SPECTRAL PICTURES OF $AB$ AND $BA$

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Abstract. The spectral pictures of products $AB$ and $BA$ of Banach space operators are compared; in particular when one of them is ‘of index zero’.

We shall describe an operator $A$ in the algebra $B(X)$ of bounded linear operators on a Banach space $X$ as being of index zero whenever there is Banach space isomorphism

$$A^{-1}(0) \cong X/\text{cl} A(X).$$

For example a Fredholm operator has this property if and only if its Fredholm index is zero. In particular, finite-dimensional operators and normal operators acting on a Hilbert space are of index zero. A ‘quasi-affinity’, in the sense of an operator which is one-to-one and of dense range, is another kind of example. D. Djordjevic [Dj] has essentially noticed that

1. Theorem. If $A \in B(X)$ is of index zero and $B \in B(X)$, there is the implication

   $$(1.1) \quad AB \text{ invertible} \iff BA \text{ invertible}$$

   and

   $$(1.2) \quad AB \text{ Fredholm} \iff BA \text{ Fredholm},$$

   in which case

   $$(1.3) \quad \text{index}(AB) = \text{index}(B) = \text{index}(BA).$$

Proof. If $BA$ is invertible, then $A$ is left invertible, in particular one-to-one with closed range; hence by index zero it has dense range, which now makes it invertible. For invertible $A$ the implication (1.1) is clear. If conversely $AB$ is invertible, then $A$ is right invertible, in particular onto; hence by index zero it is one-to-one and again invertible. This gives (1.1) both ways; towards (1.2) argue that if $BA$ is Fredholm, then $A$ is upper semi-Fredholm, having closed range and finite-dimensional null space. If also $A$ is “index zero” in the sense of (0.1), then its closed range must have finite codimension making it Fredholm, in which case (1.2) is clear. Conversely if $AB$ is Fredholm, then $A$ is lower semi-Fredholm, in the sense of having a closed range of finite codimension; hence it is also a finite-dimensional null space and again
Fredholm. This gives (1.2) both ways; finally since index$(A) = 0$ also, the usual index-of-product formula gives (1.3).

If $A \in B(X)$, write $\sigma(A)$, $\sigma_{\text{left}}(A)$, $\sigma_{\text{right}}(A)$, $\sigma_{\text{ess}}(A)$, $\sigma_{\text{ess}}^\text{left}(A)$, and $\sigma_{\text{ess}}^\text{right}(A)$ for the spectrum, the left spectrum, the right spectrum, the essential spectrum, the left essential spectrum, and the right essential spectrum, respectively, of $A$. If $A \in B(X)$, a hole in $\sigma_{\text{ess}}(A)$ is a bounded component of $\mathbb{C} \setminus \sigma_{\text{ess}}(A)$ and a pseudohole in $\sigma_{\text{ess}}(A)$ is a component of $\sigma_{\text{ess}}(A) \setminus \sigma_{\text{ess}}^\text{left}(A)$ or $\sigma_{\text{ess}}(A) \setminus \sigma_{\text{ess}}^\text{right}(A)$. The spectral picture of $A$, denoted $\mathcal{SP}(A)$, is the structure consisting of the set $\sigma_{\text{ess}}(A)$, the collection of holes and pseudoholes in $\sigma_{\text{ess}}(A)$, and the indices associated with those holes and pseudoholes. Write $K(X)$ for the ideals of compact operators on $X$.

We now have:

2. Theorem. If $A \in B(X)$ is of index zero and $B \in B(X)$, then

\begin{align*}
(2.1) & \quad \sigma(BA) = \sigma(AB); \\
(2.2) & \quad \sigma_{\text{left}}(AB) \subseteq \sigma_{\text{left}}(BA) \quad \text{and} \quad \sigma_{\text{right}}(AB) \subseteq \sigma_{\text{right}}(BA); \\
(2.3) & \quad \sigma_{\text{ess}}(BA) = \sigma_{\text{ess}}(AB); \\
(2.4) & \quad \sigma_{\text{ess}}^\text{left}(AB) \subseteq \sigma_{\text{ess}}^\text{left}(BA) \quad \text{and} \quad \sigma_{\text{ess}}^\text{right}(AB) \subseteq \sigma_{\text{ess}}^\text{right}(BA).
\end{align*}

Further if $0 \in \mathbb{C}$ is not in any pseudohole of either $AB$ or $BA$, then
\begin{equation}
(2.5) \quad \mathcal{SP}(AB) = \mathcal{SP}(BA).
\end{equation}

**Proof.** It is familiar ([Ba], [GGK], [Ha], [LYR]) that
\begin{equation}
(2.6) \quad \omega(AB) \setminus \{0\} = \omega(BA) \setminus \{0\}
\end{equation}
for the spectrum $\omega = \sigma$, as well as for the left, right, essential, left essential and right essential spectrum: thus (2.1) and (2.3) follow from (1.1) and (1.2). For the same reason (2.2) and (2.4) depend only on the fate of $0 \in \mathbb{C}$. Thus if $BA$ has a left inverse, then $A$ is left invertible, which together with having “index zero” makes it invertible, which now gives $AB$ a left inverse:
\[ CBA = I \implies CB = A^{-1} \implies ACB = I \]
means that $B$ is now left invertible, and hence also $AB$. The argument for right invertibility is exactly the same. For (2.4) suppose $BA$ is upper semi-Fredholm. Then by Atkinson’s theorem $BA$ is left invertible modulo $K(X)$ and so $I - U(BA) \in K(X)$ for some $U \in B(X)$. Note that $A$ is upper semi-Fredholm and hence by assumption, it is Fredholm of index zero. Remembering ([Ha] Theorem 6.5.2) that a Fredholm operator of index zero can be written as the sum of an invertible and a finite rank operator, write $A = V + K$, where $V$ is invertible and $K$ is of finite rank. Then
\[ I - UB(V + K) \in K(X) \implies I - UBV \in K(X) \implies I - VUB \in K(X), \]
which implies $B$ is upper semi-Fredholm and hence, so is $AB$, giving the first inclusion of (2.4), and the argument for the second is the same. Finally, to see (2.5), it is effective to remember ([GGK] p. 38) that with no restriction on $A$,
\begin{equation}
(2.7) \quad \begin{pmatrix} AB - I & 0 \\ 0 & I \end{pmatrix} = F \begin{pmatrix} BA - I & 0 \\ 0 & I \end{pmatrix} E,
\end{equation}
where $F$ is a projection, as desired.
where

\[
E := \begin{pmatrix} B & I \\ AB - I & A \end{pmatrix} \quad \text{and} \quad F := \begin{pmatrix} A & I - AB \\ -I & B \end{pmatrix}
\]

are both invertible. Thus from (2.7),

(2.8) \[ \text{index}(AB - I) = \text{index}(F) + \text{index}(BA - I) + \text{index}(E) = \text{index}(BA - I). \]

This implies that whenever \( \lambda \neq 0 \) is in a hole or pseudohole common to \( AB \) and \( BA \), then the value of the index for that pseudohole is the same for both. Thus if \( 0 \in \mathbb{C} \) is not in any pseudohole of either \( AB \) or \( BA \), then we can conclude that \( \mathcal{SP}(AB) = \mathcal{SP}(BA) \). This proves (2.5). □

We would remark that \( 0 \) can be in a pseudohole of \( AB \) but not in a pseudohole of \( BA \), or vice versa, but that if \( 0 \) is in the polynomially convex hull of a pseudohole of \( AB \), then it is also in the polynomially convex hull of a pseudohole of \( BA \), and vice versa. On the other hand, none of the inclusions in (2.2) and (2.4) can be replaced by equality:

3. Example. If \( X = \ell_2 \) and

\[
A(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) = (0, x_2, 0, x_4, 0, x_6, \ldots),
\]

\[
B(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) = (0, x_1, 0, x_2, 0, x_3, \ldots),
\]

\[
B'(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) = (x_2, x_4, x_6, x_8, x_{10}, x_{12}, \ldots),
\]

then \( A \) is of index zero, \( AB \) is left invertible but \( BA \) is not upper semi-Fredholm, while \( B'A \) is right invertible but \( AB' \) is not lower semi-Fredholm. Also \( BA \) is of index zero while \( AB \) is not.

**Proof.** Observe that

\[ BB' = A \neq I = B'B; \quad AB = B \neq BA; \quad B'A = B' \neq AB', \]

and look at the null space of \( BA \) and the closure of the range of \( AB' \). □

In Example 3, a straightforward calculation shows that \( \mathcal{SP}(AB) \) and \( \mathcal{SP}(BA) \) has only one pseudohole \( H_0 \) whose polynomially convex hull contains 0: with \( \mathbb{D} \) the open unit disk

\[
H_0(AB) = \mathbb{D} \quad \text{with index} \; H_0(AB) = -\infty;
\]

\[
H_0(BA) = \mathbb{D} \setminus \{0\} \quad \text{with index} \; H_0(BA) = -\infty.
\]

On the other hand, from (2.7) we can see that for each \( \lambda \neq 0 \),

\[
(AB - \lambda I)^{-1}(0) \cong (BA - \lambda I)^{-1}(0) \quad \text{and} \quad X/\text{cl}(AB - \lambda I)(X) \cong X/\text{cl}(BA - \lambda I)(X),
\]

which implies that with no restriction on \( A \) and \( B \),

(3.4) \[ AB - \lambda I \; \text{"of index zero"} \iff BA - \lambda I \; \text{"of index zero"}, \quad \lambda \neq 0. \]

However if \( \lambda = 0 \), Example 3 shows that (3.4) may fail though each of \( A, B, AB \) and \( BA \) has closed range and \( A \) is of index zero. As a trivial sort of dual to Theorem 1, it is clear that

(3.5) \[ A \; \text{invertible} \implies (BA \; \text{index zero} \iff AB \; \text{index zero}); \]

by Example 3 this does not extend to \( A \) of index zero.
We however have:

4. Proposition. If $X$ is separable Hilbert space and if each of $AB$, $BA$ and $B$ has closed range, then if $A$ is also Fredholm there is equivalence

\begin{equation}
BA \text{ "of index zero" } \iff AB \text{ "of index zero" }.
\end{equation}

Proof. If either $AB$ or $BA$ is Fredholm, then this is contained in (2.3), and if either $BA$ is upper semi-Fredholm or $AB$ is lower semi-Fredholm, then this is contained in (2.4). Thus we may assume that the null space of $BA$ is infinite dimensional and that the range of $AB$ is of finite codimension: on separable space $X$ this implies

\begin{equation}
(AB)^{-1}(0) \equiv X \cong X/(AB)X;
\end{equation}

we therefore have to show

\begin{equation}
(AB)^{-1}(0) \equiv X \iff X/(BA)X \cong X.
\end{equation}

We claim

\begin{equation}
(AB)^{-1}(0) \equiv X \implies B^{-1}(0) \equiv X \iff (AB)^{-1}(0) \equiv X
\end{equation}

and

\begin{equation}
X/(AB)X \cong X \iff X/B(X) \cong X \iff X/(BA)X \cong X;
\end{equation}

this is because of the isomorphisms [Ha] (6.5.4.6), (6.5.4.7)]

\begin{align}
(AB)^{-1}(0)/B^{-1}(0) & \equiv A^{-1}(0) \cap B(X); \quad AX/(AB)X \cong X/(BX + A^{-1}(0)), \\
(AB)^{-1}(0)/A^{-1}(0) & \equiv B^{-1}(0) \cap A(X); \quad BX/(BA)X \cong X/(AX + B^{-1}(0)).
\end{align}

If $A^{-1}(0)$ is finite dimensional, then the first part of (4.6) gives the second part of (4.4), while the first comes from the first part of (4.7). If $A(X)$ is of finite codimension, then the second part of (4.7) gives the second part of (4.5), while the first comes from the second part of (4.6); alternatively take adjoints in (4.4). \hfill \Box

The spectral picture $SP(T)$ determines whether an operator is “quasitriangular” [AFV], and whether it is “compalent” to another operator [BDF].

Recall ([Pe Definition 4.8]) that $T \in B(H)$ for a Hilbert space $H$ is called \textit{quasitriangular} if there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of projections of finite rank in $B(H)$ that converges strongly to 1 and satisfies $||P_nTP_n - TP_n|| \to 0$. The set of quasitriangular operators can be characterized as the set of all sums of the form $T_0 + K$, where $T_0$ is triangular and $K \in K(H)$ (cf. [Pe Corollary 4.19]). We have:

5. Corollary. If $A \in B(H)$ is of index zero, then $AB$ is quasitriangular if and only if $BA$ is quasitriangular.

Proof. By Apostol, Foias, and Voiculescu [AFV] the operator $T$ is quasitriangular if and only if $SP(T)$ contains no hole or pseudohole with negative index. \hfill \Box

Recall that $T \in B(H)$ for a Hilbert space $H$ is called \textit{essentially normal}, if $T^*T - TT^* \in K(H)$ and that operators $T_1$ and $T_2$ in $B(H)$ are said to be \textit{compalent} if there exists a unitary operator $W \in B(H)$ and a compact operator $K \in K(H)$ such that $WT_1W^* + K = T_2$. Then by the beautiful Brown-Douglas-Fillmore
6. Corollary. Let $A \in B(H)$ be of index zero. If $AB$ and $BA$ are essentially normal, then $AB$ and $BA$ are compotent.

Proof. If $AB$ and $BA$ are essentially normal, then neither of them have any pseudoholes, so that (2.5) holds. Now the result follows from the Brown-Douglas-Fillmore theorem—if $T_1$ and $T_2$ are essentially normal, then $T_1$ and $T_2$ are compotent if and only if $SP(T_1) = SP(T_2)$. □

We write $Lat(A)$ for the invariant subspace lattice of $A \in B(X)$, and recall that a “quasiaffinity” is one-to-one with dense range; obviously if $A$ is not a quasiaffinity, then either its null space or the closure of its range will be in $Lat(A)$. We observe

7. Proposition. If $A, B$ in $B(X)$ are such that $BA$ is a quasiaffinity, then

(7.1) $Lat(AB)$ nontrivial $\implies$ $Lat(AB)$ nontrivial.

Proof. By assumption $A$ is one-to-one and $B$ has dense range. We claim that if $N \in Lat(AB)$ is nontrivial, then

(7.2) $M = cl(AN) \implies M \in Lat(AB)$ with $\{0\} \neq M \neq X$.

The invariance of $M$ is clear; $M \neq \{0\}$ is because $A$ is one-to-one and $N$ is nonzero; $M \neq X$ is because $B$ is dense and $N \neq X$. □

Not everything in $Lat(AB)$ need be derived in this way from $Lat(AB)$: for example if $A$ is the forward and $B$ the backward shift, look at $(AB)^{-1}(0)$.

An operator $T \in B(H)$ for a Hilbert space $H$ has a unique polar decomposition $T = U[T]$, where $[T] = (T^*T)^{1/2}$ and $U$ is a partial isometry with the same null space as $T$. Associated with $T$, there is a useful related operator $T_\epsilon := [T]^\epsilon U[T]^{1-\epsilon}$ ($0 \leq \epsilon \leq 1$) called the generalized Aluthge transform of $T$ of order $\epsilon$ ([AI]). If $\epsilon = \frac{1}{2}$ this really is the Aluthge transform while if $\epsilon = 0$ we get back $T$ itself; if $\epsilon = 1$, then this is what Carl Pearcy has called the “Duggal transplant” of $T$.

We recapture [JKP, Corollary 1.12]:

8. Corollary. Let $T \in B(H)$. If $SP(T)$ has no pseudoholes, then $SP(T) = SP(T_\epsilon)$ for each $0 \leq \epsilon \leq 1$.

Proof. Let $T = U[T]$ be the polar decomposition of $T$. Note that $[T]^\epsilon$ is of index zero. Now applying Theorem 2 with $A := [T]^\epsilon$ and $B := U[T]^{1-\epsilon}$ gives the result. □

9. Corollary. If $T \in B(H)$ is a quasiaffinity and $0 \leq \epsilon \leq 1$, then $Lat(T)$ is nontrivial if and only if $Lat(T_\epsilon)$ is nontrivial.

Proof. Let $T = U[T]$ be the polar decomposition of $T$. Note that if $T$ is a quasiaffinity, then $T_\epsilon$ is a quasiaffinity and $U$ is a unitary operator. Write $A := [T]^\epsilon$ and $B := U[T]^{1-\epsilon}$. Now applying Proposition 7 with $T = BA$ and $T_\epsilon = AB$ gives implication one way, and for the other way reverse them. □

10. Remark. If $f(\lambda)$ is a holomorphic function on a neighbourhood of $\sigma(AB)$ with $f(0) = 0$, then for $A, B \in B(X)$ we can see that

$$f(AB) = AC \quad \text{and} \quad f(BA) = CA \quad \text{for some} \ C \in B(X)$$

(cf. [Ba, Corollary 8]). Thus the results of this paper can be extended to $f(AB)$ and $f(BA)$ with such a function $f$. 

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References


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