THE POWER OF THE TANGENT BUNDLE OF THE REAL
PROJECTIVE SPACE, ITS COMPLEXIFICATION
AND EXTENDIBILITY

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ABSTRACT. We establish the formulas on the power $\tau^k$ of the tangent bundle $\tau = \tau(RP^n)$ of the real projective $n$-space $RP^n$ and its complexification $c\tau^k$, and apply the formulas to the problem of extendibility and stable extendibility of $\tau^k$ and $c\tau^k$.

1. Introduction

Let $F$ denote either the real number field $R$ or the complex number field $C$, and let $X$ be a space and $A$ its subspace. A $t$-dimensional $F$-vector bundle $\zeta$ over $A$ is said to be extendible (respectively stably extendible) to $X$, if and only if there is a $t$-dimensional $F$-vector bundle over $X$ whose restriction to $A$ is equivalent (respectively stably equivalent) to $\zeta$ as $F$-vector bundles (cf. [9] and [3]). For simplicity, we use the same letter for a vector bundle and its equivalence class.

Let $RP^n$ denote the real projective $n$-space $RP^n$ and let $\tau = \tau(RP^n)$ stand for the tangent bundle of $RP^n$. We study the question: Determine the dimension $n$ for which an $F$-vector bundle over $RP^n$ is extendible (or stably extendible) to $RP^m$ for every $m \geq n$. We have obtained the complete answer for the tangent bundle $\tau = \tau(RP^n)$ in [6] and [8], for the complexification $c\tau$ of $\tau$ in [5], for the square $\tau^2 = \tau(RP^n) \otimes \tau(RP^n)$ in [4] and for the complexification $c\tau^2$ of $\tau^2$ in [4], where $\otimes$ denotes the tensor product. The results on $\tau$ and $\tau^2$ are as follows.

Theorem 1.1 ([6 Theorem 6.6] and [8 Theorem 4.2]). The following three conditions are equivalent:

(i) $\tau$ is extendible to $RP^m$ for every $m \geq n$.
(ii) $\tau$ is stably extendible to $RP^m$ for every $m \geq n$.
(iii) $n = 1, 3$ or 7.

Theorem 1.2 ([4 Theorem 4]). The following three conditions are equivalent:

(i) $\tau^2$ is extendible to $RP^m$ for every $m \geq n$.
(ii) $\tau^2$ is stably extendible to $RP^m$ for every $m \geq n$.
(iii) $1 \leq n \leq 16$. 
The first purpose of this paper is to obtain the complete answer for the $k$-fold power $\tau^k$. Let $\phi(n)$ be the number of integers $s$ such that $0 < s \leq n$ and $s \equiv 0, 1, 2$ or $4 \mod 8$. Then we have

**Theorem A.** For the $k$-fold power $\tau^k = \tau(RP^n) \otimes \cdots \otimes \tau(RP^n)$ (k-fold) of the tangent bundle $\tau(RP^n)$, the following three conditions are equivalent:

(i) $\tau^k$ is extendible to $RP^{m}$ for every $m \geq n$.

(ii) $\tau^k$ is stably extendible to $RP^{m}$ for every $m \geq n$.

(iii) There is an integer $a$ satisfying

\[(n + 2)^k - n^k \leq a2^{\phi(n)+1} \leq (n + 2)^k + n^k.\]

If $k = 1$, the condition (iii) is equivalent to the condition: $n = 1, 3$ or $7$, and if $k = 2$, it is equivalent to the condition: $1 \leq n \leq 16$. (Note that $2^{\phi(n)} > n + 1$ for $n \neq 1, 3, 7$, and that $2^{\phi(n)} > n^2 + 2n + 2$ for $n \geq 17$.) Hence Theorem A is a generalization of Theorems 1.1 and 1.2. The results on $c\tau$ and $c\tau^2$ are as follows.

**Theorem 1.3** ([3, Theorem 1]). The following three conditions are equivalent:

(i) $c\tau$ is extendible to $RP^{m}$ for every $m \geq n$.

(ii) $c\tau$ is stably extendible to $RP^{m}$ for every $m \geq n$.

(iii) $1 \leq n \leq 5$ or $n = 7$.

**Theorem 1.4** ([4, Theorem 5]). The following three conditions are equivalent:

(i) $c\tau^2$ is extendible to $RP^{m}$ for every $m \geq n$.

(ii) $c\tau^2$ is stably extendible to $RP^{m}$ for every $m \geq n$.

(iii) $1 \leq n \leq 17$.

The second purpose of this paper is to obtain the complete answer for the complexification $c\tau^k$ of $\tau^k$. For a real number $x$, let $[x]$ be the largest integer $n$ with $n \leq x$. Then we have

**Theorem B.** For the complexification $c\tau^k = c(\tau(RP^n) \otimes \cdots \otimes \tau(RP^n))$ of the $k$-fold power $\tau^k$, the following three conditions are equivalent:

(i) $c\tau^k$ is extendible to $RP^{m}$ for every $m \geq n$.

(ii) $c\tau^k$ is stably extendible to $RP^{m}$ for every $m \geq n$.

(iii) There is an integer $b$ satisfying

\[(n + 2)^k - n^k \leq b2^{[n/2]+1} \leq (n + 2)^k + n^k.\]

If $k = 1$, condition (iii) is equivalent to the condition $1 \leq n \leq 5$ or $n = 7$, and if $k = 2$, it is equivalent to the condition $1 \leq n \leq 17$. (Note that $2^{[n/2]} > n + 1$ for $n = 6$ or $n \geq 8$, and that $2^{[n/2]} > n^2 + 2n + 2$ for $n \geq 18$.) Hence Theorem B is a generalization of Theorems 1.3 and 1.4.

This paper is arranged as follows. In Section 2 we establish the formulas on the power $\tau^k = \tau(RP^n) \otimes \cdots \otimes \tau(RP^n)$ of the tangent bundle $\tau(RP^n)$ of the real projective $n$-space $RP^n$. In Section 3 we apply the results in Section 2 to the problem of extendibility and stable extendibility of the $k$-fold power $\tau^k$ and prove Theorem A by using Theorem 4.1 in [3]. In Section 4 we establish the formulas on the complexification $c\tau^k = c(\tau(RP^n) \otimes \cdots \otimes \tau(RP^n))$ of $\tau^k$. In Section 5 we apply the results in Section 4 to the problem of extendibility and stable extendibility of $c\tau^k$ and prove Theorem B by using Theorem 2.1 in [3].
2. The $k$-fold power of the tangent bundle of $\mathbb{RP}^n$

In this section we establish the formulas on the $k$-fold power of the tangent bundle $\tau = \tau(\mathbb{RP}^n)$. Let $\xi_n$ denote the canonical line bundle over $\mathbb{RP}^n$.

**Lemma 2.1.** Let $\tau^k = \tau(\mathbb{RP}^n) \otimes \cdots \otimes \tau(\mathbb{RP}^n)$ (k-fold) denote the $k$-fold power of the tangent bundle $\tau = \tau(\mathbb{RP}^n)$. Then, for any positive integer $r$, the following hold in the Grothendieck group $KO(\mathbb{RP}^n)$:

1. $\tau^{2r-1} = 2^{-1}\{(n + 2)^{2r-1} + n^{2r-1}\} \xi_n - 2^{-1}\{(n + 2)^{2r-1} - n^{2r-1}\}$,
2. $\tau^{2r} = -2^{-1}\{(n + 2)^{2r} - n^{2r}\} \xi_n + 2^{-1}\{(n + 2)^{2r} + n^{2r}\}$.

**Proof.** It is well known that $\tau = (n + 1)\xi_n - 1$ in $KO(\mathbb{RP}^n)$. Hence formula (1) clearly holds for $r = 1$.

Assume that formula (1) holds for $r \geq 1$. Then

$\tau^{2r} = \tau \otimes \tau^{2r-1}$

$= \{(n + 1)\xi_n - 1\}[2^{-1}\{(n + 2)^{2r-1} + n^{2r-1}\} \xi_n - 2^{-1}\{(n + 2)^{2r-1} - n^{2r-1}\}]$

$= -2^{-1}\{(n + 2)^{2r} - n^{2r}\} \xi_n + 2^{-1}\{(n + 2)^{2r} + n^{2r}\}$,

since $\xi_n \otimes \xi_n = 1$. So formula (2) holds for $r \geq 1$.

Assume that formula (2) holds for $r \geq 1$. Then

$\tau^{2r+1} = \tau \otimes \tau^{2r}$

$= \{(n + 1)\xi_n - 1\}[2^{-1}\{(n + 2)^{2r} + n^{2r}\} - 2^{-1}\{(n + 2)^{2r} - n^{2r}\} \xi_n]$

$= 2^{-1}\{(n + 2)^{2r+1} + n^{2r+1}\} \xi_n - 2^{-1}\{(n + 2)^{2r+1} - n^{2r+1}\}$,

since $\xi_n \otimes \xi_n = 1$. So formula (1) holds for $r + 1$.

Hence formulas (1) and (2) hold for any positive integer $r$ by induction on $r$. □

The following result is used in our proofs.

**Theorem 2.2** (cf. [2] Theorem 1.5, p. 100). Two $t$-dimensional $F$-vector bundles over an $n$-dimensional CW-complex which are stably equivalent are equivalent if $\langle(n + 2)/f - 1\rangle \leq t$, where $\langle x \rangle$ denotes the smallest integer $n$ with $x \leq n$ and $f = 1$ or 2 according as $F = R$ or $C$.

We establish the formula on $\tau^k$, as follows.

**Theorem 2.3.** Let $\tau^k = \tau(\mathbb{RP}^n) \otimes \cdots \otimes \tau(\mathbb{RP}^n)$ (k-fold) denote the $k$-fold power of the tangent bundle $\tau = \tau(\mathbb{RP}^n)$. Then, for any positive integer $r$, the following hold:

1. $\tau^{2r-1} \oplus 2^{-1}\{(n + 2)^{2r-1} - n^{2r-1}\} = 2^{-1}\{(n + 2)^{2r-1} + n^{2r-1}\} \xi_n$,
2. $\tau^{2r} \oplus 2^{-1}\{(n + 2)^{2r} - n^{2r}\} \xi_n = 2^{-1}\{(n + 2)^{2r} + n^{2r}\}$,

where, in the equalities (1) and (2), a positive integer $k$ denotes the $k$-dimensional trivial bundle over $\mathbb{RP}^n$ and $\oplus$ denotes the Whitney sum.

**Proof.** (1) By Lemma 2.1, we have

$\tau^{2r-1} \oplus 2^{-1}\{(n + 2)^{2r-1} - n^{2r-1}\} = 2^{-1}\{(n + 2)^{2r-1} + n^{2r-1}\} \xi_n$

in $KO(\mathbb{RP}^n)$. Since

$\dim[\tau^{2r-1} \oplus 2^{-1}\{(n + 2)^{2r-1} - n^{2r-1}\}]$

$= 2^{-1}\{(n + 2)^{2r-1} + n^{2r-1}\} > n = \dim \mathbb{RP}^n$, 

the equality
\[ \tau^{2r-1} \oplus 2^{-1}\{(n+2)^{2r-1} - n^{2r-1}\} \xi_n = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1}\} \xi_n \]
holds as R-vector bundles by Theorem 2.2.

(2) By Lemma 2.1(2), we have
\[ \tau^{2r} + 2^{-1}\{(n+2)^{2r} - n^{2r}\} \xi_n = 2^{-1}\{(n+2)^{2r} + n^{2r}\} \]
in KO(RP^n). Since \( \dim\{\tau^{2r} + 2^{-1}\{(n+2)^{2r} - n^{2r}\}\xi_n\} = 2^{-1}\{(n+2)^{2r} + n^{2r}\} > n = \dim RP^n \), the equality
\[ \tau^{2r} \oplus 2^{-1}\{(n+2)^{2r} - n^{2r}\} \xi_n = 2^{-1}\{(n+2)^{2r} + n^{2r}\} \]
holds as R-vector bundles by Theorem 2.2. \( \square \)

Moreover, the next theorem follows from Lemma 2.1.

**Theorem 2.4.** For any positive integer \( r \) and any integer \( a \), the following hold in KO(RP^n):

(1) \( \tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - a2^{\phi(n)+1}\} \xi_n + 2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r-1} + n^{2r-1}\} \)

(2) \( \tau^{2r} = 2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r} + n^{2r}\} \xi_n + 2^{-1}\{(n+2)^{2r} + n^{2r} - a2^{\phi(n)+1}\} \).

**Proof.** Subtracting \( a2^{\phi(n)}(\xi_n - 1) = 0 \) (cf. [1, Theorem 7.4]) from equality (1) in Lemma 2.1 we have equality (1) above, and adding \( a2^{\phi(n)}(\xi_n - 1) = 0 \) to equality (2) in Lemma 2.1 we have the equality (2) above. \( \square \)

3. **Extensibility and stable extensibility of the k-fold power**

\[ \tau^k = \tau(RP^n) \otimes \cdots \otimes \tau(RP^n) \]

**Theorem 3.1.** Assume that there is an integer \( a \) satisfying
\[ (n+2)^k - n^k \leq a2^{\phi(n)+1} \leq (n+2)^k + n^k. \]
Then \( \tau^k \) is extendible to RP\(^m\) for every \( m \geq n \).

**Proof.** If \( k = 1 \), the inequalities imply \( a = 1 \) and \( n = 1, 3 \) or 7, and if \( n = 1, 3 \) or 7, \( \tau(RP^n) \) is trivial. Hence the results clearly hold for \( n = 1 \) or \( k = 1 \). So we may restrict our attention to the case \( n > 1 \) and \( k > 1 \).

In case \( k \) is odd, let \( k = 2r - 1 \), where \( r \) is an integer \( > 1 \). Then, by the assumption, we have
\[ 2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - a2^{\phi(n)+1}\} \geq 0 \]
and
\[ 2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r-1} + n^{2r-1}\} \geq 0. \]
Hence Theorem 2.1(1) implies that the equality
\[ \tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^{2r-1} - a2^{\phi(n)+1}\} \xi_n \oplus 2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r-1} + n^{2r-1}\} \]
holds by Theorem 2.2 since \( \dim\tau^{2r-1} = n^{2r-1} > n = \dim RP^n \) for \( n > 1 \) and \( r > 1 \). So \( \tau^{2r-1} \) is extendible to RP\(^m\) for every \( m \geq n \), since \( \xi_n \) and the trivial bundle over RP\(^n\) are extendible to RP\(^m\) for every \( m \geq n \).

In case \( k \) is even, let \( k = 2r \), where \( r \) is a positive integer. Then, by the assumption, we have
\[ 2^{-1}\{a2^{\phi(n)+1} - (n+2)^{2r} + n^{2r}\} \geq 0 \]
and
\[2^{-1}\{(n + 2)^{2r} + n^{2r} - a2^{\phi(n) + 1}\} \geq 0.\]

Hence Theorem 2.3(2) implies that the equality
\[\tau^{2r} = 2^{-1}\{a2^{\phi(n) + 1} - (n + 2)^{2r} + n^{2r}\}e_n \oplus 2^{-1}\{(n + 2)^{2r} + n^{2r} - a2^{\phi(n) + 1}\}\]
holds by Theorem 2.2 since \(\dim \tau^{2r} = n^{2r} > n = \dim RP^n\) for \(n > 1\) and \(r > 0\). So \(\tau^{2r}\) is extendible to \(RP^m\) for every \(m \geq n\).

The following result is Theorem 4.1 in [8] which is the stably extendible version of Theorem 6.2 in [6].

**Theorem 3.2.** Let \(\xi\) be a \(t\)-dimensional \(R\)-vector bundle over \(RP^n\). Assume that there is an integer \(a\) satisfying Theorem 3.2. Assume that there is an integer \(a\) satisfying \(\tau^{2r} = n^{2r} > n = \dim RP^n\) for \(n > 1\) and \(r > 0\).

**Theorem 3.3.** Assume that there is an integer \(a\) satisfying
\[(n + 2)^k + n^k - 2^{\phi(n)+1} < a2^{\phi(n)+1} < (n + 2)^k - n^k.\]

Then \(\tau^k\) is not stably extendible to \(RP^m\) for every \(m \geq 2^{-1}\{(n + 2)^k + n^k - a2^{\phi(n) + 1}\}\) if \(k\) is odd, and for every \(m \geq 2^{-1}\{(a + 1)2^{\phi(n) + 1} - (n + 2)^k + n^k\}\) if \(k\) is even.

**Proof.** If \(k\) is odd, let \(k = 2r - 1\). Then putting
\[\zeta = \tau^{2r-1}, \quad t = n^{2r-1} \quad \text{and} \quad l = 2^{-1}\{(n + 2)^{2r-1} - n^{2r-1} - a2^{\phi(n)+1}\}\]
in Theorem 3.2 we obtain the result by Theorem 2.3(1) and Theorem 3.2 since \(t + l < 2^{\phi(n)}\) and \(l > 0\) by the assumption.

If \(k\) is even, let \(k = 2r\). Then putting
\[\zeta = \tau^{2r}, \quad t = n^{2r} \quad \text{and} \quad l = 2^{-1}\{(a + 1)2^{\phi(n)+1} - (n + 2)^{2r} - n^{2r}\}\]
in Theorem 3.2 we obtain the result by Theorem 2.3(2) and Theorem 3.2 since \(t + l < 2^{\phi(n)}\) and \(l > 0\) by the assumption.

**Proof of Theorem A.** (i) clearly implies (ii). (iii) implies (i) by Theorem 3.1. To show that (ii) implies (iii), we prove the contraposition. Assume that every integer \(a\) satisfies
\[a2^{\phi(n)+1} < (n + 2)^k - n^k \quad \text{or} \quad (n + 2)^k + n^k < a2^{\phi(n)+1}.\]

Assume that there are integers \(a\) with \(a2^{\phi(n)+1} < (n + 2)^k - n^k\). Then we define \(A\) as the maximum integer such that \(A2^{\phi(n)+1} < (n + 2)^k - n^k\). If \(A\) satisfies \(A2^{\phi(n)+1} \leq (n + 2)^k + n^k - 2^{\phi(n)+1}\), we have \((n + 2)^k - n^k \leq (A + 1)2^{\phi(n)+1} \leq (n + 2)^k + n^k\), and these are inconsistent with our assumption. Hence \(A\) satisfies \((n + 2)^k + n^k - 2^{\phi(n)+1} < A2^{\phi(n)+1} < (n + 2)^k - n^k\). So, by Theorem 3.3, \(\tau^k\) is not stably extendible to \(RP^m\) for every \(m \geq 2^{-1}\{(n + 2)^k + n^k - A2^{\phi(n)+1}\}\) if \(k\) is odd, and for every \(m \geq 2^{-1}\{(A + 1)2^{\phi(n)+1} - (n + 2)^k + n^k\}\) if \(k\) is even.

Assume that there are integers \(a\) with \((n + 2)^k + n^k < a2^{\phi(n)+1}\). Then we define \(B\) as the minimum integer such that \((n + 2)^k + n^k < B2^{\phi(n)+1}\). If \(B\) satisfies \(B2^{\phi(n)+1} \geq (n + 2)^k - n^k + 2^{\phi(n)+1}\), we have \((n + 2)^k - n^k \leq (B - 1)2^{\phi(n)+1} \leq (n + 2)^k + n^k\), and these are inconsistent with our assumption. Hence \(B\) satisfies \((n + 2)^k + n^k - 2^{\phi(n)+1} < (B - 1)2^{\phi(n)+1} < (n + 2)^k - n^k\). So, by Theorem 3.3, \(\tau^k\) is not stably extendible to \(RP^m\) for every \(m \geq 2^{-1}\{(n + 2)^k + n^k - (B - 1)2^{\phi(n)+1}\}\) if \(k\) is odd, and for every \(m \geq 2^{-1}\{(B2^{\phi(n)+1} - (n + 2)^k + n^k)\}\) if \(k\) is even. \(\square\)
4. The complexification of the k-fold power of $\tau(\mathbb{RP}^n)$

Complexifying the equalities (1) and (2) in Lemma 2.1, we immediately have

**Lemma 4.1.** Let $\tau^k = c(\tau(\mathbb{RP}^n) \otimes \cdots \otimes \tau(\mathbb{RP}^n))$ denote the complexification of the k-fold power $\tau^k$ of the tangent bundle $\tau = \tau(\mathbb{RP}^n)$. Then, for any positive integer $k$, the following hold in the Grothendieck group $K(\mathbb{RP}^n)$:

1. $\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^2r-1\}c_{\xi_n} - 2^{-1}\{(n+2)^{2r-1} - n^2r-1\}$,
2. $\tau^{2r} = -2^{-1}\{(n+2)^{2r} - n^2r\}c_{\xi_n} + 2^{-1}\{(n+2)^{2r} + n^2r\}$.

Complexifying the equalities (1) and (2) in Theorem 2.3, we immediately have

**Theorem 4.2.** For the complexification $\tau^k = c(\tau(\mathbb{RP}^n) \otimes \cdots \otimes \tau(\mathbb{RP}^n))$ of the k-fold power $\tau^k$ of the tangent bundle $\tau = \tau(\mathbb{RP}^n)$, the following hold:

1. $\tau^{2r-1} \otimes 2^{-1}\{(n+2)^{2r-1} - n^2r-1\} = 2^{-1}\{(n+2)^{2r-1} + n^2r-1\}c_{\xi_n}$,
2. $\tau^{2r} \otimes 2^{-1}\{(n+2)^{2r} - n^2r\}c_{\xi_n} = 2^{-1}\{(n+2)^{2r} + n^2r\}$.

Furthermore, the next theorem follows from Lemma 4.1.

**Theorem 4.3.** For any positive integer $r$ and any integer $b$, the following hold in $K(\mathbb{RP}^n)$:

1. $\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^2r-1 - b2^{[n/2]+1}\}c_{\xi_n} + 2^{-1}\{b2^{[n/2]+1} - (n+2)^{2r-1} + n^2r-1\}$,
2. $\tau^{2r} = 2^{-1}\{b2^{[n/2]+1} - (n+2)^{2r} + n^2r\}c_{\xi_n} + 2^{-1}\{(n+2)^{2r} + n^2r - b2^{[n/2]+1}\}$.

*Proof.* Subtracting $b2^{[n/2]}(\xi_n - 1) = 0$ (cf. [1] Theorem 7.3) from equality (1) in Lemma 4.1 we have equality (1) above, and adding $b2^{[n/2]}(\xi_n - 1) = 0$ to equality (2) in Lemma 4.1 we have equality (2) above.

5. Extendibility and Stable Extendibility of the Complexification $\tau^k = c(\tau(\mathbb{RP}^n) \otimes \cdots \otimes \tau(\mathbb{RP}^n))$

The proofs of the following Theorems 5.1 and 5.3 are parallel to those of Theorems 4.1 and 4.3, respectively.

**Theorem 5.1.** Assume that there is an integer $b$ satisfying

$$(n+2)^k - n^k \leq b2^{[n/2]+1} \leq (n+2)^k + n^k.$$  

Then $\tau^k$ is extendible to $\mathbb{RP}^m$ for every $m \geq n$.

*Proof.* In case $k$ is odd, let $k = 2r - 1$, where $r$ is a positive integer. Then, by the assumption, we have

$2^{-1}\{(n+2)^{2r-1} + n^2r-1 - b2^{[n/2]+1}\} \geq 0$

and

$2^{-1}\{b2^{[n/2]+1} - (n+2)^{2r-1} + n^2r-1\} \geq 0$.

Hence Theorem 4.3(1) implies that the equality

$\tau^{2r-1} = 2^{-1}\{(n+2)^{2r-1} + n^2r-1 - b2^{[n/2]+1}\}c_{\xi_n} + 2^{-1}\{b2^{[n/2]+1} - (n+2)^{2r-1} + n^2r-1\}$

holds by Theorem 2.2, since $\dim \tau^k = n^{2r-1} \geq (n+2)/2 - 1 = \langle n/2 \rangle$. So $\tau^{2r-1}$ is extendible to $\mathbb{RP}^m$ for every $m \geq n$, since $c_{\xi_n}$ and the trivial bundle over $\mathbb{RP}^n$ are extendible to $\mathbb{RP}^m$ for every $m \geq n$.  


In case $k$ is even, let $k = 2r$, where $r$ is a positive integer. Then, by the assumption, we have

$$2^{-1}\{b^{[n/2]+1} - (n + 2)^{2r} + n^{2r}\} \geq 0$$

and

$$2^{-1}\{(n + 2)^{2r} + n^{2r} - b^{[n/2]+1}\} \geq 0.$$ 

Hence Theorem 4.2) implies that the equality

$$c_\tau^{2r} = 2^{-1}\{b^{[n/2]+1} - (n + 2)^{2r} + n^{2r}\}c_\xi_n \oplus 2^{-1}\{(n + 2)^{2r} + n^{2r} - b^{[n/2]+1}\}$$

holds by Theorem 2.2, since $\dim c_\tau^{2r} = n^{2r} \geq \langle (n + 2)/2 - 1 \rangle = \langle n/2 \rangle$. So $c_\tau^{2r}$ is extendible to $RP^m$ for every $m \geq n$. □

The following result is Theorem 2.1 in [8] which is the stably extendible version of Theorem 4.2 for $d = 1$ in [7].

**Theorem 5.2.** Let $\zeta$ be a $t$-dimensional $C$-vector bundle over $RP^n$. Assume that there is a positive integer $l$ such that $\zeta$ is stably equivalent to $(t + l)c_\xi_n$, and $l + t < 2^{[n/2]}$. Then $n < 2t + 2l$ and $\zeta$ is not stably extendible to $RP^m$ for every $m \geq 2t + 2l$.

**Theorem 5.3.** Assume that there is an integer $b$ satisfying

$$(n + 2)^2 + b^k - 2^{[n/2]+1} < b^{[n/2]+1} < (n + 2)^k - n^k.$$ 

Then $c_\tau^k$ is not stably extendible to $RP^m$ for every $m \geq (n + 2)^k + n^k - b^{[n/2]+1}$ if $k$ is odd, and for every $m \geq (b + 1)2^{[n/2]+1} - (n + 2)^k + n^k$ if $k$ is even.

**Proof.** If $k$ is odd, let $k = 2r - 1$. Then putting

$$\zeta = c_\tau^{2r-1}, \quad t = n^{2r-1} \quad \text{and} \quad l = 2^{-1}\{(n + 2)^{2r-1} - n^{2r-1} - b^{[n/2]+1}\}$$

in Theorem 4.2, we obtain the result by Theorem 4.3(1) and Theorem 5.2 since $t + l < 2^{[n/2]}$ and $l > 0$ by the assumption.

If $k$ is even, let $k = 2r$. Then putting

$$\zeta = c_\tau^{2r}, \quad t = n^{2r} \quad \text{and} \quad l = 2^{-1}\{(b + 1)2^{[n/2]+1} - (n + 2)^{2r} - n^{2r}\}$$

in Theorem 4.2, we obtain the result by Theorem 4.3(2) and Theorem 5.2 since $t + l < 2^{[n/2]}$ and $l > 0$ by the assumption. □

**Proof of Theorem B.** (i) implies (ii) clearly. (iii) implies (i) by Theorem 5.1. To show that (ii) implies (iii), we prove the contraposition. Assume that every integer $b$ satisfies

$$b^{[n/2]+1} < (n + 2)^k - n^k \quad \text{or} \quad (n + 2)^k + n^k < b^{[n/2]+1}.$$ 

Assume that there are integers $b$ with $b^{[n/2]+1} < (n + 2)^k - n^k$. Then we define $C$ as the maximum integer such that $C^{[n/2]+1} < (n + 2)^k - n^k$. If $C$ satisfies $C^{[n/2]+1} < (n + 2)^k - n^k - 2^{[n/2]+1}$, we have $(n + 2)^k - n^k \leq (C + 1)2^{[n/2]+1} < (n + 2)^k + n^k$, and these are inconsistent with our assumption. Hence $C$ satisfies $(n + 2)^k + n^k - 2^{[n/2]+1} < C^{[n/2]+1} < (n + 2)^k - n^k$. So, by Theorem 5.3 $c_\tau^k$ is not stably extendible to $RP^m$ for every $m \geq (n + 2)^k + n^k - C^{[n/2]+1}$ if $k$ is odd, and for every $m \geq (C + 1)2^{[n/2]+1} - (n + 2)^k + n^k$ if $k$ is even.

Assume that there are integers $b$ with $(n + 2)^k + n^k < b^{[n/2]+1}$. Then we define $D$ as the minimum integer such that $(n + 2)^k + n^k < D^{[n/2]+1}$. If $D$ satisfies $D^{[n/2]+1} \geq (n + 2)^k - n^k + 2^{[n/2]+1}$, we have $(n + 2)^k - n^k \leq (D - 1)2^{[n/2]+1} \leq (n + 2)^k + n^k$, and these are inconsistent with our assumption. Hence $D$ satisfies
\( (n + 2)^k + n^k - 2^{\lfloor n/2 \rfloor + 1} < (D - 1)2^{\lfloor n/2 \rfloor + 1} < (n + 2)^k - n^k \). So, by Theorem 5.3 \( c_T^k \) is not stably extendible to \( RP^m \) for every \( m \geq (n + 2)^k + n^k - (D - 1)2^{\lfloor n/2 \rfloor + 1} \) if \( k \) is odd, and for every \( m \geq D2^{\lfloor n/2 \rfloor + 1} - (n + 2)^k + n^k \) if \( k \) is even. \( \square \)

**References**


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