

## THE POWER OF THE TANGENT BUNDLE OF THE REAL PROJECTIVE SPACE, ITS COMPLEXIFICATION AND EXTENDIBILITY

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ABSTRACT. We establish the formulas on the power  $\tau^k$  of the tangent bundle  $\tau = \tau(RP^n)$  of the real projective  $n$ -space  $RP^n$  and its complexification  $c\tau^k$ , and apply the formulas to the problem of extendibility and stable extendibility of  $\tau^k$  and  $c\tau^k$ .

### 1. INTRODUCTION

Let  $F$  denote either the real number field  $R$  or the complex number field  $C$ , and let  $X$  be a space and  $A$  its subspace. A  $t$ -dimensional  $F$ -vector bundle  $\zeta$  over  $A$  is said to be *extendible* (respectively *stably extendible*) to  $X$ , if and only if there is a  $t$ -dimensional  $F$ -vector bundle over  $X$  whose restriction to  $A$  is equivalent (respectively stably equivalent) to  $\zeta$  as  $F$ -vector bundles (cf. [9] and [3]). For simplicity, we use the same letter for a vector bundle and its equivalence class.

Let  $RP^n$  denote the real projective  $n$ -space  $RP^n$  and let  $\tau = \tau(RP^n)$  stand for the tangent bundle of  $RP^n$ . We study the question: Determine the dimension  $n$  for which an  $F$ -vector bundle over  $RP^n$  is extendible (or stably extendible) to  $RP^m$  for every  $m \geq n$ . We have obtained the complete answer for the tangent bundle  $\tau = \tau(RP^n)$  in [6] and [8], for the complexification  $c\tau$  of  $\tau$  in [5], for the square  $\tau^2 = \tau(RP^n) \otimes \tau(RP^n)$  in [4] and for the complexification  $c\tau^2$  of  $\tau^2$  in [4], where  $\otimes$  denotes the tensor product. The results on  $\tau$  and  $\tau^2$  are as follows.

**Theorem 1.1** ([6, Theorem 6.6] and [8, Theorem 4.2]). *The following three conditions are equivalent:*

- (i)  $\tau$  is extendible to  $RP^m$  for every  $m \geq n$ .
- (ii)  $\tau$  is stably extendible to  $RP^m$  for every  $m \geq n$ .
- (iii)  $n = 1, 3$  or  $7$ .

**Theorem 1.2** ([4, Theorem 4]). *The following three conditions are equivalent:*

- (i)  $\tau^2$  is extendible to  $RP^m$  for every  $m \geq n$ .
- (ii)  $\tau^2$  is stably extendible to  $RP^m$  for every  $m \geq n$ .
- (iii)  $1 \leq n \leq 16$ .

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The first purpose of this paper is to obtain the complete answer for the  $k$ -fold power  $\tau^k$ . Let  $\phi(n)$  be the number of integers  $s$  such that  $0 < s \leq n$  and  $s \equiv 0, 1, 2$  or  $4 \pmod{8}$ . Then we have

**Theorem A.** *For the  $k$ -fold power  $\tau^k = \tau(RP^n) \otimes \cdots \otimes \tau(RP^n)$  ( $k$ -fold) of the tangent bundle  $\tau(RP^n)$ , the following three conditions are equivalent:*

- (i)  $\tau^k$  is extendible to  $RP^m$  for every  $m \geq n$ .
- (ii)  $\tau^k$  is stably extendible to  $RP^m$  for every  $m \geq n$ .
- (iii) There is an integer  $a$  satisfying

$$(n+2)^k - n^k \leq a2^{\phi(n)+1} \leq (n+2)^k + n^k.$$

If  $k = 1$ , the condition (iii) is equivalent to the condition:  $n = 1, 3$  or  $7$ , and if  $k = 2$ , it is equivalent to the condition:  $1 \leq n \leq 16$ . (Note that  $2^{\phi(n)} > n + 1$  for  $n \neq 1, 3, 7$ , and that  $2^{\phi(n)} > n^2 + 2n + 2$  for  $n \geq 17$ .) Hence Theorem A is a generalization of Theorems 1.1 and 1.2. The results on  $c\tau$  and  $c\tau^2$  are as follows.

**Theorem 1.3** ([5, Theorem 1]). *The following three conditions are equivalent:*

- (i)  $c\tau$  is extendible to  $RP^m$  for every  $m \geq n$ .
- (ii)  $c\tau$  is stably extendible to  $RP^m$  for every  $m \geq n$ .
- (iii)  $1 \leq n \leq 5$  or  $n = 7$ .

**Theorem 1.4** ([4, Theorem 5]). *The following three conditions are equivalent:*

- (i)  $c\tau^2$  is extendible to  $RP^m$  for every  $m \geq n$ .
- (ii)  $c\tau^2$  is stably extendible to  $RP^m$  for every  $m \geq n$ .
- (iii)  $1 \leq n \leq 17$ .

The second purpose of this paper is to obtain the complete answer for the complexification  $c\tau^k$  of  $\tau^k$ . For a real number  $x$ , let  $[x]$  be the largest integer  $n$  with  $n \leq x$ . Then we have

**Theorem B.** *For the complexification  $c\tau^k = c(\tau(RP^n) \otimes \cdots \otimes \tau(RP^n))$  of the  $k$ -fold power  $\tau^k$ , the following three conditions are equivalent:*

- (i)  $c\tau^k$  is extendible to  $RP^m$  for every  $m \geq n$ .
- (ii)  $c\tau^k$  is stably extendible to  $RP^m$  for every  $m \geq n$ .
- (iii) There is an integer  $b$  satisfying

$$(n+2)^k - n^k \leq b2^{\lfloor n/2 \rfloor + 1} \leq (n+2)^k + n^k.$$

If  $k = 1$ , condition (iii) is equivalent to the condition  $1 \leq n \leq 5$  or  $n = 7$ , and if  $k = 2$ , it is equivalent to the condition  $1 \leq n \leq 17$ . (Note that  $2^{\lfloor n/2 \rfloor} > n + 1$  for  $n = 6$  or  $n \geq 8$ , and that  $2^{\lfloor n/2 \rfloor} > n^2 + 2n + 2$  for  $n \geq 18$ .) Hence Theorem B is a generalization of Theorems 1.3 and 1.4.

This paper is arranged as follows. In Section 2 we establish the formulas on the power  $\tau^k = \tau(RP^n) \otimes \cdots \otimes \tau(RP^n)$  of the tangent bundle  $\tau(RP^n)$  of the real projective  $n$ -space  $RP^n$ . In Section 3 we apply the results in Section 2 to the problem of extendibility and stable extendibility of the  $k$ -fold power  $\tau^k$  and prove Theorem A by using Theorem 4.1 in [8]. In Section 4 we establish the formulas on the complexification  $c\tau^k = c(\tau(RP^n) \otimes \cdots \otimes \tau(RP^n))$  of  $\tau^k$ . In Section 5 we apply the results in Section 4 to the problem of extendibility and stable extendibility of  $c\tau^k$  and prove Theorem B by using Theorem 2.1 in [8].

2. THE  $k$ -FOLD POWER OF THE TANGENT BUNDLE OF  $\mathbb{R}P^n$

In this section we establish the formulas on the  $k$ -fold power of the tangent bundle  $\tau = \tau(RP^n)$ . Let  $\xi_n$  denote the canonical line bundle over  $RP^n$ .

**Lemma 2.1.** *Let  $\tau^k = \tau(RP^n) \otimes \cdots \otimes \tau(RP^n)$  ( $k$ -fold) denote the  $k$ -fold power of the tangent bundle  $\tau = \tau(RP^n)$ . Then, for any positive integer  $r$ , the following hold in the Grothendick group  $KO(RP^n)$ :*

- (1)  $\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}\xi_n - 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\}$ ,
- (2)  $\tau^{2r} = -2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n + 2^{-1}\{(n+2)^{2r} + n^{2r}\}$ .

*Proof.* It is well known that  $\tau = (n+1)\xi_n - 1$  in  $KO(RP^n)$ . Hence formula (1) clearly holds for  $r = 1$ .

Assume that formula (1) holds for  $r \geq 1$ . Then

$$\begin{aligned} \tau^{2r} &= \tau \otimes \tau^{2r-1} \\ &= \{(n+1)\xi_n - 1\}[2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}\xi_n - 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\}] \\ &= -2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n + 2^{-1}\{(n+2)^{2r} + n^{2r}\}, \end{aligned}$$

since  $\xi_n \otimes \xi_n = 1$ . So formula (2) holds for  $r \geq 1$ .

Assume that formula (2) holds for  $r \geq 1$ . Then

$$\begin{aligned} \tau^{2r+1} &= \tau \otimes \tau^{2r} \\ &= \{(n+1)\xi_n - 1\}[2^{-1}\{(n+2)^{2r} + n^{2r}\} - 2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n] \\ &= 2^{-1}\{(n+2)^{2r+1} + n^{2r+1}\}\xi_n - 2^{-1}\{(n+2)^{2r+1} - n^{2r+1}\}, \end{aligned}$$

since  $\xi_n \otimes \xi_n = 1$ . So formula (1) holds for  $r + 1$ .

Hence formulas (1) and (2) hold for any positive integer  $r$  by induction on  $r$ .  $\square$

The following result is used in our proofs.

**Theorem 2.2** (cf. [2, Theorem 1.5, p. 100]). *Two  $t$ -dimensional  $F$ -vector bundles over an  $n$ -dimensional CW-complex which are stably equivalent are equivalent if  $\langle (n+2)/f - 1 \rangle \leq t$ , where  $\langle x \rangle$  denotes the smallest integer  $n$  with  $x \leq n$  and  $f = 1$  or  $2$  according as  $F = R$  or  $C$ .*

We establish the formula on  $\tau^k$ , as follows.

**Theorem 2.3.** *Let  $\tau^k = \tau(RP^n) \otimes \cdots \otimes \tau(RP^n)$  ( $k$ -fold) denote the  $k$ -fold power of the tangent bundle  $\tau = \tau(RP^n)$ . Then, for any positive integer  $r$ , the following hold:*

- (1)  $\tau^{2r-1} \oplus 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}\xi_n$ ,
- (2)  $\tau^{2r} \oplus 2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n = 2^{-1}\{(n+2)^{2r} + n^{2r}\}$ ,

where, in the equalities (1) and (2), a positive integer  $k$  denotes the  $k$ -dimensional trivial bundle over  $RP^n$  and  $\oplus$  denotes the Whitney sum.

*Proof.* (1) By Lemma 2.1(1), we have

$$\tau^{2r-1} + 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}\xi_n$$

in  $KO(RP^n)$ . Since

$$\begin{aligned} \dim[\tau^{2r-1} \oplus 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\}] \\ = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\} > n = \dim RP^n, \end{aligned}$$

the equality

$$\tau^{2r-1} \oplus 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}\xi_n$$

holds as  $R$ -vector bundles by Theorem 2.2.

(2) By Lemma 2.1(2), we have

$$\tau^{2r} + 2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n = 2^{-1}\{(n+2)^{2r} + n^{2r}\}$$

in  $KO(RP^n)$ . Since  $\dim[\tau^{2r} \oplus 2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n] = 2^{-1}\{(n+2)^{2r} + n^{2r}\} > n = \dim RP^n$ , the equality

$$\tau^{2r} \oplus 2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n = 2^{-1}\{(n+2)^{2r} + n^{2r}\}$$

holds as  $R$ -vector bundles by Theorem 2.2. □

Moreover, the next theorem follows from Lemma 2.1.

**Theorem 2.4.** *For any positive integer  $r$  and any integer  $a$ , the following hold in  $KO(RP^n)$ :*

- (1)  $\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - a2^{\phi(n)+1}\}\xi_n + 2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r-1} + n^{2r-1}\}$ ,
- (2)  $\tau^{2r} = 2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r} + n^{2r}\}\xi_n + 2^{-1}\{(n+2)^{2r} + n^{2r} - a2^{\phi(n)+1}\}$ .

*Proof.* Subtracting  $a2^{\phi(n)}(\xi_n - 1) = 0$  (cf. [1, Theorem 7.4]) from equality (1) in Lemma 2.1, we have equality (1) above, and adding  $a2^{\phi(n)}(\xi_n - 1) = 0$  to equality (2) in Lemma 2.1, we have the equality (2) above. □

3. EXTENDIBILITY AND STABLE EXTENDIBILITY OF THE  $k$ -FOLD POWER

$$\tau^k = \tau(\mathbb{R}P^n) \otimes \cdots \otimes \tau(\mathbb{R}P^n)$$

**Theorem 3.1.** *Assume that there is an integer  $a$  satisfying*

$$(n+2)^k - n^k \leq a2^{\phi(n)+1} \leq (n+2)^k + n^k.$$

*Then  $\tau^k$  is extendible to  $RP^m$  for every  $m \geq n$ .*

*Proof.* If  $k = 1$ , the inequalities imply  $a = 1$  and  $n = 1, 3$  or  $7$ , and if  $n = 1, 3$  or  $7$ ,  $\tau(RP^n)$  is trivial. Hence the results clearly hold for  $n = 1$  or  $k = 1$ . So we may restrict our attention to the case  $n > 1$  and  $k > 1$ .

In case  $k$  is odd, let  $k = 2r - 1$ , where  $r$  is an integer  $> 1$ . Then, by the assumption, we have

$$2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - a2^{\phi(n)+1}\} \geq 0$$

and

$$2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r-1} + n^{2r-1}\} \geq 0.$$

Hence Theorem 2.4(1) implies that the equality

$$\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - a2^{\phi(n)+1}\}\xi_n \oplus 2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r-1} + n^{2r-1}\}$$

holds by Theorem 2.2, since  $\dim \tau^{2r-1} = n^{2r-1} > n = \dim RP^n$  for  $n > 1$  and  $r > 1$ . So  $\tau^{2r-1}$  is extendible to  $RP^m$  for every  $m \geq n$ , since  $\xi_n$  and the trivial bundle over  $RP^n$  are extendible to  $RP^m$  for every  $m \geq n$ .

In case  $k$  is even, let  $k = 2r$ , where  $r$  is a positive integer. Then, by the assumption, we have

$$2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r} + n^{2r}\} \geq 0$$

and

$$2^{-1}\{(n+2)^{2r} + n^{2r} - a2^{\phi(n)+1}\} \geq 0.$$

Hence Theorem 2.4(2) implies that the equality

$$\tau^{2r} = 2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r} + n^{2r}\}\xi_n \oplus 2^{-1}\{(n+2)^{2r} + n^{2r} - a2^{\phi(n)+1}\}$$

holds by Theorem 2.2, since  $\dim \tau^{2r} = n^{2r} > n = \dim RP^n$  for  $n > 1$  and  $r > 0$ . So  $\tau^{2r}$  is extendible to  $RP^m$  for every  $m \geq n$ . □

The following result is Theorem 4.1 in [8] which is the stably extendible version of Theorem 6.2 in [6].

**Theorem 3.2.** *Let  $\zeta$  be a  $t$ -dimensional  $R$ -vector bundle over  $RP^n$ . Assume that there is a positive integer  $l$  such that  $\zeta$  is stably equivalent to  $(t+l)\xi_n$ , and  $t+l < 2^{\phi(n)}$ . Then  $n < t+l$  and  $\zeta$  is not stably extendible to  $RP^m$  for every  $m \geq t+l$ .*

**Theorem 3.3.** *Assume that there is an integer  $a$  satisfying*

$$(n+2)^k + n^k - 2^{\phi(n)+1} < a2^{\phi(n)+1} < (n+2)^k - n^k.$$

*Then  $\tau^k$  is not stably extendible to  $RP^m$  for every  $m \geq 2^{-1}\{(n+2)^k + n^k - a2^{\phi(n)+1}\}$  if  $k$  is odd, and for every  $m \geq 2^{-1}\{(a+1)2^{\phi(n)+1} - (n+2)^k + n^k\}$  if  $k$  is even.*

*Proof.* If  $k$  is odd, let  $k = 2r - 1$ . Then putting

$$\zeta = \tau^{2r-1}, \quad t = n^{2r-1} \quad \text{and} \quad l = 2^{-1}\{(n+2)^{2r-1} - n^{2r-1} - a2^{\phi(n)+1}\}$$

in Theorem 3.2, we obtain the result by Theorem 2.4(1) and Theorem 3.2, since  $t+l < 2^{\phi(n)}$  and  $l > 0$  by the assumption.

If  $k$  is even, let  $k = 2r$ . Then putting

$$\zeta = \tau^{2r}, \quad t = n^{2r} \quad \text{and} \quad l = 2^{-1}\{(a+1)2^{\phi(n)+1} - (n+2)^{2r} - n^{2r}\}$$

in Theorem 3.2, we obtain the result by Theorem 2.4(2) and Theorem 3.2, since  $t+l < 2^{\phi(n)}$  and  $l > 0$  by the assumption. □

*Proof of Theorem A.* (i) clearly implies (ii). (iii) implies (i) by Theorem 3.1. To show that (ii) implies (iii), we prove the contraposition. Assume that every integer  $a$  satisfies

$$a2^{\phi(n)+1} < (n+2)^k - n^k \quad \text{or} \quad (n+2)^k + n^k < a2^{\phi(n)+1}.$$

Assume that there are integers  $a$  with  $a2^{\phi(n)+1} < (n+2)^k - n^k$ . Then we define  $A$  as the maximum integer such that  $A2^{\phi(n)+1} < (n+2)^k - n^k$ . If  $A$  satisfies  $A2^{\phi(n)+1} \leq (n+2)^k + n^k - 2^{\phi(n)+1}$ , we have  $(n+2)^k - n^k \leq (A+1)2^{\phi(n)+1} \leq (n+2)^k + n^k$ , and these are inconsistent with our assumption. Hence  $A$  satisfies  $(n+2)^k + n^k - 2^{\phi(n)+1} < A2^{\phi(n)+1} < (n+2)^k - n^k$ . So, by Theorem 3.3,  $\tau^k$  is not stably extendible to  $RP^m$  for every  $m \geq 2^{-1}\{(n+2)^k + n^k - A2^{\phi(n)+1}\}$  if  $k$  is odd, and for every  $m \geq 2^{-1}\{(A+1)2^{\phi(n)+1} - (n+2)^k + n^k\}$  if  $k$  is even.

Assume that there are integers  $a$  with  $(n+2)^k + n^k < a2^{\phi(n)+1}$ . Then we define  $B$  as the minimum integer such that  $(n+2)^k + n^k < B2^{\phi(n)+1}$ . If  $B$  satisfies  $B2^{\phi(n)+1} \geq (n+2)^k - n^k + 2^{\phi(n)+1}$ , we have  $(n+2)^k - n^k \leq (B-1)2^{\phi(n)+1} \leq (n+2)^k + n^k$ , and these are inconsistent with our assumption. Hence  $B$  satisfies  $(n+2)^k + n^k - 2^{\phi(n)+1} < (B-1)2^{\phi(n)+1} < (n+2)^k - n^k$ . So, by Theorem 3.3,  $\tau^k$  is not stably extendible to  $RP^m$  for every  $m \geq 2^{-1}\{(n+2)^k + n^k - (B-1)2^{\phi(n)+1}\}$  if  $k$  is odd, and for every  $m \geq 2^{-1}\{B2^{\phi(n)+1} - (n+2)^k + n^k\}$  if  $k$  is even. □

4. THE COMPLEXIFICATION OF THE  $k$ -FOLD POWER OF  $\tau(\mathbb{R}P^n)$ 

Complexifying the equalities (1) and (2) in Lemma 2.1, we immediately have

**Lemma 4.1.** *Let  $c\tau^k = c(\tau(\mathbb{R}P^n) \otimes \cdots \otimes \tau(\mathbb{R}P^n))$  denote the complexification of the  $k$ -fold power  $\tau^k$  of the tangent bundle  $\tau = \tau(\mathbb{R}P^n)$ . Then, for any positive integer  $r$ , the following hold in the Grothendieck group  $K(\mathbb{R}P^n)$ :*

- (1)  $c\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}c\xi_n - 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\},$
- (2)  $c\tau^{2r} = -2^{-1}\{(n+2)^{2r} - n^{2r}\}c\xi_n + 2^{-1}\{(n+2)^{2r} + n^{2r}\}.$

Complexifying the equalities (1) and (2) in Theorem 2.3, we immediately have

**Theorem 4.2.** *For the complexification  $c\tau^k = c(\tau(\mathbb{R}P^n) \otimes \cdots \otimes \tau(\mathbb{R}P^n))$  of the  $k$ -fold power  $\tau^k$  of the tangent bundle  $\tau = \tau(\mathbb{R}P^n)$ , the following hold:*

- (1)  $c\tau^{2r-1} \oplus 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\}c\xi_n,$
- (2)  $c\tau^{2r} \oplus 2^{-1}\{(n+2)^{2r} - n^{2r}\}c\xi_n = 2^{-1}\{(n+2)^{2r} + n^{2r}\}.$

Furthermore, the next theorem follows from Lemma 4.1.

**Theorem 4.3.** *For any positive integer  $r$  and any integer  $b$ , the following hold in  $K(\mathbb{R}P^n)$ :*

- (1)  $c\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - b2^{\lfloor n/2 \rfloor + 1}\}c\xi_n$   
 $+ 2^{-1}\{b2^{\lfloor n/2 \rfloor + 1} - (n+2)^{2r-1} + n^{2r-1}\},$
- (2)  $c\tau^{2r} = 2^{-1}\{b2^{\lfloor n/2 \rfloor + 1} - (n+2)^{2r} + n^{2r}\}c\xi_n + 2^{-1}\{(n+2)^{2r} + n^{2r} - b2^{\lfloor n/2 \rfloor + 1}\}.$

*Proof.* Subtracting  $b2^{\lfloor n/2 \rfloor}(\xi_n - 1) = 0$  (cf. [1, Theorem 7.3]) from equality (1) in Lemma 4.1, we have equality (1) above, and adding  $b2^{\lfloor n/2 \rfloor}(\xi_n - 1) = 0$  to equality (2) in Lemma 4.1, we have equality (2) above.  $\square$

## 5. EXTENDIBILITY AND STABLE EXTENDIBILITY OF THE COMPLEXIFICATION

$$c\tau^k = c(\tau(\mathbb{R}P^n) \otimes \cdots \otimes \tau(\mathbb{R}P^n))$$

The proofs of the following Theorems 5.1 and 5.3 are parallel to those of Theorems 3.1 and 3.3, respectively.

**Theorem 5.1.** *Assume that there is an integer  $b$  satisfying*

$$(n+2)^k - n^k \leq b2^{\lfloor n/2 \rfloor + 1} \leq (n+2)^k + n^k.$$

*Then  $c\tau^k$  is extendible to  $\mathbb{R}P^m$  for every  $m \geq n$ .*

*Proof.* In case  $k$  is odd, let  $k = 2r - 1$ , where  $r$  is a positive integer. Then, by the assumption, we have

$$2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - b2^{\lfloor n/2 \rfloor + 1}\} \geq 0$$

and

$$2^{-1}\{b2^{\lfloor n/2 \rfloor + 1} - (n+2)^{2r-1} + n^{2r-1}\} \geq 0.$$

Hence Theorem 4.3(1) implies that the equality

$$c\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - b2^{\lfloor n/2 \rfloor + 1}\}c\xi_n \oplus 2^{-1}\{b2^{\lfloor n/2 \rfloor + 1} - (n+2)^{2r-1} + n^{2r-1}\}$$

holds by Theorem 2.2, since  $\dim c\tau^{2r-1} = n^{2r-1} \geq \langle (n+2)/2 - 1 \rangle = \langle n/2 \rangle$ . So  $c\tau^{2r-1}$  is extendible to  $\mathbb{R}P^m$  for every  $m \geq n$ , since  $c\xi_n$  and the trivial bundle over  $\mathbb{R}P^n$  are extendible to  $\mathbb{R}P^m$  for every  $m \geq n$ .

In case  $k$  is even, let  $k = 2r$ , where  $r$  is a positive integer. Then, by the assumption, we have

$$2^{-1}\{b2^{[n/2]+1} - (n + 2)^{2r} + n^{2r}\} \geq 0$$

and

$$2^{-1}\{(n + 2)^{2r} + n^{2r} - b2^{[n/2]+1}\} \geq 0.$$

Hence Theorem 4.3(2) implies that the equality

$$c\tau^{2r} = 2^{-1}\{b2^{[n/2]+1} - (n + 2)^{2r} + n^{2r}\}c\xi_n \oplus 2^{-1}\{(n + 2)^{2r} + n^{2r} - b2^{[n/2]+1}\}$$

holds by Theorem 2.2, since  $\dim c\tau^{2r} = n^{2r} \geq \langle (n + 2)/2 - 1 \rangle = \langle n/2 \rangle$ . So  $c\tau^{2r}$  is extendible to  $RP^m$  for every  $m \geq n$ .  $\square$

The following result is Theorem 2.1 in [8] which is the stably extendible version of Theorem 4.2 for  $d = 1$  in [7].

**Theorem 5.2.** *Let  $\zeta$  be a  $t$ -dimensional  $C$ -vector bundle over  $RP^n$ . Assume that there is a positive integer  $l$  such that  $\zeta$  is stably equivalent to  $(t + l)c\xi_n$ , and  $t + l < 2^{[n/2]}$ . Then  $n < 2t + 2l$  and  $\zeta$  is not stably extendible to  $RP^m$  for every  $m \geq 2t + 2l$ .*

**Theorem 5.3.** *Assume that there is an integer  $b$  satisfying*

$$(n + 2)^k + n^k - 2^{[n/2]+1} < b2^{[n/2]+1} < (n + 2)^k - n^k.$$

*Then  $c\tau^k$  is not stably extendible to  $RP^m$  for every  $m \geq (n + 2)^k + n^k - b2^{[n/2]+1}$  if  $k$  is odd, and for every  $m \geq (b + 1)2^{[n/2]+1} - (n + 2)^k + n^k$  if  $k$  is even.*

*Proof.* If  $k$  is odd, let  $k = 2r - 1$ . Then putting

$$\zeta = c\tau^{2r-1}, \quad t = n^{2r-1} \quad \text{and} \quad l = 2^{-1}\{(n + 2)^{2r-1} - n^{2r-1} - b2^{[n/2]+1}\}$$

in Theorem 5.2, we obtain the result by Theorem 4.3(1) and Theorem 5.2, since  $t + l < 2^{[n/2]}$  and  $l > 0$  by the assumption.

If  $k$  is even, let  $k = 2r$ . Then putting

$$\zeta = c\tau^{2r}, \quad t = n^{2r} \quad \text{and} \quad l = 2^{-1}\{(b + 1)2^{[n/2]+1} - (n + 2)^{2r} - n^{2r}\}$$

in Theorem 5.2, we obtain the result by Theorem 4.3(2) and Theorem 5.2, since  $t + l < 2^{[n/2]}$  and  $l > 0$  by the assumption.  $\square$

*Proof of Theorem B.* (i) implies (ii) clearly. (iii) implies (i) by Theorem 5.1. To show that (ii) implies (iii), we prove the contraposition. Assume that every integer  $b$  satisfies

$$b2^{[n/2]+1} < (n + 2)^k - n^k \quad \text{or} \quad (n + 2)^k + n^k < b2^{[n/2]+1}.$$

Assume that there are integers  $b$  with  $b2^{[n/2]+1} < (n + 2)^k - n^k$ . Then we define  $C$  as the maximum integer such that  $C2^{[n/2]+1} < (n + 2)^k - n^k$ . If  $C$  satisfies  $C2^{[n/2]+1} \leq (n + 2)^k + n^k - 2^{[n/2]+1}$ , we have  $(n + 2)^k - n^k \leq (C + 1)2^{[n/2]+1} \leq (n + 2)^k + n^k$ , and these are inconsistent with our assumption. Hence  $C$  satisfies  $(n + 2)^k + n^k - 2^{[n/2]+1} < C2^{[n/2]+1} < (n + 2)^k - n^k$ . So, by Theorem 5.3,  $c\tau^k$  is not stably extendible to  $RP^m$  for every  $m \geq (n + 2)^k + n^k - C2^{[n/2]+1}$  if  $k$  is odd, and for every  $m \geq (C + 1)2^{[n/2]+1} - (n + 2)^k + n^k$  if  $k$  is even.

Assume that there are integers  $b$  with  $(n + 2)^k + n^k < b2^{[n/2]+1}$ . Then we define  $D$  as the minimum integer such that  $(n + 2)^k + n^k < D2^{[n/2]+1}$ . If  $D$  satisfies  $D2^{[n/2]+1} \geq (n + 2)^k - n^k + 2^{[n/2]+1}$ , we have  $(n + 2)^k - n^k \leq (D - 1)2^{[n/2]+1} \leq (n + 2)^k + n^k$ , and these are inconsistent with our assumption. Hence  $D$  satisfies

$(n+2)^k + n^k - 2^{\lfloor n/2 \rfloor + 1} < (D-1)2^{\lfloor n/2 \rfloor + 1} < (n+2)^k - n^k$ . So, by Theorem 5.3,  $c\tau^k$  is not stably extendible to  $RP^m$  for every  $m \geq (n+2)^k + n^k - (D-1)2^{\lfloor n/2 \rfloor + 1}$  if  $k$  is odd, and for every  $m \geq D2^{\lfloor n/2 \rfloor + 1} - (n+2)^k + n^k$  if  $k$  is even.  $\square$

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