

SPECTRAL PICTURES OF AB AND BA

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(Communicated by David R. Larson)

ABSTRACT. The spectral pictures of products AB and BA of Banach space operators are compared; in particular when one of them is ‘of index zero’.

We shall describe an operator A in the algebra $B(X)$ of bounded linear operators on a Banach space X as being *of index zero* whenever there is Banach space isomorphism

$$(0.1) \quad A^{-1}(0) \cong X/\text{cl } A(X).$$

For example a Fredholm operator has this property if and only if its Fredholm index is zero. In particular, finite-dimensional operators and normal operators acting on a Hilbert space are of index zero. A ‘quasi-affinity’, in the sense of an operator which is one-to-one and of dense range, is another kind of example. D. Djordjevic [Dj] has essentially noticed that

1. Theorem. *If $A \in B(X)$ is of index zero and $B \in B(X)$, there is the implication*

$$(1.1) \quad AB \text{ invertible, } \iff BA \text{ invertible}$$

and

$$(1.2) \quad AB \text{ Fredholm } \iff BA \text{ Fredholm,}$$

in which case

$$(1.3) \quad \text{index}(AB) = \text{index}(B) = \text{index}(BA).$$

Proof. If BA is invertible, then A is left invertible, in particular one-to-one with closed range; hence by index zero it has dense range, which now makes it invertible. For invertible A the implication (1.1) is clear. If conversely AB is invertible, then A is right invertible, in particular onto; hence by index zero it is one-to-one and again invertible. This gives (1.1) both ways; towards (1.2) argue that if BA is Fredholm, then A is upper semi-Fredholm, having closed range and finite-dimensional null space. If also A is “index zero” in the sense of (0.1), then its closed range must have finite codimension making it Fredholm, in which case (1.2) is clear. Conversely if AB is Fredholm, then A is lower semi-Fredholm, in the sense of having a closed range of finite codimension; hence it is also a finite-dimensional null space and again

Received by the editors November 7, 2003.

2000 *Mathematics Subject Classification.* Primary 47A10, 47A53, 47A66.

Key words and phrases. Spectral pictures, index zero, quasitriangular, essentially normal operators, compalant, generalized Aluthge transforms.

This work was supported by a grant (R14-2003-006-01000-0) from the Korea Science and Engineering Foundation, and by Enterprise Ireland Basic Research Grant SC/2002/0266.

Fredholm. This gives (1.2) both ways; finally since $\text{index}(A) = 0$ also, the usual index-of-product formula gives (1.3). \square

If $A \in B(X)$, write $\sigma(A)$, $\sigma^{\text{left}}(A)$, $\sigma^{\text{right}}(A)$, $\sigma_{\text{ess}}(A)$, $\sigma_{\text{ess}}^{\text{left}}(A)$, and $\sigma_{\text{ess}}^{\text{right}}(A)$ for the spectrum, the left spectrum, the right spectrum, the essential spectrum, the left essential spectrum, and the right essential spectrum, respectively, of A . If $A \in B(X)$, a *hole* in $\sigma_{\text{ess}}(A)$ is a bounded component of $\mathbb{C} \setminus \sigma_{\text{ess}}(A)$ and a *pseudohole* in $\sigma_{\text{ess}}(A)$ is a component of $\sigma_{\text{ess}}(A) \setminus \sigma_{\text{ess}}^{\text{left}}(A)$ or $\sigma_{\text{ess}}(A) \setminus \sigma_{\text{ess}}^{\text{right}}(A)$. The *spectral picture* of A , denoted $\mathcal{SP}(A)$, is the structure consisting of the set $\sigma_{\text{ess}}(A)$, the collection of holes and pseudoholes in $\sigma_{\text{ess}}(A)$, and the indices associated with those holes and pseudoholes. Write $K(X)$ for the ideals of compact operators on X .

We now have:

2. Theorem. *If $A \in B(X)$ is of index zero and $B \in B(X)$, then*

$$(2.1) \quad \sigma(BA) = \sigma(AB);$$

$$(2.2) \quad \sigma^{\text{left}}(AB) \subseteq \sigma^{\text{left}}(BA) \quad \text{and} \quad \sigma^{\text{right}}(BA) \subseteq \sigma^{\text{right}}(AB);$$

$$(2.3) \quad \sigma_{\text{ess}}(BA) = \sigma_{\text{ess}}(AB);$$

$$(2.4) \quad \sigma_{\text{ess}}^{\text{left}}(AB) \subseteq \sigma_{\text{ess}}^{\text{left}}(BA) \quad \text{and} \quad \sigma_{\text{ess}}^{\text{right}}(BA) \subseteq \sigma_{\text{ess}}^{\text{right}}(AB) .$$

Further if $0 \in \mathbb{C}$ is not in any pseudohole of either AB or BA , then

$$(2.5) \quad \mathcal{SP}(AB) = \mathcal{SP}(BA) .$$

Proof. It is familiar ([Ba], [GGK], [Ha], [LYR]) that

$$(2.6) \quad \omega(AB) \setminus \{0\} = \omega(BA) \setminus \{0\}$$

for the spectrum $\omega = \sigma$, as well as for the left, right, essential, left essential and right essential spectrum: thus (2.1) and (2.3) follow from (1.1) and (1.2). For the same reason (2.2) and (2.4) depend only on the fate of $0 \in \mathbb{C}$. Thus if BA has a left inverse, then A is left invertible, which together with having “index zero” makes it invertible, which now gives AB a left inverse:

$$CBA = I \implies CB = A^{-1} \implies ACB = I$$

means that B is now left invertible, and hence also AB . The argument for right invertibility is exactly the same. For (2.4) suppose BA is upper semi-Fredholm. Then by Atkinson’s theorem BA is left invertible modulo $K(X)$ and so $I - U(BA) \in K(X)$ for some $U \in B(X)$. Note that A is upper semi-Fredholm and hence by assumption, it is Fredholm of index zero. Remembering ([Ha, Theorem 6.5.2]) that a Fredholm operator of index zero can be written as the sum of an invertible and a finite rank operator, write $A = V + K$, where V is invertible and K is of finite rank. Then

$$I - UB(V + K) \in K(X) \implies I - UBV \in K(X) \implies I - VUB \in K(X),$$

which implies B is upper semi-Fredholm and hence, so is AB , giving the first inclusion of (2.4), and the argument for the second is the same. Finally, to see (2.5), it is effective to remember ([GGK, p. 38]) that with no restriction on A ,

$$(2.7) \quad \begin{pmatrix} AB - I & 0 \\ 0 & I \end{pmatrix} = F \begin{pmatrix} BA - I & 0 \\ 0 & I \end{pmatrix} E,$$

where

$$E := \begin{pmatrix} B & I \\ AB - I & A \end{pmatrix} \quad \text{and} \quad F := \begin{pmatrix} A & I - AB \\ -I & B \end{pmatrix}$$

are both invertible. Thus from (2.7),

$$(2.8) \quad \text{index}(AB - I) = \text{index}(F) + \text{index}(BA - I) + \text{index}(E) = \text{index}(BA - I).$$

This implies that whenever $\lambda \neq 0$ is in a hole or pseudohole common to AB and BA , then the value of the index for that pseudohole is the same for both. Thus if $0 \in \mathbb{C}$ is not in any pseudohole of either AB or BA , then we can conclude that $\mathcal{SP}(AB) = \mathcal{SP}(BA)$. This proves (2.5). \square

We would remark that 0 can be in a pseudohole of AB but not in a pseudohole of BA , or vice versa, but that if 0 is in the polynomially convex hull of a pseudohole of AB , then it is also in the polynomially convex hull of a pseudohole of BA , and vice versa. On the other hand, none of the inclusions in (2.2) and (2.4) can be replaced by equality:

3. Example. If $X = \ell_2$ and

$$(3.1) \quad A(x_1, x_2, x_3, x_4, x_5, x_6, \dots) = (0, x_2, 0, x_4, 0, x_6, \dots),$$

$$(3.2) \quad B(x_1, x_2, x_3, x_4, x_5, x_6, \dots) = (0, x_1, 0, x_2, 0, x_3, \dots),$$

$$(3.3) \quad B'(x_1, x_2, x_3, x_4, x_5, x_6, \dots) = (x_2, x_4, x_6, x_8, x_{10}, x_{12}, \dots),$$

then A is of index zero, AB is left invertible but BA is not upper semi-Fredholm, while $B'A$ is right invertible but AB' is not lower semi-Fredholm. Also BA is of index zero while AB is not.

Proof. Observe that

$$BB' = A \neq I = B'B; \quad AB = B \neq BA; \quad B'A = B' \neq AB',$$

and look at the null space of BA and the closure of the range of AB' . \square

In Example 3, a straightforward calculation shows that $\mathcal{SP}(AB)$ and $\mathcal{SP}(BA)$ has only one pseudohole H_0 whose polynomially convex hull contains 0 : with \mathbb{D} the open unit disk

$$\begin{aligned} H_0(AB) &= \mathbb{D} \quad \text{with index } H_0(AB) = -\infty; \\ H_0(BA) &= \mathbb{D} \setminus \{0\} \quad \text{with index } H_0(BA) = -\infty. \end{aligned}$$

On the other hand, from (2.7) we can see that for each $\lambda \neq 0$,

$$(AB - \lambda I)^{-1}(0) \cong (BA - \lambda I)^{-1}(0) \quad \text{and} \quad X/\text{cl}(AB - \lambda I)(X) \cong X/\text{cl}(BA - \lambda I)(X),$$

which implies that with no restriction on A and B ,

$$(3.4) \quad AB - \lambda I \text{ "of index zero"} \iff BA - \lambda I \text{ "of index zero"}, \quad \lambda \neq 0.$$

However if $\lambda = 0$, Example 3 shows that (3.4) may fail though each of A , B , AB and BA has closed range and A is of index zero. As a trivial sort of dual to Theorem 1, it is clear that

$$(3.5) \quad A \text{ invertible} \implies (BA \text{ index zero} \iff AB \text{ index zero});$$

by Example 3 this does not extend to A of index zero.

We however have:

4. Proposition. *If X is separable Hilbert space and if each of AB , BA and B has closed range, then if A is also Fredholm there is equivalence*

$$(4.1) \quad BA \text{ “of index zero”} \iff AB \text{ “of index zero”}.$$

Proof. If either AB or BA is Fredholm, then this is contained in (2.3), and if either BA is upper semi-Fredholm or AB is lower semi-Fredholm, then this is contained in (2.4). Thus we may assume that the null space of BA is infinite dimensional and that the range of AB is of finite codimension: on separable space X this implies

$$(4.2) \quad (BA)^{-1}(0) \cong X \cong X/(AB)X;$$

we therefore have to show

$$(4.3) \quad (AB)^{-1}(0) \cong X \iff X/(BA)X \cong X.$$

We claim

$$(4.4) \quad (BA)^{-1}(0) \cong X \implies B^{-1}(0) \cong X \iff (AB)^{-1}(0) \cong X$$

and

$$(4.5) \quad X/(AB)X \cong X \implies X/B(X) \cong X \iff X/(BA)X \cong X;$$

this is because of the isomorphisms [Ha, (6.5.4.6), (6.5.4.7)]

$$(4.6) \quad (AB)^{-1}(0)/B^{-1}(0) \cong A^{-1}(0) \cap B(X); \quad AX/(AB)X \cong X/(BX + A^{-1}(0)),$$

$$(4.7) \quad (BA)^{-1}(0)/A^{-1}(0) \cong B^{-1}(0) \cap A(X); \quad BX/(BA)X \cong X/(AX + B^{-1}(0)).$$

If $A^{-1}(0)$ is finite dimensional, then the first part of (4.6) gives the second part of (4.4), while the first comes from the first part of (4.7). If $A(X)$ is of finite codimension, then the second part of (4.7) gives the second part of (4.5), while the first comes from the second part of (4.6); alternatively take adjoints in (4.4). \square

The spectral picture $\mathcal{SP}(T)$ determines whether an operator is “quasitriangular” [AFV], and whether it is “compalent” to another operator [BDF].

Recall ([Pe, Definition 4.8]) that $T \in B(H)$ for a Hilbert space H is called *quasitriangular* if there exists a sequence $\{P_n\}_{n=1}^\infty$ of projections of finite rank in $B(H)$ that converges strongly to 1 and satisfies $\|P_n T P_n - T P_n\| \rightarrow 0$. The set of quasitriangular operators can be characterized as the set of all sums of the form $T_0 + K$, where T_0 is triangular and $K \in K(H)$ (cf. [Pe, Corollary 4.19]). We have:

5. Corollary. *If $A \in B(H)$ is of index zero, then AB is quasitriangular if and only if BA is quasitriangular.*

Proof. By Apostol, Foias, and Voiculescu [AFV] the operator T is quasitriangular if and only if $\mathcal{SP}(T)$ contains no hole or pseudohole with negative index. \square

Recall that $T \in B(H)$ for a Hilbert space H is called *essentially normal*, if $T^*T - TT^* \in K(H)$ and that operators T_1 and T_2 in $B(H)$ are said to be *compalent* if there exists a unitary operator $W \in B(H)$ and a compact operator $K \in K(H)$ such that $WT_1W^* + K = T_2$. Then by the beautiful Brown-Douglas-Fillmore

theorem ([BDF]) we have:

6. Corollary. *Let $A \in B(H)$ be of index zero. If AB and BA are essentially normal, then AB and BA are compalant.*

Proof. If AB and BA are essentially normal, then neither of them have any pseudoholes, so that (2.5) holds. Now the result follows from the Brown-Douglas-Fillmore theorem—if T_1 and T_2 are essentially normal, then T_1 and T_2 are compalant if and only if $\mathcal{SP}(T_1) = \mathcal{SP}(T_2)$. \square

We write $\text{Lat}(A)$ for the invariant subspace lattice of $A \in B(X)$, and recall that a “quasiaffinity” is one-to-one with dense range; obviously if A is not a quasiaffinity, then either its null space or the closure of its range will be in $\text{Lat}(A)$. We observe

7. Proposition. *If A, B in $B(X)$ are such that BA is a quasiaffinity, then*

$$(7.1) \quad \text{Lat}(BA) \text{ nontrivial} \implies \text{Lat}(AB) \text{ nontrivial.}$$

Proof. By assumption A is one-to-one and B has dense range. We claim that if $N \in \text{Lat}(BA)$ is nontrivial, then

$$(7.2) \quad M = \text{cl}(AN) \implies M \in \text{Lat}(AB) \text{ with } \{0\} \neq M \neq X.$$

The invariance of M is clear; $M \neq \{0\}$ is because A is one-to-one and N is nonzero; $M \neq X$ is because B is dense and $N \neq X$. \square

Not everything in $\text{Lat}(AB)$ need be derived in this way from $\text{Lat}(BA)$: for example if A is the forward and B the backward shift, look at $(AB)^{-1}(0)$.

An operator $T \in B(H)$ for a Hilbert space H has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry with the same null space as T . Associated with T , there is a useful related operator $T_\epsilon := |T|^\epsilon U |T|^{1-\epsilon}$ ($0 \leq \epsilon \leq 1$) called the *generalized Aluthge transform* of T of order ϵ ([Al]). If $\epsilon = \frac{1}{2}$ this really is the Aluthge transform while if $\epsilon = 0$ we get back T itself; if $\epsilon = 1$, then this is what Carl Pearcy has called the “Duggal transplant” of T .

We recapture [JKP, Corollary 1.12]:

8. Corollary. *Let $T \in B(H)$. If $\mathcal{SP}(T)$ has no pseudoholes, then $\mathcal{SP}(T) = \mathcal{SP}(T_\epsilon)$ for each $0 \leq \epsilon \leq 1$.*

Proof. Let $T = U|T|$ be the polar decomposition of T . Note that $|T|^\epsilon$ is of index zero. Now applying Theorem 2 with $A := |T|^\epsilon$ and $B := U|T|^{1-\epsilon}$ gives the result. \square

9. Corollary. *If $T \in B(H)$ is a quasiaffinity and $0 \leq \epsilon \leq 1$, then $\text{Lat}(T)$ is nontrivial if and only if $\text{Lat}(T_\epsilon)$ is nontrivial.*

Proof. Let $T = U|T|$ be the polar decomposition of T . Note that if T is a quasiaffinity, then T_ϵ is a quasiaffinity and U is a unitary operator. Write $A := |T|^\epsilon$ and $B := U|T|^{1-\epsilon}$. Now applying Proposition 7 with $T = BA$ and $T_\epsilon = AB$ gives implication one way, and for the other way reverse them. \square

10. Remark. If $f(\lambda)$ is a holomorphic function on a neighbourhood of $\sigma(AB)$ with $f(0) = 0$, then for $A, B \in B(X)$ we can see that

$$f(AB) = AC \quad \text{and} \quad f(BA) = CA \quad \text{for some } C \in B(X)$$

(cf. [Ba, Corollary 8]). Thus the results of this paper can be extended to $f(AB)$ and $f(BA)$ with such a function f .

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