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EVERY REIDEMEISTER MOVE IS NEEDED FOR EACH KNOT TYPE

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ABSTRACT. We show that every knot type admits a pair of diagrams that cannot be made identical without using Reidemeister Ω_2-moves. The proof is compatible with known results for the other move types, in the sense that every knot type admits a pair of diagrams that cannot be made identical without using all of the move types.

1. INTRODUCTION

Reidemeister proved [3] that given two diagrams of ambient isotopic links, there is a sequence of transformations on one of the diagrams that gives an explicit isotopy. Each transformation is either a planar isotopy, a cusp move (class Ω_1), a self-tangency move (class Ω_2), or a triple point move (class Ω_3). Every introductory knot theory textbook describes these move types; alternatively, see [2].

Two diagrams are *equivalent* if such a sequence exists, and the sequence is a *Reidemeister sequence* for the pair. Sometimes a Reidemeister sequence is specified without mention of the second diagram; in this case the second diagram is the result of applying the moves in the sequence. If, for a given diagram pair and $n \in \{1, 2, 3\}$, there is a Reidemeister sequence for the pair that does not contain an Ω_n-move, the pair is Ω_n*-independent*. Otherwise the pair is Ω_n*-dependent*.

Since Ω_1-moves are the only moves that change the winding number of a diagram, it is clear that every link type admits Ω_1-dependent diagram pairs. Olof-Petter Östlund [2] has shown that every link type admits Ω_3-dependent diagram pairs, as well as pairs that are simultaneously Ω_1-dependent and Ω_3-dependent. In the case of links with at least two components, Ω_2-moves are the only moves that change the number of intersections between components. Thus such links admit Ω_2-dependent diagram pairs. Vassily Manturov [1] has recently shown that a connected sum of any four distinct prime knots admits Ω_2-dependent diagram pairs. We consider Ω_2-moves in more generality, and construct Ω_2-dependent diagram pairs for every knot type. Our approach is similar to Manturov's, but was developed independently. We conclude by showing that for each knot type one can construct a diagram pair that is simultaneously Ω_1-dependent, Ω_2-dependent, and Ω_3-dependent.

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2. Main theorem

Briefly, the structure of the main argument is as follows. We give conditions on a knot diagram that severely limit what one can accomplish without Ω_2-moves. The only transformations possible amount to replacing the edges in the original diagram with unknotted $(1,1)$-tangles (by a $(1,1)$-tangle we mean a single stranded tangle). We show that there is a diagram resulting from a single Ω_2-move that cannot be attained without Ω_2-moves. The result is then generalized slightly to construct Ω_2-dependent diagram pairs for every knot type. Finally, taking pairwise connected sums with diagrams from Östlund's argument gives a pair of diagrams for each knot type that is simultaneously Ω_1-dependent, Ω_2-dependent, and Ω_3-dependent.

Let D be a planar knot diagram in general position. A *polygon* p in D is the boundary of a connected component of the complement D^c of D in the plane. Call p a 0*-gon* if D contains no crossing points. Call p an n*-gon* if, when all crossing points of D that lie on p are removed, the remainder consists of n connected components homeomorphic to an open interval. Call the points so removed the *vertices* of p, and the connected components the *edges*. Note that p can have fewer vertices than edges, and a 0-gon does not have any edges.

Theorem 1. *Let D be a diagram for which the following hold:*

(1) *There are no* 0*-gons,* 1*-gons, or* 2*-gons,*
(2) *The first* Ω_n*-move cannot be an* Ω_3*-move.*

Then any Reidemeister sequence not containing an Ω_2*-move does nothing more up to planar isotopy than replace the edges of* D *with unknotted* $(1,1)$*-tangles. Furthermore, one can apply a single* Ω_2*-move that crosses two distinct edges of* D*, giving a diagram* D' *such that the pair* $\{D, D'\}$ *is* Ω_2*-dependent.*

Figure 1 satisfies the above preconditions. Figure 2 gives an unknotted example. There are other examples, such as alternating diagrams with no 0-gons, 1-gons, or 2-gons.

Proof. Suppose that Ω_2-moves are disallowed. Let $\{k_i\}$ be the set of crossing points in D. In order for one of the k_i to take part in an Ω_1-move (in the sense that k_i appears in at least one of the pictures that locally describe the move), it must be the vertex of a 1-gon. In order for a k_i to take part in an Ω_3-move, it must be one

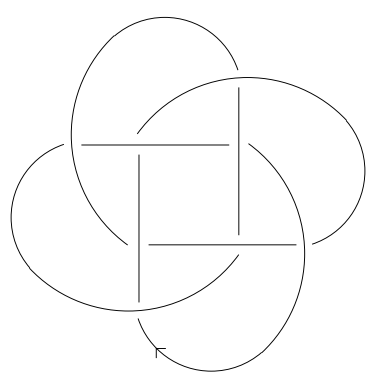

Figure 1. A simple diagram satisfying the preconditions of Theorem 1.

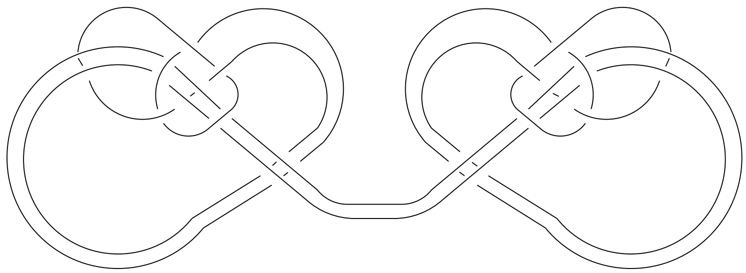

FIGURE 2. A diagram of the unknot satisfying the preconditions of Theorem 1.

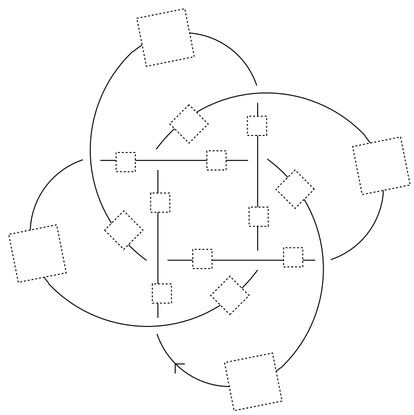

FIGURE 3. Figure 1, with edges replaced by unknotted $(1,1)$-tangles. Each dotted box represents an unknotted $(1,1)$-tangle.

of the vertices of a 3-gon. Since D contains no 1-gons, and none of its 3-gons admit an Ω_3-move, no k_i can take part in the first Ω_n-move.

Suppose that a sequence of moves has been made such that no k_i has taken part in an Ω_1-move or an Ω_3-move. Then the diagram is isotopic to the original diagram with each of the edges replaced by an unknotted $(1,1)$-tangle. (Figure 3 provides an illustration, where each dotted box contains a $(1,1)$-tangle.) The $(1,1)$-tangles cannot intersect because any Ω_1-move in which no k_i takes part is just a kink on a single $(1,1)$-tangle, and any Ω_3-move in which no k_i takes part cannot cause the intersection of two $(1,1)$-tangles that did not intersect before the move.

Now, it is still impossible for one of the k_i to take part in an Ω_1-move or an Ω_3-move. For, given a k_i, every polygon p that contains k_i as a vertex looks like one of the polygons in D that contain k_i, possibly with some extra edges due to the $(1,1)$-tangles. Thus, p has at least three edges, and p has exactly three edges only when it is one of the 3-gons in D (here it is necessary that there are no 2-gons in D). Since none of the 3-gons in D admit Ω_3-moves, no k_i can take part in an Ω_1-move or an Ω_3-move.

Thus, up to isotopy, if Ω_2-moves are not allowed, then a sequence of moves on D will fix the k_i and replace the edges of D with unknotted $(1,1)$-tangles. This gives the first part of the theorem.

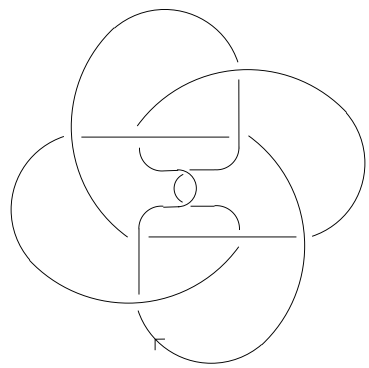

FIGURE 4. This diagram cannot be derived from Figure 1 without an Ω_2-move.

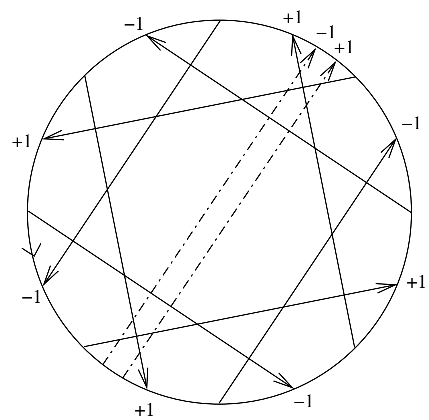

FIGURE 5. The Gauss diagram for Figure 4. Removing the dashed arrows gives the Gauss diagram for Figure 1.

For the second part of the theorem, let the diagram D' be created by performing a single Ω_2-move that crosses two distinct edges of D. Figure 4 provides an example. Let E be a diagram such that (D, E) is an Ω_2-independent diagram pair. There are easy ways to show that D' and E cannot be isotopic. For instance, one could show that if D' and E have the same number of crossings, E must contain a 1-gon, but D' contains no 1-gons. The following argument is more complicated but straightforwardly generalizes to the full result.

Every oriented knot diagram B is given by a smooth immersion ϕ from the oriented circle S^1 to the plane $\mathbb{R}^2$ with decorated crossing points. This immersion is unique up to orientation-preserving self-diffeomorphisms of S^1. The map ϕ is one to one except at crossing points, where it is two to one. The *Gauss diagram* G_B for B is constructed from S^1 by drawing a signed arrow between the two elements of $\phi^{-1}(k)$ for each crossing point k of B. Each arrow points toward the under strand of the crossing. The sign of each arrow is the sign of that crossing, either $+1$ or -1 according to the standard convention. Figures 5 and 7 give examples. For a slightly lengthier introduction to Gauss diagrams, see [2].

Let D be given an orientation. This gives an orientation for E. Consider the Gauss diagrams G_D and G_E of D and E. One can see that G_E is just G_D with some extra arrows representing the $(1,1)$-tangles. It is easy to verify that none of the extra arrows intersect the arrows of G_D.

The Ω_2-move that takes D to D' adds two arrows a_1 and a_2 to G_D, giving the Gauss diagram $G_{D'}$. The heads of a_1 and a_2 are adjacent on S^1, as are the tails. Thus, any arrow that intersects a_1 or a_2 must intersect both a_1 and a_2. One can easily show that both a_1 and a_2 must intersect at least one of the arrows in G_D.

Any Gauss diagram containing a pair of arrows with adjacent heads and tails represents a knot diagram that contains a 2-gon. Since D contains no 2-gons, any copy of G_D in $G_{D'}$ intersecting no other arrows cannot contain a_1 and a_2, and so contains neither a_1 nor a_2. Thus the addition of the arrows a_1 and a_2 causes G_D to intersect other arrows without creating any new copies of G_D that do not intersect other arrows. This reduces the number of copies of G_D intersecting no other arrows from one to zero. Since E has at least one copy of G_D intersecting no other arrows, D' and E are not isotopic. This proves the theorem. □

Theorem 2. *Every knot type admits an Ω_2-dependent diagram pair.*

Proof. Figure 2 is a diagram of the unknot and so Theorem 1 gives an Ω_2-dependent unknotted diagram pair. Removing a small closed line segment from one of the edges of a diagram A results in a $(1,1)$-tangle T having the same knot type as A. If B is another knot diagram, then $A\#B$ is obtained by replacing an edge in B with a $(1,1)$-tangle planar isotopic to T.

Let D be the diagram in Figure 2. Let F be a diagram given by replacing one or more of the edges of D with arbitrary $(1,1)$-tangles. Then by the same argument as in Theorem 1, move sequences not containing Ω_2-moves can only replace these $(1,1)$-tangles with other $(1,1)$-tangles of the same knot types. Now, G_F must contain at least one copy of G_D that intersects no other arrows. It may contain more (for instance if $F = D\#D$). However, none of these copies can be altered or intersected with other arrows by a Reidemeister sequence not containing Ω_2-moves. Let F' be the result of a single Ω_2-move that crosses two distinct edges of F. Then the created arrows on $G_{F'}$ intersect a copy of G_D that intersected nothing else in G_F. This reduces the number of copies of G_D that intersect no other arrows, just as in Theorem 1. Thus (F, F') is an Ω_2-dependent diagram pair. □

We wish to construct diagram pairs that are simultaneously Ω_1-dependent, Ω_2-dependent, and Ω_3-dependent. In order to do this we briefly summarize a portion of the proof given by Östlund in [2], enough to prove the existence of Ω_3-dependent diagram pairs. The reader should see [2] for details.

Theorem 3. *Every knot type admits an Ω_1-dependent, Ω_2-dependent, Ω_3-dependent diagram pair.*

Proof. Östlund's proof counts the signed number of instances of the Gauss subdiagram given in Figure 6. The sign of each subdiagram is given by the product of the signs of the crossings in the subdiagram. Östlund shows that this count is invariant under Ω_1-moves and Ω_2-moves, but can vary under Ω_3-moves. Östlund uses this count to prove that for every knot type there is a diagram pair that is simultaneously Ω_1-dependent and Ω_3-dependent. As an example, the Gauss diagram in Figure 7 represents a figure eight knot. It contains one copy of Figure 6 as a

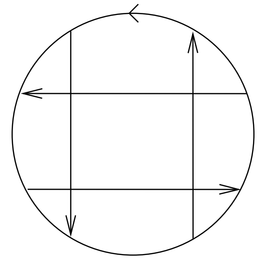

FIGURE 6. The signed sum of subdiagrams of this form is invariant under Ω_1-moves and Ω_2-moves, but not Ω_3-moves.

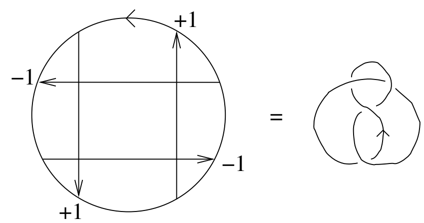

FIGURE 7. A Gauss diagram for a figure eight knot.

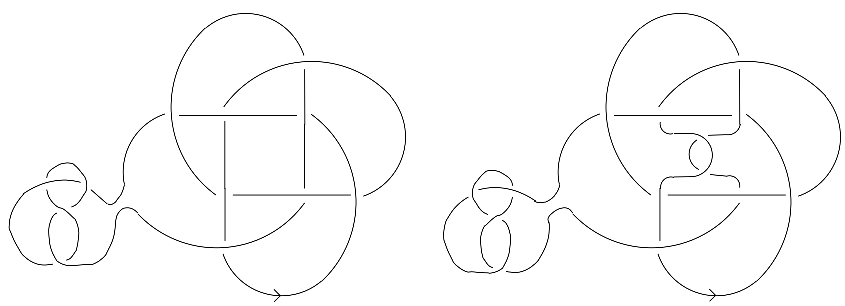

FIGURE 8. These diagrams have different winding numbers. The pair is Ω_1-dependent, Ω_2-dependent, and Ω_3-dependent.

subdiagram with sign $1^2(-1)^2 = 1$. Its mirror image has the same Gauss diagram with the arrows and signs (but not the orientation of S^1) reversed, and this does not contain any copies of Figure 6 as a subdiagram. Figure 7 and its mirror image have different winding numbers, so the pair is Ω_1-dependent and Ω_3-dependent.

Östlund's count is also additive under a connected sum of diagrams. Thus, if (D, D') is an Ω_2-dependent diagram pair constructed according to Theorem 2, where D and D' differ only by an Ω_2-move, and (E, E') is one of Östlund's Ω_1-dependent, Ω_3-dependent diagram pairs, then the diagrams $D\#E$ and $D'\#E'$ will form an Ω_1-dependent, Ω_2-dependent, Ω_3-dependent diagram pair. □

Figure 8 gives an example.

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