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# EIGENVALUE INEQUALITIES IN AN EMBEDDABLE FACTOR

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Abstract. We provide a characterization of the possible eigenvalues of the sum of two selfadjoint elements of a $\mathrm{II}_1$ factor which can be embedded in the ultrapower $\mathcal{R}^\omega$ of the hyperfinite $\mathrm{II}_1$ factor.

## 1. Introduction

Consider a $\mathrm{II}_1$ factor $\mathcal{M}$ with normal faithful trace $\tau$ such that $\tau(1) = 1$. Each selfadjoint element $a \in \mathcal{M}$ can be written as

$$a = \int_0^1 u(t)\, de(t)$$

where $u$ is a nonincreasing function on $[0, 1)$, and $e$ is a spectral measure on $[0, 1)$ such that $\tau(e([0, t])) = t$ for $t \in [0, 1)$. The function $u$ is uniquely determined if it is also assumed right-continuous; in this case we will use the notation $\lambda_a$ for $u$, and call it the eigenvalue function of $a$.

We want to address the following question: given nonincreasing right-continuous functions $u, v, w$ on $[0, 1)$, under what conditions do there exist elements $a, b \in \mathcal{M}$ such that $\lambda_a = u$, $\lambda_b = v$, and $\lambda_{a+b} = w$?

The finite-dimensional analogue of this question was completely solved; the solution will be described in some detail later (see [9] and [6] for a detailed discussion). In our context, some necessary conditions on $u, v, w$ were found in [7], [4], and [5]. These conditions are subsumed by an analogue of a result of Freede and Thompson [10] which we proved in [1].

In this paper we focus on the case in which $\mathcal{M}$ is an ultrapower $\mathcal{R}^\omega$ of the hyperfinite $\mathrm{II}_1$ factor $\mathcal{R}$. In this case we produce necessary and sufficient conditions on $u, v, w$ for the existence of $a, b \in \mathcal{M}$ with the required properties. Our conditions are derived from those found in the finite-dimensional situation. Indeed, our methods can be applied to more general questions of this type (such as, e.g., conditions on $\lambda_a$, $\lambda_b$, $\lambda_c$, and $\lambda_{a+b+c}$). We do not know whether our conditions on $(u, v, w)$ are satisfied by $(\lambda_a, \lambda_b, \lambda_{a+b})$ if $a, b$ are elements in a factor which does not embed in $\mathcal{R}^\omega$ (but note that the results in [7, 4, 1] are proved in complete generality). The

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answer would of course be affirmative if every $\mathrm{II}_1$ factor could be embedded in $\mathcal{R}^\omega$, a problem first formulated by Connes [3].

## 2. Finite-dimensional algebras

In this section we describe the solution to the eigenvalue problem for sums of finite selfadjoint matrices. The solution involves sums of eigenvalues indexed by some subsets of $\{1, 2, \ldots, n\}$. Fix a natural number $n$ and, for a subset $I \subset \{1, 2, \ldots, n\}$, write $\Sigma_I = \sum_{i\in I} i$. Following the notation in [6], we consider for every $r \le n$ the following collection of triples $(I, J, K)$ of subsets of cardinality $r$ in $\{1, 2, \ldots, n\}$:

$$U_r^n = \{(I, J, K) : \Sigma_I + \Sigma_J = \Sigma_K + \frac{r(r+1)}{2}\}.$$

It will also be convenient to view a subset $I$ of $r$ elements in $\{1, 2, \ldots, n\}$ as an increasing function $I : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, n\}$, i.e.,

$$I = \{I(1), I(2), \ldots, I(r)\}.$$

We then define inductively on $r$ a subset $T_r^n$ of $U_r^n$ as follows: $T_1^n = U_1^n$ and, for $1 < r \le n$,

$$\begin{aligned} T_r^n = \quad & \{(I, J, K) \in U_r^n : \Sigma_{I\circ I'} + \Sigma_{J\circ J'} \le \Sigma_{K\circ K'} + \tfrac{p(p+1)}{2} \\ & \text{for every } \ p < r \ \text{ and } \ (I', J', K') \in T_p^r\}. \end{aligned}$$

It is easily verified that $T_r^n \subset T_r^N$ if $N > n$.

The following result was conjectured by Horn [8], and proved as a consequence of work by Klyachko, Totaro, Knudson and Tao (cf. [6] for an exposition).

2.1. **Theorem.** *Consider nonincreasing sequences $\alpha, \beta, \gamma \in \mathbb{R}^n$. The following are equivalent:*

(1) *there exist selfadjoint $n \times n$ matrices $A, B$ such that the eigenvalues of $A$ (resp. $B$, resp. $A+B$) repeated according to their multiplicities are the components of $\alpha$ (resp. $\beta$, resp. $\gamma$).*

(2) $\sum_{i=1}^n \alpha_i + \sum_{j=1}^n \beta_j = \sum_{k=1}^n \gamma_k$ *and for all $r < n$ and all $(I, J, K) \in T_r^n$, we have*

$$\sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j \ge \sum_{k\in K} \gamma_k.$$

It will be convenient to introduce the eigenvalue function of a selfadjoint $n \times n$ matrix $A$. Thus, if the eigenvalues of $A$ are $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, we will write

$$\lambda_A(t) = \lambda_i \ \text{ for } \ t \in \left[\frac{i-1}{n}, \frac{i}{n}\right).$$

Let us note that $\lambda_{A\otimes I} = \lambda_A$, where $I$ denotes an identity matrix of arbitrary size.

Now, given a subset $I \subset \{1, 2, \ldots, n\}$, we can construct a subset $\sigma_I^n \subset [0, 1)$ as follows:

$$\sigma_I^n = \bigcup_{i\in I} \left[\frac{i-1}{n}, \frac{i}{n}\right).$$

The conditions in Theorem 2.1 can now be expressed as

$$\int_0^1 \lambda_A(t)\, dt + \int_0^1 \lambda_B(t)\, dt = \int_0^1 \lambda_{A+B}(t)\, dt$$

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and

$$\int_{\sigma_I^n} \lambda_A(t)\,dt + \int_{\sigma_J^n} \lambda_B(t)\,dt \geq \int_{\sigma_K^n} \lambda_{A+B}(t)\,dt$$

whenever $(I,J,K) \in T_r^n$ and $r < n$. Note that under this form, one can ask whether these inequalities hold for $(I,J,K) \in T_r^N$ with arbitrary $N$ and $r < N$. This is in fact true. Indeed, let us observe that, given a subset $I$ with $r$ elements of $\{1,2,\ldots,N\}$ and an integer $n$, there exists a subset $I_n$ with $nr$ elements of $\{1,2,\ldots,nN\}$ such that $\sigma_I^N = \sigma_{I_n}^{nN}$. In fact

$$I_n = \bigcup_{i \in I} \{n(i-1)+1, n(i-1)+2, \ldots, ni\}.$$

Moreover, properties of the Littlewood-Richardson coefficients (cf., for instance, [6] or [2]) show that $(I_n, J_n, K_n)$ belongs to $T_{nr}^{nN}$ whenever $(I,J,K) \in T_r^N$. This leads immediately to the following result.

2.2. **Proposition.** *Consider selfadjoint matrices $A, B \in \mathcal{M}_n$, and natural numbers $N, r$ such that $r < N$. Then*

$$\int_{\sigma_I^N} \lambda_A(t)\,dt + \int_{\sigma_J^N} \lambda_B(t)\,dt \geq \int_{\sigma_K^N} \lambda_{A+B}(t)\,dt$$

*whenever $(I,J,K) \in T_r^N$.*

*Proof.* As noted above, replacing $A$ and $B$ by $A \otimes I_N, B \otimes I_N \in \mathcal{M}_{nN}$ does not change the eigenvalue functions of $A$, $B$, or $A+B$. The result follows now because $\sigma_I^N = \sigma_{I_n}^{nN}$, $\sigma_K^N = \sigma_{K_n}^{nN}$, and $(I_n, J_n, K_n) \in T_{nr}^{nN}$. □

Let us use the notation

$$\mathcal{T} = \bigcup_{n=1}^{\infty} \bigcup_{r=1}^{n-1} \{(\sigma_I^n, \sigma_J^n, \sigma_K^n) : (I,J,K) \in T_r^n\}.$$

The preceding observation allows us to reformulate Theorem 2.1 as follows:

2.3. **Theorem.** *Consider nonincreasing sequences $\alpha, \beta, \gamma \in \mathbb{R}^n$, and form nonincreasing functions $u, v, w$ on $[0,1)$ such that $u(t) = \alpha_i, v(t) = \beta_i, w(t) = \gamma_i$ for $t \in [\frac{i-1}{n}, \frac{i}{n})$. The following are equivalent:*

(1) *there exist selfadjoint $n \times n$ matrices $A, B$ such that the eigenvalues of $A$ (resp. $B$, resp. $A+B$) repeated according to their multiplicities are the components of $\alpha$ (resp. $\beta$, resp. $\gamma$).*

(2) $\int_0^1 u(t)dt + \int_0^1 v(t)dt = \int_0^1 w(t)dt$ *and* $\int_{\omega_1} u(t)dt + \int_{\omega_2} v(t)dt \geq \int_{\omega_3} w(t)dt$ *for every triple $(\omega_1, \omega_2, \omega_3) \in \mathcal{T}$.*

## 3. Embeddable factors

Let us denote by $\tau_n$ the normalized trace on the algebra $\mathcal{M}_n$ of complex $n \times n$ matrices, i.e., $\tau_n(T) = \frac{1}{n}Tr(T)$ for $T \in \mathcal{M}_n$. The characteristic property of families of selfadjoint elements in the ultrapower $\mathcal{R}^\omega$ is the existence of finite-dimensional matrix approximants. More precisely, given selfadjoint elements $x_1, x_2, \ldots, x_k \in \mathcal{R}^\omega$, there exist integers $n_1 < n_2 < \cdots$ and selfadjoint matrices $X_1^{(m)}, X_2^{(m)}, \ldots, X_k^{(m)} \in \mathcal{M}_{n_m}$ such that

$$\tau(p(x_1, x_2, \ldots x_k)) = \lim_{m \to \infty} \tau_{n_m}(p(X_1^{(m)}, X_2^{(m)}, \ldots, X_k^{(m)}))$$

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for every polynomial $p$ in $k$ noncommuting variables. We will call such a sequence of $k$-tuples $(X_1^{(m)}, X_2^{(m)}, \ldots, X_k^{(m)})$ a sequence of matricial approximants of $(x_1, x_2, \ldots, x_k)$. The matrices $X_k^{(m)}$ can always be taken to have uniformly bounded norm. In this case we will speak of bounded matricial approximants.

3.1. **Lemma.** *Consider a selfadjoint element $x$ of a $\mathrm{II}_1$ factor $\mathcal{M}$, and let $(X_m)_{m=1}^\infty$ be a bounded sequence of matricial approximants of $x$. Them $\lim_{m\to\infty} \lambda_{X_m}(t) = \lambda_x(t)$ for all but at most countably many values $t \in [0,1)$.*

*Proof.* Since we have

$$\tau(x^n) = \int_0^1 (\lambda_x(t))^n \, dt,$$

we deduce that

$$\lim_{m\to\infty} \int_0^1 (\lambda_{X_m}(t))^n \, dt = \int_0^1 (\lambda_x(t))^n \, dt$$

for all positive integers $n$. Now, the functions $\lambda_{X_m}$ are uniformly bounded, hence Helly's selection theorem insures the existence of subsequences of $(\lambda_{X_m})_{m=1}^\infty$ which converge at all but countably many points $t \in [0,1)$. To conclude the proof, we must show that any such pointwise limit coincides with $\lambda_x$ at all but countably many points. Assume therefore that the limit $\lambda(t) = \lim_{m\to\infty} \lambda_{X_m}(t)$ exists for all but countably many points $t$, and $\lambda$ is right continuous. We then have $\int_0^1 \lambda(t)^n \, dt = \int_0^1 \lambda_x(t)^n \, dt$, $n = 1, 2, \ldots$. By the Stone-Weierstrass theorem, $\int_0^1 u(\lambda(t)) \, dt = \int_0^1 u(\lambda_x(t)) \, dt$ for every continuous function $u$ on $\mathbb{R}$, and this equality can be extended by the monotone convergence theorem to all lower semicontinuous functions $u$. The conclusion $\lambda = \lambda_x$ is now easily reached by taking $u = \chi_{(\alpha,\infty)}$, $\alpha \in \mathbb{R}$. □

We can now prove the main result of this paper.

3.2. **Theorem.** *Consider bounded nonincreasing right-continuous functions $u, v, w$ defined on $[0,1)$. The following are equivalent:*

(1) *there exist $a, b \in \mathcal{R}^\omega$ such that $u = \lambda_a$, $v = \lambda_b$, and $w = \lambda_{a+b}$;*

(2) *$\int_0^1 u(t)\, dt + \int_0^1 v(t)\, dt = \int_0^1 w(t)\, dt$ and $\int_{\omega_1} u(t)\, dt + \int_{\omega_2} v(t)\, dt \ge \int_{\omega_3} w(t)\, dt$ for every triple $(\omega_1, \omega_2, \omega_3) \in \mathcal{T}$.*

*Proof.* Assume first that $u = \lambda_a$, $v = \lambda_b$, and $w = \lambda_{a+b}$ for some $a, b \in \mathcal{R}^\omega$. As noted before, there exists a bounded sequence $(A_m, B_m)_{m=1}^\infty$ of matricial approximants of $(a, b)$, and Lemma 3.1 shows that, with countably many exceptions, $\lim_{m\to\infty} \lambda_{A_m}(t) = u(t)$, $\lim_{m\to\infty} \lambda_{B_m}(t) = v(t)$, and $\lim_{m\to\infty} \lambda_{A_m+B_m}(t) = w(t)$. The relations in (2) now follow from the corresponding relations for $\lambda_{A_m}$, $\lambda_{B_m}$, $\lambda_{A_m+B_m}$ via the dominated convergence theorem. Conversely, assume that $u, v, w$ satisfy (2). For each integer $n$ define $n$-tuples $\alpha^{(n)}, \beta^{(n)}, \gamma^{(n)} \in \mathbb{R}^n$ by

$$\alpha_i^{(n)} = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} u(t)\, dt, \quad \beta_i^{(n)} = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} v(t)\, dt, \quad \gamma_i^{(n)} = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} w(t)\, dt,$$

for $i = 1, 2, \ldots, n$. The hypothesis (2) implies that $\alpha^{(n)}, \beta^{(n)}, \gamma^{(n)}$ satisfy the conditions of Theorem 2.1(2), and therefore there exist selfadjoint matrices $A_n, B_n \in \mathcal{M}_n$ such that the eigenvalues of $A_n, B_n, A_n + B_n$ are the components of $\alpha^{(n)}, \beta^{(n)}$, $\gamma^{(n)}$, respectively. Clearly, the norms of $A_n$ and $B_n$ are uniformly bounded, and

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therefore we can find integers $n_1 < n_2 < \cdots$ such that $\lim_{m\to\infty} p(A_{n_m}, B_{n_m})$ exists for every polynomial $p$ in two noncommuting variables. Since $\mathcal{M}_n$ has a trace-preserving embedding into $\mathcal{R}$, we can find elements $a_m, b_m \in \mathcal{R}$ such that $\lambda_{a_m} = \lambda_{A_{n_m}}$, $\lambda_{b_m} = \lambda_{B_{n_m}}$, $\lambda_{a_m+b_m} = \lambda_{A_{n_m}+B_{n_m}}$. Denote by $a$ and $b$ the elements of $\mathcal{R}^\omega$ determined by the sequences $(a_m)_{m=1}^\infty$ and $(b_m)_{m=1}^\infty$. Clearly then $(A_{n_m}, B_{n_m})$ are matrix approximants for $a$ and $b$, so that $\lim_{m\to\infty} \lambda_{A_{n_m}}(t) = \lambda_a(t)$, $\lim_{m\to\infty} \lambda_{B_{n_m}}(t) = \lambda_b(t)$, and $\lim_{m\to\infty} \lambda_{A_{n_m}+B_{n_m}}(t) = \lambda_{a+b}(t)$, for all but countably many values of $t$. On the other hand, $\lim_{n\to\infty} \lambda_{A_n}(t) = u(t)$ at all points of continuity for $u$, so that $\lambda_a(t) = u(t)$ at all but countably many points. Thus $\lambda_a = u$ by right continuity. Analogously, $\lambda_b = v$ and $\lambda_{a+b} = w$, thus proving (1). □

## 4. Concluding remarks

As mentioned in the Introduction, we do not have a proof of the implication (1)⇒(2) for arbitrary $\mathrm{II}_1$ factors. Any counterexample would solve in the negative Connes' question on the embeddability of $\mathrm{II}_1$ factors into $\mathcal{R}^\omega$.

At the other extreme, it may be interesting to see whether the implication (2)⇒(1) holds for a given embeddable factor, such as $\mathcal{R}$ itself.

The inequalities in Theorem 3.2 are not the only ones of the form

$$\int_{\omega_1} u(t)\,dt + \int_{\omega_2} v(t)\,dt \geq \int_{\omega_3} w(t)\,dt$$

which are true for $u = \lambda_a$, $v = \lambda_b$, $w = \lambda_{a+b}$. Thus, for instance, Grothendieck's inequality

$$\int_0^\alpha u(t)\,dt + \int_0^\alpha v(t)\,dt \geq \int_0^\alpha w(t)\,dt$$

only follows from Theorem 3.2 for rational values of $\alpha$. Let us also note that each triple $(\omega_1, \omega_2, \omega_3) \in \mathcal{T}$ can be written in infinitely many ways as $(\sigma_I^n, \sigma_J^n, \sigma_K^n)$ for some $(I, J, K) \in T_r^n$. More precisely, $(\sigma_I^n, \sigma_J^n, \sigma_K^n) = (\sigma_{I_N}^{nN}, \sigma_{J_N}^{nN}, \sigma_{K_N}^{nN})$, as seen earlier.

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