M} \sum_{n=m}^{\infty} u_{nm} - y \in \mathcal{V}$$

whenever $M \geq M_1$. Next choose $M_2 > M_1$ so that

$$\sum_{m=k}^{M_1} \sum_{n=M_2+1}^{\infty} u_{nm} \in \mathcal{V}.$$

We continue in an obvious manner to obtain a sequence of indices $k \leq M_1 < M_2 < M_3 < \dots$ so that

$$\sum_{m=k}^{M_j} \sum_{n=M_{j+1}+1}^{\infty} u_{nm} \in \mathcal{V}, \quad j \in \mathbb{N},$$

and therefore

$$\begin{aligned} \sum_{m=k}^{M_j} \sum_{n=m}^{M_{j+1}} u_{nm} - y &= \sum_{m=k}^{M_j} \sum_{n=m}^{\infty} u_{nm} - y - \sum_{m=k}^{M_j} \sum_{n=M_{j+1}+1}^{\infty} u_{nm} \\ &\in \mathcal{V} - \mathcal{V} \subset \mathcal{U}, \quad j \in \mathbb{N}. \end{aligned} \tag{8}$$

We can choose inductively a rearrangement (v_i) of the matrix (u_{nm}) so that

$$\sum_{m=k}^{M_j} \sum_{n=m}^{M_{j+1}} u_{nm} = \sum_{i=1}^{N_j} v_i, \quad j \in \mathbb{N}, \tag{9}$$

for some sequence $N_1 < N_2 < \dots$ of integers. Indeed, we start by picking the elements u_{nm} with $m \in \{k, \dots, M_1\}$ and $n \in \{m, \dots, M_2\}$. Next we add the elements u_{nm} with $m \in \{M_1 + 1, \dots, M_2\}$ and $n \in \{m, \dots, M_3\}$, and also those with $(m, n) \in \{k, \dots, M_1\} \times \{M_2 + 1, \dots, M_3\}$. We continue in an obvious way.

Since

$$\lim_{j\to\infty} \sum_{i=1}^{N_j} v_i = x,$$

we get from (8) and (9) that $x - y \in \mathcal{U}$. This means that $x - y$ is contained in all neighbourhoods of 0. Hence $x = y$ as desired. □

Remark 2. Lemma 3 holds in the stronger form with $z_m \in C_k$ (instead of $z_m \in C_m$). (In fact, an obvious modification of the above proof (look at $n \geq k$ instead of $n \geq m$) shows that $y = x$, where x is the uniquely determined sum of the rearrangements (as in the above proof). Let $z = \sum_{n=k}^{\infty} \sum_{m=k}^{\infty} a_{nm} x_n$. Then $z = x$, by the same proof. Hence $y = z$. Since clearly $z \in C_k$, we have $y \in C_k$ as desired.) This form of Lemma 3 extends a result by Galán and Simons [GS, Lemma 2] from real sequentially complete Hausdorff locally convex spaces to general Hausdorff topological vector spaces.

Proof of Theorem 3. Denote

$$\sigma(z) = \sup_{s \in S} f(z, s), \quad z \in \boldsymbol{\mathcal{C}},$$

and notice that $\sigma(z) \in \mathbb{R}$ because if $z = \alpha x$ with $\alpha \geq 1$ and $x \in C$, then $\sigma(z) = \alpha\sigma(x)$ and the function f is bounded on $C \times S$. We also notice that, by (6),

$$\sigma(\lambda z) = \lambda\, \sigma(z), \quad \lambda \geq 1,\, z \in \boldsymbol{\mathcal{C}}.$$

To establish inequality (5), that is,

$$\inf_{x \in C_1} \sigma(x) \leq \sup_{t \in T} \limsup_n f(x_n, t) =: \sigma_T,$$

it clearly suffices to prove that, for any $\epsilon > 0$, there exist $v \in C_1$, $y_m \in C_{m+1}$ with $m \in \mathbb{N}$, and $t \in T$ satisfying

$$\sigma(v) - \epsilon \leq f(y_m, t), \quad m \in \mathbb{N}. \tag{10}$$

Indeed, by Lemma 2, we would then have

$$\begin{aligned} \inf_{x \in C_1} \sigma(x) - \epsilon &\leq \sigma(v) - \epsilon \\ &= \lim_m \sup_{n \geq m+1} f(x_n, t) \\ &= \limsup_m f(x_m, t) \leq \sigma_T, \end{aligned}$$

yielding the desired inequality because $\epsilon > 0$ is arbitrary.

Let $\epsilon > 0$. Developing the idea of the proof of Simons' inequality in [O1], we shall define $v \in C_1$ and $y_m \in C_{m+1}$, $m \in \mathbb{N}$, by using some inductively constructed sequences $(v_k)_{k=0}^{\infty}$ and $(z_k)_{k=1}^{\infty}$. Their construction will be based on the simple observation that

$$\inf_{z \in 2^k C} \sigma(z) \in \mathbb{R}, \quad k = 0, 1, \ldots$$

(since f is bounded on the sets $2^k\, C \times S$).

We start by taking $v_0 = 0$. Using the fact that

$$\inf_{z \in C_1} \sigma(z) \in \mathbb{R}$$

(because $C_1 \subset C$), we choose $z_1 \in C_1$ so that

$$\sigma(z_1) \leq \inf_{z \in C_1} \sigma(z) + \frac{\epsilon}{2}.$$

Once $z_k \in C_k$, $k \in \mathbb{N}$, has been chosen, we put

$$v_k = \sum_{n=1}^{k} \frac{z_n}{2^n}.$$

Then, for all $z \in C$, in particular, for all $z \in C_{k+1}$, we have

$$2^k v_k + z = 2^k \left(v_k + \frac{z_k}{2^k} \right) \in 2^k \left(\sum_{n=1}^{k} \left(\frac{1}{2^n}\, C \right) + \frac{1}{2^k}\, C \right) \subset 2^k\, C$$

(because C is convex) and therefore

$$\inf_{z \in C_{k+1}} \sigma(2^k v_k + z) \in \mathbb{R}.$$

This enables us to choose $z_{k+1} \in C_{k+1}$ so that

$$\sigma(2^k v_k + z_{k+1}) \le \inf_{z \in C_{k+1}} \sigma(2^k v_k + z) + \frac{\epsilon}{2^{k+1}}. \tag{11}$$

Write

$$y_k = \sum_{n=k+1}^{\infty} 2^k \frac{z_n}{2^n}, \quad k = 0, 1, \dots ,$$

and

$$v = y_0 = \sum_{n=1}^{\infty} \frac{z_n}{2^n}.$$

Since $z_n \in C_n$, $n \in \mathbb{N}$, we have, by Lemma 3, that $v \in C_1$ and $y_k \in C_{k+1}$, $k \in \mathbb{N}$, as needed, and by assumption, we have a $t \in T$ satisfying $f(v, t) = \sigma(v)$.

To verify condition (10), we first observe that

$$2^k v_k + z_{k+1} = 2^k v_k + 2^{k+1} (v_{k+1} - v_k) = 2^{k+1} v_{k+1} - 2^k v_k, \quad k = 0, 1, \dots ,$$

and

$$2^k v_k + y_k = 2^k v, \quad k = 0, 1, \dots .$$

Therefore, by (11),

$$\sigma(2^{k+1} v_{k+1} - 2^k v_k) \le \sigma(2^k v) + \frac{\epsilon}{2^{k+1}} = 2^k \sigma(v) + \frac{\epsilon}{2^{k+1}}, \quad k = 0, 1, \dots .$$

Now, using the subadditivity of $f(\,\cdot\,, t) : \boldsymbol{\mathcal{C}} \longrightarrow \mathbb{R}$ (see Lemma 1), we can infer that

$$\begin{aligned}
f(2^m v_m, t) &= f\Big(\sum_{k=0}^{m-1} \big(2^{k+1} v_{k+1} - 2^k v_k\big), t \Big) \\
&\le \sum_{k=0}^{m-1} \sigma\big(2^{k+1} v_{k+1} - 2^k v_k\big) < \sum_{k=0}^{m-1} 2^k \sigma(v) + \epsilon \\
&= (2^m - 1)\, \sigma(v) + \epsilon = f(2^m v, t) - \sigma(v) + \epsilon \\
&= f(2^m v_m + y_m, t) - \sigma(v) + \epsilon \\
&\le f(2^m v_m, t) + f(y_m, t) - \sigma(v) + \epsilon, \quad m \in \mathbb{N}.
\end{aligned}$$

This means that (10) holds and thus completes the proof of the main assertion of Theorem 3.

For the *moreover part*, let us assume that $-C_1 \subset C$ and show that

$$\sigma_0 := \inf_{x \in C_0} \sigma(x) = \inf_{x \in C_1} \sigma(x) =: \sigma_1,$$

where $C_0 = \operatorname{conv}\{x_1, x_2 \dots\}$, or equivalently, since $C_0 \subset C_1$, that

$$\sigma_0 \le \sigma_1.$$

We shall use the observation that

$$\sigma(x) = \sup_{t \in T} f(x, t)$$

for every $x \in C_1$, which is clear from the assumption.

Let

$$y = \sum_{n=1}^{\infty} \lambda_n x_n, \quad \lambda_n \ge 0,\ \sum_{n=1}^{\infty} \lambda_n = 1,$$

be any element of $C_1 \setminus C_0$ and let $\epsilon \in (0, 1/2)$. Fix $l \in \mathbb{N}$ so that

$$\lambda := \sum_{n=1}^{l} \lambda_n > 1 - \epsilon.$$

As in the proof of Lemma 2, we can write

$$\frac{y}{1-\lambda} = \frac{\lambda}{1-\lambda} x_0 + z$$

for some $x_0 \in C_0$ and $z \in C_1$. Since

$$\frac{\lambda}{1-\lambda} x_0 = \frac{y}{1-\lambda} + (-z), \quad x_0,\, y,\, (-z) \in C \subset \boldsymbol{\mathcal{C}},\ \frac{\lambda}{1-\lambda},\, \frac{1}{1-\lambda} > 1,$$

by Lemma 1,

$$\frac{\lambda}{1-\lambda} f(x_0,\, t) \leq \frac{1}{1-\lambda} f(y,\, t) + M$$

for every $t \in T$, where $M = \sup\{|\ f(x,\, t)\ |:\ x \in C,\, t \in T\}$. Hence,

$$\lambda f(x_0,\, t) \leq f(y,\, t) + \epsilon M \leq \sigma(y) + \epsilon M$$

for every $t \in T$, implying

$$\lambda \sigma_0 \leq \lambda \sigma(x_0) \leq \sigma(y) + \epsilon M.$$

Therefore

$$\min\{(1-\epsilon)\sigma_0,\, \sigma_0\} \leq \sigma(y) + \epsilon M.$$

This implies, since $\epsilon \in (0, 1/2)$ is arbitrary, that $\sigma_0 \leq \sigma(y)$ whenever $y \in C_1 \setminus C_0$. Since $\sigma_0 \leq \sigma(y)$ also for all $y \in C_0$, we conclude that $\sigma_0 \leq \sigma_1$ as needed. □

3. APPLICATIONS

It is straightforward to verify that if C is a bounded closed convex subset of a Banach space, then C is invariant under infinite convex combinations and Theorem 3 applies.

We shall first apply Theorem 3 to study pointwise convergence of bounded sequences in Banach spaces.

Corollary 1. *Let (x_n) be a bounded sequence in a Banach space and let C denote its closed absolutely convex hull. Suppose that S, T, and f satisfy the conditions of Theorem 3. If $\sup_{s \in S} f(x,\, s) \geq 0$ for all $x \in C$ and $\lim_n f(x_n,\, t) = 0$ for all $t \in T$, then there exists a sequence (y_n) of convex combinations of (x_n) such that*

$$\lim_n \sup_{s \in S} f(y_n,\, s) = 0.$$

Proof. Since $-C_1 \subset C$ (the notation for C_1 is as in Theorem 3), we get from Theorem 3 that

$$0 \leq \inf_{x \in C_0} \sup_{s \in S} f(x,\, s) \leq \sup_{t \in T} \lim_n f(x_n,\, t) = 0,$$

where $C_0 = \operatorname{conv}\{x_1,\, x_2,\, \ldots\}$. Therefore the desired sequence $(y_n) \subset C_0$ exists. □

Corollary 2. *Let S be a compact topological space and X a Banach space. Let (φ_n) be a bounded sequence in the Banach space $C(S, X)$ of continuous functions $\varphi : S \longrightarrow X$. Let $T \subset S$ be such that, for every $\varphi \in C(S, X)$, there exists $t \in T$ satisfying*

$$\|\varphi\| = \|\varphi(t)\|_X.$$

If the sequence (φ_n) converges to zero pointwise on T, then there exists a sequence (ψ_n) of convex combinations of (φ_n) which is uniformly convergent to zero on S (that is, $\|\psi_n\| \to 0$).

Proof. The result immediately follows from Corollary 1 if one recalls that

$$\|\varphi\| = \sup_{s\in S} \|\varphi(s)\|_X, \quad \varphi \in C(S, X),$$

and sets

$$f(\varphi, s) = \|\varphi(s)\|_X, \quad \varphi \in C,\, s \in S.$$

□

Remark 3. The particular case of Corollary 2 with $T = S$ was deduced from Theorem 2 in [DF, Proposition]. The particular case of Corollary 2 with $X = \mathbb{R}$ is immediate from [S1, Corollary 9].

Let S be a convex subset of a linear space. Recall that a point t of S is an *extreme point* of S if $t = \frac{1}{2}(s_1 + s_2)$ and $s_1, s_2 \in S$ imply that $t = s_1 = s_2$. Recall also that a function φ from S to a linear space is *affine* if $\varphi(\lambda s_1 + (1-\lambda)s_2) = \lambda\varphi(s_1) + (1-\lambda)\varphi(s_2)$ for all $s_1, s_2 \in S$ and all $\lambda \in (0, 1)$.

Corollary 3. *Let S be a compact convex subset of a Hausdorff locally convex topological vector space and let T be the set of extreme points of S. Let X be a Banach space. If a bounded sequence of affine functions $(\varphi_n) \subset C(S, X)$ converges to zero pointwise on T, then there exists a sequence (ψ_n) of convex combinations of (φ_n) which is uniformly convergent to zero on S.*

Proof. Put $f(\varphi, s) = \|\varphi(s)\|_X$ for all affine functions $\varphi \in C(S, X)$ and all $s \in S$. The claim is immediate from Corollary 1 and Bauer's maximum principle: if $g : S \longrightarrow \mathbb{R}$ is a continuous convex function (we take $g(s) = \|\varphi(s)\|_X$ for affine functions $\varphi \in C(S, X)$), then there exists $t \in T$ such that $g(t) = \sup\{g(s) : s \in S\}$. □

We conclude with a corollary of Theorem 3 that can be efficiently applied in situations encountered for instance in key results of [L, Theorem 4.2] or [O3, Lemma 2.1]. Let $L(X, Y)$ denote the Banach space of bounded linear operators from a Banach space X to a Banach space Y. Recall that the functionals $x \otimes y^* \in (L(X, Y))^*$, where $x \in X$ and $y^* \in Y^*$, are defined by $(x \otimes y^*)(A) = y^*(Ax)$, $A \in L(X, Y)$.

Corollary 4. *Let X and Y be Banach spaces and let (A_n) be a bounded sequence in $L(X, Y)$. If*

$$\limsup_n \operatorname{Re} g(A_n) \le \lambda,$$

for some $\lambda \ge 0$ and for all functionals g contained in the weak closure of the subset $\{x \otimes y^* : x \in X, y^* \in Y^*, \|x\| = \|y^*\| = 1\}$ of $(L(X, Y))^*$, then there exists a sequence (B_n) of convex combinations of (A_n) such that*

$$\lim_n \|B_n\| \le \lambda.$$

Proof. For applying Theorem 3, let C denote the closed absolutely convex hull of (A_n) in $L(X, Y)$. Further, let S be the closed unit ball of $(L(X, Y))^*$ and let T be the weak* closure of $\{x \otimes y^* :\ x \in X,\ y^* \in Y^*,\ \|x\| = \|y^*\| = 1\}$. Note that $T \subset S$ and, by Alaoglu's theorem, T is weak* compact.

Define $f : C \times S \to \mathbb{R}$ by

$$f(A, g) = \operatorname{Re} g(A), \quad A \in C \subset L(X, Y),\ g \in S \subset (L(X, Y))^*.$$

Then conditions (3) and (4) are clearly satisfied. Moreover, for any $A \in C$ (in fact, for any $A \in L(X, Y)$), using the weak* compactness of T, we have

$$\begin{aligned}\sup_{g \in S} f(A, g) \quad &= \quad \|A\| = \sup\big\{\operatorname{Re}(x \otimes y^*)(A) :\ x \in X,\ y^* \in Y^*,\\ &\qquad \|x\| = \|y^*\| = 1\big\} = \max_{g \in T} f(A, g).\end{aligned}$$

Since $-C_1 \subset C$ (the notation for C_1 is as in Theorem 3), we get from Theorem 3 that

$$\inf_{A \in C_0} \|A\| \le \sup_{g \in T} \limsup_n \operatorname{Re} g(A_n) \le \lambda,$$

where $C_0 = \operatorname{conv}\{A_1, A_2, \ldots\}$. To obtain the desired sequence, choose $(B_n) \subset C_0$ so that

$$\lim_n \|B_n\| = \inf_{A \in C_0} \|A\|.$$

□

Remark 4. The particular case of Corollary 4 with $\lambda = 0$ follows from the generalization of Rainwater's theorem due to Simons [S1, Corollary 11].

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