TOPOLOGICAL ENTROPY AND AF SUBALGEBRAS OF GRAPH $C^*$-ALGEBRAS

JA A JEONG AND GI HYUN PARK

Abstract. Let $A_E$ be the canonical AF subalgebra of a graph $C^*$-algebra $C^*(E)$ associated with a locally finite directed graph $E$. For Brown and Voiculescu's topological entropy $ht(\Phi_E)$ of the canonical completely positive map $\Phi_E$ on $C^*(E)$, $ht(\Phi_E) = ht(\Phi_E|A_E) = h_l(E)$ is known to hold for a finite graph $E$, where $h_l(E)$ is the loop entropy of Gurevic and $h_b(E)$ is the block entropy of Salama. For an irreducible infinite graph $E$, the inequality $h_l(E) \leq ht(\Phi_E|A_E)$ has recently been known. It is shown in this paper that $ht(\Phi_E|A_E) \leq \max\{h_b(E), h_b(\overline{E})\}$, where $\overline{E}$ is the graph $E$ with the direction of the edges reversed. Some irreducible infinite graphs $E_p (p > 1)$ with $ht(\Phi_E|A_{Ep}) = \log p$ are also examined.

1. Introduction

Voiculescu [22] introduced a notion of topological entropy $ht(\alpha)$ for an automorphism $\alpha$ of a nuclear unital $C^*$-algebra $A$ to measure the growth of $\alpha^n$ as $n \to \infty$ using the fact that a nuclear $C^*$-algebra has the completely positive approximation property. The definition extends very well to automorphisms of exact $C^*$-algebras (as done by Brown in [4]) due to the deep result by Kirchberg [13] that exact $C^*$-algebras are nuclearly embeddable. But without effort one can define $ht(\Phi)$ even for a completely positive (cp) map on an exact $C^*$-algebra as described in [2]. Since a $C^*$-subalgebra of an exact $C^*$-algebra is always exact, if $\Phi: A \to A$ is a cp map on an exact $C^*$-algebra $A$ and $B$ is a $\Phi$-invariant $C^*$-subalgebra of $A$, then $ht(\Phi|_B)$ can be defined and the monotonicity $ht(\Phi|_B) \leq ht(\Phi)$ holds.

The topological entropy has been computed in several cases; for example, the equality $ht(\alpha * \beta) = \max\{ht(\alpha), ht(\beta)\}$ for the reduced free product automorphism $\alpha * \beta$ was proved in [1]. when the free product is with amalgamation over a finite-dimensional $C^*$-algebra. Also Dykema [9] showed that $ht(\alpha) = 0$ for certain classes of automorphisms $\alpha$ of reduced amalgamated free products of $C^*$-algebras, which turns out to extend Størmer's result [21] that the Connes-Størmer entropy of the free shift automorphism of the $I_1$-factor $L(F_\infty)$ is zero.

In this paper we are concerned with the topological entropy of the shift type cp maps on $C^*$-algebras arising from directed graphs. A typical one is the canonical
cp map \( \Phi_A : \mathcal{O}_A \to \mathcal{O}_A \) of the Cuntz-Krieger algebra \( \mathcal{O}_A \) given by

\[
\Phi_A(x) = \sum_{i=1}^{n} s_i x s_i^* ,
\]

where \( s_1, \ldots, s_n \) are the partial isometries that generate \( \mathcal{O}_A \). The reason we call \( \Phi_A \) shift type is that \( \mathcal{O}_A \) contains a \( \Phi_A \)-invariant commutative \( C^* \)-subalgebra \( \mathcal{D}_A \) which is isomorphic to \( C(X_A) \) in such a way that the restriction \( \Phi_A|_{\mathcal{D}_A} \) corresponds to the shift map \( \sigma_{X_A} \) on the (compact) shift space \( X_A \) associated with the incidence matrix \( A \). The topological entropy of \( \Phi_A \) is then computed (see [5, 2, 11, 19]) as

\[
ht(\Phi_A) = \log r(A) \quad (r(A) \text{ is the spectral radius of } A).
\]

But \( \log r(A) = h_{\text{top}}(X_A) \) is a well-known fact, so that one can deduce by [8] that \( ht(\Phi_A) = ht(\Phi_A|_{\mathcal{D}_A}) \). On the other hand, \( \mathcal{O}_A \) also contains another important \( \Phi_A \)-invariant \( C^* \)-subalgebra \( \mathcal{A}_A \) which is an AF algebra with \( \mathcal{D}_A \subset \mathcal{A}_A \). Thus by monotonicity of entropy, we have

\[
ht(\Phi_A) = ht(\Phi_A|_{\mathcal{D}_A}) = ht(\Phi_A|_{\mathcal{A}_A}).
\]

The Cuntz-Krieger algebras \( \mathcal{O}_A \) are now well understood as graph \( C^* \)-algebras \( C^*(E) = C^*(s_e, p_e) \) associated with finite directed graphs \( E \), and the cp map \( \Phi_A \) of \( \mathcal{O}_A \) is interpreted as the map \( \Phi_E : C^*(E) \to C^*(E) \) given by \( \Phi_E(x) = \sum_{e \in E^1} s_e x s_e^* \).

Hence if \( E \) is a finite graph (possibly with sinks) which contains an infinite path, it follows that \( ht(\Phi_E) = ht(\Phi_E|_{\mathcal{A}_E}) = ht(\Phi_E|_{\mathcal{D}_E}) = \log r(A_E) \), where \( \mathcal{A}_E \) is the AF subalgebra of \( C^*(E) \) corresponding to \( \mathcal{A}_A \) in \( \mathcal{O}_A \) and \( A_E \) is the edge matrix of \( E \) (see [11]).

If \( E \) is infinite but locally finite, then the map \( \Phi_E \) is known to be a contractive cp map, and furthermore if \( E \) is irreducible and \( \mathcal{A}_E \) is the canonical AF subalgebra of \( C^*(E) \), the inequality \( h_1(E) \leq ht(\Phi_E|_{\mathcal{A}_E}) \) is known to hold [11]. The purpose of the present paper is then to give an upper bound for \( ht(\Phi_E|_{\mathcal{A}_E}) \), and we actually prove the following (see Theorem 3.10):

\[
ht(\Phi_E|_{\mathcal{A}_E}) \leq \max\{h_3(E), h_2(E)\}.
\]

In particular, for an irreducible infinite graph \( E_p \) constructed in [20] so that \( h_1(E_p) = h_0(E_p) = \log p \) \((p > 1)\), we have \( ht(\Phi_{E_p}) = h_2(E_p) = \log p \).

We believe that the result would be helpful to compute the entropy \( ht(\Phi_E) \) of \( \Phi_E \) on the whole graph \( C^* \)-algebra \( C^*(E) \).

2. Preliminaries

2.1. Graph \( C^* \)-algebras. A (directed) graph is a quadruple \( E = (E^0, E^1, r, s) \) of the vertex set \( E^0 \), the edge set \( E^1 \), and the range, source maps \( r, s : E^1 \to E^0 \). A family \( \{p_v, s_e \mid v \in E^0, e \in E^1\} \) of mutually orthogonal projections \( p_v \) and partial isometries \( s_e \) is called a Cuntz-Krieger \( E \)-family if the following relations hold:

\[
s_e^* s_e = p_{r(e)}, \quad s_e s_e^* \leq p_{s(e)}.
\]

\[
p_v = \sum_{s(e) = v} s_e s_e^*, \quad \text{if } 0 < |s^{-1}(v)| < \infty.
\]

The graph \( C^* \)-algebra \( C^*(E) \) is then defined to be a \( C^* \)-algebra generated by a universal Cuntz-Krieger \( E \)-family (see [13, 17, 3]). If \( E \) is row-finite, that is, each vertex emits only finitely many vertices, the relations can be written as (with the
edge matrix $A_E$ of $E$)
\[ s_e^*s_e = \sum_{f \in E^1} A_E(e, f)s_fs_f^*; \]

hence the family is also called a Cuntz-Krieger $A_E$-family.

Given a $\{0, 1\}$ matrix $B$ such that each row has only finitely many non-zero entries (row-finite), let $E$ be the graph with the vertex matrix $B$. Then by definition $C^*(E)$ is generated by a Cuntz-Krieger $A_E$-family. But it is also generated by a Cuntz-Krieger $B$-family [17, Proposition 4.1]. Hence many results on $C^*$-algebras of $\{0, 1\}$ matrices can be applied to graph $C^*$-algebras even though not all $\{0, 1\}$ matrices can occur as edge matrices of some graphs.

We call a graph $E$ locally finite if each vertex receives and emits only finitely many edges. Throughout this paper we consider only locally finite graphs and adopt the notation in [16]. If a finite path $\alpha \in E^*$ of length $|\alpha| > 0$ is a return path, that is, $s(\alpha) = r(\alpha)$, then $\alpha$ is called a loop at $v = s(\alpha)$. A graph $E$ is said to be irreducible if for any two vertices $v, w$ there is a finite path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. It is known that if $E$ is irreducible and every loop has an exit, then $C^*(E)$ is simple ([16]).

2.2. Topological entropy of cp maps. Let $A$ be a $C^*$-algebra, $\pi : A \to B(H)$ a faithful $*$-representation, and $Pf(A)$ the set of all finite subsets of $A$. For $\omega \in Pf(A)$ and $\delta > 0$, put

\[ CPA(\pi, A) = \{ (\phi, \psi, B) \mid \phi : A \to B, \psi : B \to B(H) \text{ are contractive cp maps} \]

and $B$ is a $C^*$-algebra with $\dim B < \infty\}$,

\[ rcp(\pi, \omega, \delta) = \inf \{ \text{rank}(B) \mid (\phi, \psi, B) \in CPA(\pi, A), \|\psi \circ \phi(x) - \pi(x)\| < \delta, \]

for all $x \in \omega\},

where $\text{rank}(B)$ denotes the dimension of a maximal abelian subalgebra of $B$.

Since the cp $\delta$-rank $rcp(\pi, \omega, \delta)$ is independent of the choice of $\pi$ ([2] [4]) and graph $C^*$-algebras $C^*(E)$ are nuclear ([15]) we may write $rcp(\omega, \delta)$ for $rcp(\pi, \omega, \delta)$ assuming that $C^*(E) \subseteq B(H)$ for a Hilbert space $H$.

**Definition 2.1** ([2] [4] [22]). Let $A \subseteq B(H)$ be a $C^*$-algebra and let $\Phi : A \to A$ be a cp map. Put \n
\[ ht(\Phi, \omega, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log \left( rcp \left( \bigcup_{i=0}^{n-1} \Phi^i(\omega, \delta) \right) \right), \]

\[ ht(\Phi, \omega) = \sup_{\delta > 0} ht(\Phi, \omega, \delta). \]

Then $ht(\Phi) := \sup_{\omega \in Pf(A)} ht(\Phi, \omega)$ is called the topological entropy of $\Phi$.

**Remark 2.2.** We refer the reader to [2] and [4] for the following useful properties and their proofs. Let $A$ be an exact $C^*$-algebra and let $\Phi : A \to A$ be a cp map.

(a) If $\theta : A \to B$ is a $C^*$-isomorphism, then $ht(\Phi) = ht(\Phi) \theta^{-1}$.

(b) Let $A$ be the unital $C^*$-algebra obtained by adjoining a unit and let $\tilde{\Phi} : A \to A$ be the extension of $\Phi$. Then $ht(\tilde{\Phi}) = ht(\Phi)$.

(c) If $A_0 \subseteq A$ is a $\Phi$-invariant $C^*$-subalgebra of $A$, $ht(\Phi|_{A_0}) \leq ht(\Phi)$. 

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We will use the following Arveson’s extension theorem several times.

**Arveson Extension Theorem** (see [4]). Let $A$ be a unital $C^*$-algebra, $S \subset A$ a unital subspace with $S = S^*$, and $\phi : S \to B$ a contractive cp map where $B = B(H)$ or $\dim(B) < \infty$. Then $\phi$ extends to a cp map $\tilde{\phi} : A \to B$. If $S$ is a $C^*$-subalgebra of $A$, then we obtain a unital cp extension of $\phi$ even when $S$ does not contain the unit of $A$.

If $E$ is a locally finite graph, the map $\Phi_E : C^*(E) \to C^*(E)$, defined by

$$\Phi_E(x) = \sum_{e \in E} s_e x s_e^*,$$

is well defined, contractive and completely positive [11]. For a finite graph $E$, the topological entropy $ht(\Phi_E)$ has been obtained as follows (see [2], [5], [19], or [11]).

**Theorem 2.3.** Let $E$ be a finite graph possibly with sinks and let $A_E$ be the edge matrix of $E$. If $E$ contains an infinite path, then

$$ht(\Phi_E) = \log r(A_E),$$

where $r(A_E)$ is the spectral radius of $A_E$.

By $h_{top}(X)$ we denote the topological entropy of a compact space $(X, T)$ together with a continuous map $T : X \to X$ (for a definition, see [23] Chapter 7). Let $E$ be a locally finite infinite graph with no sinks and $X_E$ the locally compact shift space of (one-sided) infinite paths with the one point compactification $\bar{X}_E$. The first identity in the following theorem is shown for the doubly infinite path space of $E$ by Gurevic [10]. See Remark 3.6(a) for a definition of the entropy $h(X_E)$ for a finite graph $E$.

**Theorem 2.4** ([11] Theorem 4.4]). Let $E$ be a locally finite irreducible infinite graph. Then

$$h_{top}(\bar{X}_E) = \sup_{E'} h(X_{E'}) \leq ht(\Phi_E),$$

where the supremum is taken over all the finite (irreducible) subgraphs of $E$.

## 3. Main results

Throughout this section $E$ will denote a locally finite infinite graph unless stated otherwise. For a path $\alpha \in E^*$, let $\alpha^0$ be the set of vertices lying on $\alpha = \alpha_1 \cdots \alpha_n$, that is, $\alpha^0 = \{s(\alpha_1), r(\alpha_1), \ldots, r(\alpha_n)\}$. For a fixed vertex $v$ we consider the following subsets of finite paths $E^n$ of length $n$:

(i) $E^n(v) = \{\alpha \in E^n \mid v \in \alpha^0\}$,

(ii) $E^n_\alpha(v) = \{\alpha \in E^n \mid s(\alpha) = v\}$,

(iii) $E^n_\alpha^*(v) = \{\alpha \in E^n_\alpha(v) \mid r(\alpha_i) \neq v, \ 1 \leq i \leq n\}$,

(iv) $E^n(v) = \{\alpha \in E^n \mid \alpha$ is a loop at $v\}$.

Similarly we can think of $E^n_\alpha(v)$ and $E^n_\alpha^*(v)$.

**Definition 3.1.** Let $E$ be a graph and $v \in E^0$. Put

$$h_l(E, v) = \limsup_n \frac{1}{n} \log |E^n_\alpha(v)| \quad \text{and} \quad h_b(E, v) = \limsup_n \frac{1}{n} \log |E^n_\alpha^*(v)|.$$

The loop entropy $h_l(E)$ and the block entropy $h_b(E)$ of $E$ are defined by

$$h_l(E) := \sup_{v \in E^0} h_l(E, v) \quad \text{and} \quad h_b(E) := \sup_{v \in E^0} h_b(E, v).$$
If $E$ is irreducible, $h_l(E,v)$ and $h_b(E,v)$ are independent of the choice of a vertex $v$ [20]; hence $h_l(E) = h_l(E,v)$ and $h_b(E) = h_b(E,v)$ for any $v \in E^0$. Let $tE$ denote the graph $E$ with the direction of all edges reversed. Then $h_l(E) = h_l(tE)$ is immediate while $h_b(E) \neq h_b(tE)$ in general as we will see in Example 3.3.

We will use the following notation for the infinite series with coefficients from (i)-(iv) above:

(i)' $E(v,z) := \sum |E^n(v)|z^n$,
(ii)' $E_*(v,z) := \sum |E^n_*(v)|z^n$,
(iii)' $E^n_*(v,z) := \sum |E^n_*(v)|^*z^n$,
(iv)' $E_l(v,z) := \sum |E^n_l(v)|z^n$.

We denote the radius of convergence of the series $E_*(v,z)$ by $R_{E_2}$. Thus

$$R_{E_2}^{-1} = \limsup_{n \to \infty} |E^n_*(v)|^{1/n}.$$ 

Similarly $R_{E^n_2}$ denotes the radius of convergence of $E^n_*(v,z) := \sum |E^n_2(v)|^*z^n$. As in [20] p.331, if $C_v(n)$ is the number of sequences $v_i \cdots v_{i_{n-1}}$ of vertices such that $v_j \neq v$ for $j = i_1, \ldots, i_{n-1}$, $|C_v(n)| = |E^{n-1}_*(v)|$ and so the radius of convergence of $E^n_*(v,z)$ coincides with that (denoted by $Q_0$ in [20]) of $\sum_n C_v(n)z^n$. The following is Lemma (3.1) of [20].

**Proposition 3.2 ([20]).** If $E$ is an irreducible graph, then

$$h_b(E) = \max\{\log (R_{E_2}^{-1}), h_l(E)\}.$$ 

Note that if $E$ is irreducible, then $h_b(tE) = \limsup \frac{1}{n}\log |E^n_2(v)|$ and so from the above proposition we have

(1) $$h_b(tE) = \max\{\log (R_{E_2}^{-1}), h_l(E)\}.$$ 

The following example shows that $h_b(E) \neq h_b(tE)$ in general.

**Example 3.3.** For each pair of positive real numbers $1 < p \leq q$, Salama [20] constructed an irreducible infinite graph $E_{p,q}$ with

$$h_l(E_{p,q}) = \log p \quad \text{and} \quad h_b(E_{p,q}) = \log q.$$ 

As mentioned in the proof of [20] Theorem (3.9)], $E_{p,q}$ may be constructed to be a (uniformly) locally finite graph using the idea in [20] Example (3.7).

For example, the following graph $E := E_{2,8}$ satisfies $h_l(E) = \log 2$ and $h_b(E) = \log 8$. There are 8 edges from the vertex $n$ to the vertex $n + 1$ for each $n \geq 0$.

![Graph E](image)

Now we first show that

$$\log (R_{E_2}^{-1}) < h_l(E).$$
For a fixed vertex 0 we have

\[
R_{E_r}^{-1} = \limsup_{n \to \infty} |E_r^n(0)^*|^{1/n} \\
= \limsup_{n \to \infty} \left| \{ \alpha \in E_r^n(0) \mid s(\alpha_i) \neq 0, \text{ for } 1 \leq i \leq n \} \right|^{1/n}.
\]

Since

\[
|E_r^{4k}(0)^*| = 1 + 8^{k-1} + 8^{k-4} + 8^{k-7} + \cdots,
\]

it follows that

\[
\limsup_{k \to \infty} |E_r^{4k}(0)^*|^{1/4} = 8^{1/4}.
\]

But it is not hard to see that

\[
\limsup_{n \to \infty} |E_r^n(0)^*|^{1/4} = \limsup_{k \to \infty} |E_r^{4k}(0)^*|^{1/4}.
\]

Hence \( \log(R_{E_r}^{-1}) = \log 8^{1/4} < \log 2 = h_l(E) \). By (1), \( h_b(t(E)) = h_l(E) = \log 2 \) and so we conclude that \( h_b(t(E)) < h_b(E) \).

**Lemma 3.4.** If \( E \) is an irreducible graph, then the value

\[
\limsup_{n \to \infty} \frac{1}{n} \log |E^n(v)|
\]

is independent of the choice of a vertex \( v \).

**Proof.** Let \( v, w \) be two vertices of \( E \). Then there exist two paths \( \mu \in E^k, \nu \in E^m \) with \( s(\mu) = r(\nu) = v, s(\nu) = r(\mu) = w \) because \( E \) is irreducible. We assume that \( \mu \) and \( \nu \) have the smallest length, respectively. If \( \alpha = \alpha_1 \alpha_2 \cdots \alpha_n \in E^n(v) \), then with \( i_0 = \min \{ i \mid s(\alpha_i) = v \} \) write \( \alpha = \alpha' \alpha'' \), where \( \alpha' = \alpha_1 \cdots \alpha_{i_0} - 1 \) and \( \alpha'' = \alpha_{i_0} \cdots \alpha_n \) (if \( i_0 = 1 \), \( \alpha = \alpha'' \)). Then the map

\[
E^n(v) \to E^{n+k+m}(w), \quad \alpha = \alpha' \alpha'' \mapsto \alpha' \nu \alpha''
\]

is injective; hence \( |E^n(v)| \leq |E^{n+k+m}(w)| \) for each \( n \). Therefore

\[
\limsup_{n \to \infty} \frac{1}{n} \log |E^n(v)| \leq \limsup_{n \to \infty} \frac{1}{n} \log |E^{n+k+m}(w)|
\]

\[
\leq \limsup_{n \to \infty} \frac{1}{n} \log |E^n(w)|.
\]

\( \square \)

**Proposition 3.5.** Let \( E \) be an irreducible graph and \( v_0 \in E^0 \).

(a) If \( E \) is finite, then

\[
\limsup_{n \to \infty} \frac{1}{n} \log |E^n(v_0)| = \limsup_{n \to \infty} \frac{1}{n} \log |E^n|.
\]

In particular, \( h_l(E) = h_b(E) = h_b(t(E)) \).

(b) If \( E \) is infinite, then

\[
\limsup_{n \to \infty} \frac{1}{n} \log |E^n(v_0)| = \max \{ h_b(E), h_b(t(E)) \}.
\]
Proof. (a) Let $E^0 = \{v_0, v_1, \ldots, v_{k-1}\}$. Since $E$ is irreducible there exist finite paths $\{\mu_i, v_i \mid 0 \leq i \leq k-1\}$ such that $s(\mu_i) = r(v_i) = v_0$, $r(\mu_i) = v_i = s(v_i)$.

Suppose $|\mu_i| = m_i$, $|v_j| = l_j$. If $\alpha \in E^n$ is a path with $s(\alpha) = v_i$, $r(\alpha) = v_j$ then $\mu_i \alpha v_j \in E^{n+m_i+l_j}(v_0)$ is a loop at $v_0$. The map $\alpha \mapsto \mu_i \alpha v_j$ is not necessarily injective, but there exist at most $k_0$ paths in $E^n$ that have the same image in $E^{n+m_i+l_j}(v_0)$ under the map, where $k_0 = \max_{i,j} \{m_i + l_j\}$. Hence we have

$$|E^n| \leq k_0 \cdot \bigcup_{0 \leq i,j \leq k-1} E^{n+m_i+l_j}(v_0) \leq k_0 k^2 \max_{i,j} |E^{n+m_i+l_j}(v_0)|.$$ 

On the other hand, for each $n$, there exists a $k_n \in \{0, \ldots, k_0\}$ such that

$$|E^{n+k_n}(v_0)| = \max_{i,j} |E^{n+m_i+l_j}(v_0)|.$$ 

Then $|E^n| \leq k_0 k^2 |E^{n+k_n}(v_0)|$ and it follows that

$$\limsup_{n \to \infty} \frac{1}{n} \log |E^n| \leq \limsup_{n \to \infty} \frac{1}{n} \log |E^n(v_0)|.$$ 

(b) Note first that

$$|E^n(v_0)| = \big| \bigcup_{k=0}^n \{\alpha \beta \mid \alpha \in E^k_v(v_0)^*, \beta \in E^{n-k}_s(v_0)\} \big|$$

$$= \sum_{k=0}^n |E^k_v(v_0)^*| |E^{n-k}_s(v_0)| = \sum_{k=0}^n \left|\left(\sum_{n} |(tE)^k_s(v_0)^*| |E^{n-k}_s(v_0)| \right)\right|.$$ 

Then

$$E(v_0, z) = \sum_{n} \left(\sum_{k=0}^n |(tE)^k_s(v_0)^*| |E^{n-k}_s(v_0)| \right)z^n$$

$$= \left(\sum_{n} |(tE)_s^n(v_0)^*| z^n \right) \left(\sum_{n} |E^n_s(v_0)| z^n \right)$$

$$= (tE)^s_*(v_0, z) \cdot E_s(v_0, z),$$

so that the radius of convergence $R_E$ of $E(v_0, z)$ is equal to min $\{R_{(tE)}^s, R_{E_s}\}$. Thus

$$R^{-1}_E = \max \{R_{(tE)}^{-1}, R_{E_s}^{-1}\}.$$ 

But Proposition 3.2 gives

$$\log \left(R_{(tE)}^{-1}\right) \leq h_b(tE),$$

and also by definition $\log(R_{E_s}^{-1}) = h_b(E)$. Therefore

$$\limsup_{n \to \infty} \frac{1}{n} \log |E^n(v_0)| = \log(R^{-1}_E) \leq \max\{h_b(tE), h_b(E)\}.$$ 

$$\square$$
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is the Perron eigenvalue of the edge matrix \( A_E \) of \( E \) is \( \lambda_E = \max_{1 \leq i \leq \ell} \{ \lambda_{E_i} \} \), where \( \lambda_{E_i} \) is the Perron eigenvalue of the edge matrix of the irreducible graph \( E_i \)
(see [18, 4.4]).

(a) Assuming \( \lambda_E = \lambda_{E_1} \) without loss of generality, we have from [18, Theorem 4.4.4] that
\[
\log \lambda_E = \log \lambda_{E_1},
\]
where \( h(X_E) = \lim_n \frac{1}{n} \log |E^n| \) is the topological entropy of \( X_E \) (or \( \Sigma_E \),
the two-sided edge shift space). See [18, Definition 4.1.1] or [14, p.23] for
the definition of \( h(X_E) \). Since \( \log \lambda_{E_1} = h(X_{E_1}) \) (18, Theorem 4.3.1) and
\( h(X_{E_1}) = h_t(E_1) = h_b(E_1) \) by Proposition 3.5(a) (\( E_1 \) is irreducible), we
see that all the entropies \( h_t(E), h_b(E) \) and \( h(X_E) \) are the same and equal
to \( \log \lambda_E \) because log \( \lambda_E = \log \lambda_{E_1} = h_t(E_1) \leq h_t(E) \leq h_b(E) \leq h(X_E) = \log \lambda_E \).

(b) Since the eigenvalues of \( A_E \) are exactly the eigenvalues of the \( A_{E_1} \), by [18, Lemma 4.4.3] it follows that \( \log \lambda_E = \log \sigma \). Thus by Theorem 2.3,
\( h_t(\Phi_E) = h_t(E) = h_b(E) = h(X_E) \) for any finite graph \( E \) which contains
an infinite path (or a loop). If \( E \) has no infinite paths, \( h(X_E) = -\infty \) while
\( h_t(\Phi_E) \geq 0 \).

Let \( E \) be an irreducible infinite graph and let \( D_E \) be the commutative \( C^* \)-
subalgebra of \( C^*(E) \) generated by the projections \( \{ p_\alpha = s_\alpha s_\alpha^* | \alpha \in E^* \} \). Then
\( D_E = \overline{\text{span}} \{ p_\alpha | \alpha \in E^* \} \) and the map
\[
w : D_E \to C_0(X_E), \quad w(p_\alpha) = \chi_{[\alpha]},
\]
is a \( C^* \)-isomorphism such that \( w(\Phi_E|_{D_E})w^{-1} = \sigma_E^\ast \). Here \( \chi_{[\alpha]} \) is the char-
acteristic function on the cylinder set \( [\alpha] = \{ \beta \in X_E | \beta = \alpha \} \) which is both
open and closed, and \( \sigma_E^\ast : C_0(X_E) \to C_0(X_E) \) is the \( \ast \)-homomorphism induced by
the shift map \( \sigma_E \) on \( X_E \), that is, \( \sigma_E^\ast(f) = f \circ \sigma_E \) for \( f \in C_0(X_E) \). By Remark
2.2(a), \( h_t(\Phi_E|_{D_E}) = h_t(\sigma_E^\ast) \). But \( h_t(\sigma_E^\ast) = h_t(\sigma_E^\ast) \) by Remark 2.2(b)
and \( h_t(\sigma_E^\ast) = h_{top}(X_E) \) by [8, Proposition 1.2]. On the other hand, \( h_{top}(X_E) = \sup_{E' \subset E} h(X_{E'}) \)
is proved in [10], where the supremum is taken over all the finite subgraphs \( E' \) of \( E \)
(or equivalently, over all the irreducible finite subgraphs). If \( \sup_{E' \subset E} h(X_{E'}) < \infty \), \( h_t(E) = \sup_{E' \subset E} h(X_{E'}) \) is known (see [18, p.465]). If \( \sup_{E' \subset E} h(X_{E'}) = \infty \),
clearly \( h_t(E) = \infty \) since \( h(X_{E'}) = h_t(E') \) for any finite graph \( E' \) (Remark 3.6(a))
and \( h_t(E') \leq h_t(E) \). Thus \( h_{top}(X_E) = \sup_{E' \subset E} h(X_{E'}) = h_t(E) \) always holds for a
locally finite irreducible infinite graph \( E \). Hence we have
\[
(2) \quad h_t(\Phi_E|_{D_E}) = h_t(E).
\]

Put
\[
A_E := \overline{\text{span}} \{ s_\alpha s_\beta^* | \alpha, \beta \in E^*, |\alpha| = |\beta| \}.
\]
Then \( A_E \) is a \( \Phi_E \)-invariant AF \( C^* \)-subalgebra of \( C^*(E) \) with \( D_E \subset A_E \); hence it
follows from (2) that
\[
(3) \quad h_t(E) \leq h_t(\Phi_E|_{A_E}).
\]
Lemma 3.7. Let $v$ be a vertex of an irreducible graph $E$ with at least two vertices and let $n \geq 1$. Then the elements in the set
\[
\omega(n, v) = \{ s_\alpha x^*_\beta \mid r(\alpha) = r(\beta) = v, |\alpha| = |\beta| \leq n \}
\]
are linearly independent.

Proof: We prove the assertion by induction on $n$. For $n = 1$, suppose
\[
x = \sum_{e,f \in E^1} \lambda_{ef} s_e x^*_f + \lambda_0 p_v = 0.
\]
If $e_0$ and $f_0$ are edges with $r(e_0) = r(f_0) = v$ and either $s(e_0) \neq v$ or $s(f_0) \neq v$, then $s_{e_0}^* p_v s_{f_0} = 0$; hence
\[
0 = s_{e_0} x s_{f_0} = \lambda_{e_0 f_0} (s_{e_0}^* s_{e_0})(s_{f_0}^* s_{f_0}) = \lambda_{e_0 f_0} p_v;
\]
thus $\lambda_{e_0 f_0} = 0$. Similarly, $\lambda_{ef} = 0$ if $e$ and $f$ are loops at $v$ and $e \neq f$. Then $x$ becomes
\[
x = \sum_{e \in E_1^1(v)} \lambda_{ee} s_e x^*_e + \lambda_0 p_v = 0.
\]
By irreducibility of $E$ and the assumption that $|E^0| > 1$, there exists an edge $f$ with $r(f) = v, r(f) \neq v$. Then $s_f x^*_f x = \lambda_0 s_f x^*_f = 0$, so that $\lambda_0 = 0$ and we have $x = \sum_{e \in E_1^1(v)} \lambda_{ee} s_e x^*_e = 0$. Since the projections $\{ s_e x^*_e \mid e \in E_1^1(v) \}$ are mutually orthogonal, it follows that $\lambda_{ee} = 0$ for each $e \in E_1^1(v)$.

Now suppose that the assertion is true for $n - 1$. If
\[
x = \sum_{|\alpha| = |\beta| \leq n - 1} \lambda_{\alpha\beta} s_\alpha x^*_\beta = 0, \; \lambda_{\alpha\beta} \in \mathbb{C},
\]
then for an edge $e \in E_1^1$ we have
\[
0 = s_e x^*_e s_e = \sum_{\alpha = \alpha', \beta = \beta'} \lambda_{\alpha\beta} s_\alpha x^*_\beta s_e = \sum_{|\alpha'| = |\beta'| \leq n - 1} \lambda_{(\alpha\alpha')(\beta\beta')} s_\alpha (s_{\beta'})^*.
\]
Note that the elements $s_\alpha (s_{\beta'})^*$ appearing in the sum are distinct. Thus by the induction hypothesis, one sees that $\lambda_{(\alpha\alpha')(\beta\beta')} = 0$. But the edge $e$ was arbitrary, and so we conclude that the coefficients $\lambda_{\alpha\beta}$ are all zero. $\square$

Using the same idea as in the proof of [4] Proposition 2.6] one can prove the following, which is stated in [2] without a proof in the case where $\{ \omega_\lambda \}$ is an increasing sequence. We provide a proof only for the reader’s convenience.

Proposition 3.8. Let $\Phi : A \to A$ be a contractive cp map of an exact $C^*$-algebra $A$. If $\{ \omega_\lambda \}_{\lambda \in \Lambda}$ is a net (partially ordered by inclusion) of finite subsets in $A$ such that the linear span of $\bigcup_{\lambda \in \mathbb{Z}^+} \Phi^i(\omega_\lambda)$ is dense in $A$, then
\[
ht(\Phi) = \sup_{\lambda} ht(\Phi, \omega_\lambda).
\]

Proof. Let $\omega = \{ a_1, a_2, \ldots, a_m \}$ be a finite subset in $A$ and $\delta > 0$. Then there exists a $\lambda \in \Lambda$ and $p \in \mathbb{N}$ such that if $\bigcup_{0 \leq i \leq p} \Phi^i(\omega_\lambda) = \{ x_1, \ldots, x_k \}$, then
\[
\left\| a_i - \sum_{i,j} \lambda_{ij} x_j \right\| < \delta
\]
for some $\lambda_{ij} \in \mathbb{C}$. Put $C := \max_{i,j} |\lambda_{ij}|$. Choose $(\phi, \psi, B) \in CPA(id, A)$ with $\text{rank}(B) = rcp(\omega \cup \cdots \cup \Phi^{p+n}(\omega, C^{-1}\delta))$. Then for $0 \leq l \leq p+n$,

$$\|\psi \circ \Phi^l(a_i) - \Phi^l(a_i)\|$$

$$\leq \|\psi \circ \Phi^l(a_i) - \Phi^l(\sum_{ij} \lambda_{ij} x_{ij})\| + \|\psi(\sum_{ij} \lambda_{ij} x_{ij}) - \Phi^l(a_i)\|$$

$$\leq 2\delta + \max_{i,j} |\lambda_{ij}| \cdot C^{-1}\delta = 3\delta.$$  

Thus for any $n \in \mathbb{N}$,

$$rcp(\omega \cup \cdots \cup \Phi^{p+n}(\omega, 3\delta)) \leq rcp(\omega \cup \cdots \cup \Phi^{p+n}(\omega, C^{-1}\delta)),$$

which implies that

$$ht(\Phi, \omega, 3\delta) \leq ht(\Phi, \omega, C^{-1}\delta).$$

Therefore we have $ht(\Phi, \omega) \leq ht(\Phi, \omega, \lambda)$.

The AF algebra $A_E$ contains $\Phi_E$-invariant AF subalgebras $A_E(v), v \in E^0$,

$$A_E(v) := \overline{\text{span}}\{ s_\alpha s_\beta^* | \ r(\alpha) = r(\beta) = v, |\alpha| = |\beta| \}.$$  

We show that the topological entropy of the restriction map $\Phi_E|_{A_E(v)}$ has an upper bound $h_0(l_E)$ which might be strictly smaller than the upper bound for $ht(\Phi_E|_{A_E})$ given in Theorem 3.10.

**Proposition 3.9.** Let $E$ be an irreducible infinite graph. Then for each $v \in E^0$,

$$ht(\Phi_E|_{A_E(v)}) \leq h_0(l_E).$$

**Proof.** Let $A_n(v)$ be the $C^*$-subalgebra of $A_E(v)$ generated by $\omega(n, v)$. Then from

$$s_\alpha s_\beta^* \cdot s_\mu s_\nu^* = \begin{cases} s_\alpha s_\beta^* s_\mu^* s_\nu^*, & \text{if } \mu = \beta \mu', \\ s_\alpha s_\beta^* s_\mu^* s_\nu^*, & \text{if } \beta = \mu \beta', \\ 0, & \text{otherwise,} \end{cases}$$

we see that $A_n(v) = \text{span}(\omega(n, v))$ is finite dimensional.

Since $\{\omega(n, v)\}$ is an increasing sequence of finite subsets in $A_E(v)$ such that the linear span of $\bigcup_n \omega(n, v)$ is dense in $A_E(v)$, by Proposition 3.8 it suffices to show that

$$ht(\Phi_E, \omega(n, v)) \leq h_0(l_E), \text{ for } n \in \mathbb{N}.$$  

Set $E^*_r(v) := \bigcup_{l \geq 0} E^*_l(v)$ and $r(n) := |\bigcup_{k=0}^n E^*_r(v)|$. Fix $n_0 \in \mathbb{N}$, and define a map $\phi : \omega(n_0, v) \to M_{r(n_0)}$ by

$$\phi(s_\alpha s_\beta^*) = \sum_{|\alpha\gamma| \leq n_0, \gamma \in E^*_r(v)} e_{(\alpha\gamma)(\beta\gamma)},$$

where $\{e_{\mu\nu}\}$ are the standard matrix units of the matrix algebra $M_{r(n_0)}$. Since the elements in $\omega(n_0, v)$ are linearly independent by Lemma 3.7, one can extend the map to the linear map $\phi : A_{n_0}(v) \to M_{r(n_0)}$. Now we show that $\phi$ is in fact a $*$-isomorphism. To prove that it is a $*$-homomorphism, we only need to see that

$$\phi(s_\alpha s_\beta^*) = \phi(s_\alpha s_\beta^*) \phi(s_\mu s_\nu^*).$$
If $\beta = \mu\beta'$, then $s_\alpha s_\beta^*s_\mu s_\nu^* = s_\alpha (s_{\nu\beta'})^*$ and

$$\phi(s_\alpha s_\beta^*)\phi(s_\mu s_\nu^*) = \sum_{|\alpha\gamma| \leq n_0 \atop \gamma \in E_r^1(v)} e_{(\alpha\gamma)(\mu\beta'\gamma)} \cdot \sum_{|\mu\delta| \leq n_0 \atop \delta \in E_r^1(v)} e_{(\mu\delta)(\nu\delta)}$$

$$= \sum_{|\alpha\gamma| \leq n_0 \atop \gamma \in E_r^1(v)} e_{(\alpha\gamma)(\nu\beta'\gamma)} = \phi(s_\alpha (s_{\nu\beta'})^*) = \phi(s_\alpha s_\beta^*s_\mu s_\nu^*).$$

If $\mu = \beta\mu'$, a similar proof works. Otherwise, we have $\phi((s_\alpha s_\beta^*)(s_\mu s_\nu^*)) = 0 = \phi(s_\alpha s_\beta^*)\phi(s_\mu s_\nu^*)$. In order to show that $\phi$ is injective, let $\phi(\sum_{\alpha,\beta} \lambda_{\alpha\beta} s_\alpha s_\beta^*) = 0$. Then

$$\sum_{\alpha,\beta} \lambda_{\alpha\beta} \phi(s_\alpha s_\beta^*) = \sum_{\alpha,\beta} \lambda_{\alpha\beta} \left( \sum_{|\alpha\gamma| \leq n_0 \atop \gamma \in E_r^1(v)} e_{(\alpha\gamma)(\beta\gamma)} \right) = 0.$$

But the vectors $\sum_{|\alpha\gamma| \leq n_0 \atop \gamma \in E_r^1(v)} e_{(\alpha\gamma)(\beta\gamma)} \ (r(\alpha) = r(\beta) = v, \ |\alpha| = |\beta| \leq n_0)$ are linearly independent in $M_r(n_0)$. In fact, if $A := \sum_{\alpha,\beta} \lambda_{\alpha\beta} \left( \sum_{|\alpha\gamma| \leq n_0 \atop \gamma \in E_r^1(v)} e_{(\alpha\gamma)(\beta\gamma)} \right) = 0$, then $e_{\nu\nu} A e_{\nu\nu} = \lambda_{\nu\nu} e_{\nu\nu} = 0$, that is, $\lambda_{\nu\nu} = 0$, and for any $\alpha, \beta \in E_r^1(v)$, $e_{\alpha\alpha} A e_{\beta\beta} = \lambda_{\alpha\beta} e_{\alpha\beta} = 0$; hence $\lambda_{\alpha\beta} = 0$. Repeating the process one has $\lambda_{\alpha\beta} = 0$ for any $\alpha, \beta \in \bigcup_{k=0}^{n_0} E_r^k(v)$. Therefore $\sum_{\alpha,\beta} \lambda_{\alpha\beta} s_\alpha s_\beta^* = 0$, and the map $\phi$ is injective. The surjectivity of $\phi$ follows from $\dim(A_{n_0}(v)) = r(n_0)^2$. We simply write $\phi$ for $\phi : A_{n_0+1}(v) \to M_r(n_0+1) \ (l \geq 0)$, and $\tilde{\phi}$ for its contractive cp extension to $A_E(v)$ that exists by Arveson's extension theorem.

For each $n \in \mathbb{N}$ and $0 \leq l \leq n - 1$, note that

$$\bigcup_{l=0}^{n-1} \Phi^E_l(\omega(n_0, v)) \subseteq \text{span}(\omega(n_0 + n - 1, v)).$$

Then the element

$$(\tilde{\phi}, \psi := \phi^{-1}, M_r(n_0+n-1)) \in CPA(id, A_E(v))$$

satisfies $\psi \circ \tilde{\phi}|_{\omega(n_0+n-1, v)} = id_{\omega(n_0+n-1, v)}$. Thus for each $\delta > 0$,

$$rcp(id, \omega(n_0 + n - 1, v), \delta) \leq r(n_0 + n - 1),$$

and so

$$ht(\Phi_E|_{A_E(v)}, \omega(n_0, v), \delta) \leq \limsup_{n \to \infty} \frac{1}{n} \log(r(n_0 + n - 1))$$

$$= \limsup_{n \to \infty} \frac{1}{n} \log(r(n))$$

$$= \limsup_{n \to \infty} \frac{1}{n} \log \left| \bigcup_{k=0}^{n} E_r^k(v) \right|$$

$$= h_r(E).$$

For the last equality, note that if $k \leq n$, then $\left| E_r^k(v) \right| \leq \left| E_r^n(v) \right|$; hence

$$\left| \bigcup_{k=0}^{n} E_r^k(v) \right| \leq (n + 1) \cdot \left| E_r^n(v) \right|.$$
The following theorem gives an upper bound for $ht(\Phi_E|\mathcal{A}_E)$.

**Theorem 3.10.** Let $E$ be an irreducible infinite graph and let $\mathcal{A}_E$ be the AF subalgebra of $C^*(E)$ generated by the partial isometries $\{s_\alpha s_\beta^* | \alpha, \beta \in E^*, |\alpha| = |\beta|\}$. Then

$$ht(\Phi_E|\mathcal{A}_E) \leq \max\{h_b(t^*E), h_b(E)\}.$$  

**Proof.** Let $E^0 = \{v_1, v_2, \cdots \}$. For each $n_0 \in \mathbb{N}$ and $n_1 \in \mathbb{Z}^+ = \{0\} \cup \mathbb{N}$, put

$$\omega(n_0, n_1) := \left\{ s_\alpha s_\beta^* | \alpha, \beta \in E^{n_1}, r(\alpha) = r(\beta) \in \{v_1, \cdots, v_{n_0}\} \right\},$$

$$\omega_\Sigma(n_0, n_1) := \left\{ \sum s_\alpha s_\beta^* s_\alpha s_\beta^* = \sum s_\alpha s_\beta^* s_\beta s_\alpha^* \in \omega(n_0, n_1) \right\}.$$  

Note that $\omega_\Sigma(n_0, n_1)$ is not the linear span of $\omega(n_0, n_1)$. Then $\{\omega_\Sigma(n_0, n_1) | n_0 \in \mathbb{N}, n_1 \in \mathbb{Z}^+\}$ is a net of finite subsets in $\mathcal{A}_E$ which is partially ordered by inclusion. In fact, given two finite sets $\omega_\Sigma(n_0, n_1), \omega_\Sigma(m_0, m_1)$ $(n_1 \leq m_1)$, one may write each element $s_\alpha s_\beta^* \in \omega(n_0, n_1)$ as

$$s_\alpha s_\beta^* = s_\alpha \left( \sum_{|\mu| = m_0-n_1} s_\mu s_\mu^* \right) s_\beta s_\beta^* = \sum s_\alpha s_\beta^* s_\beta s_\alpha^* \in \omega_\Sigma(m_0, m_1),$$

where $m_2 > \max\{n_0, m_0\}$ is an integer large enough so that $r(\alpha \mu) \in \{v_1, \cdots, v_{m_2}\}$ for any $\alpha \mu$ appearing in the last sum. Then clearly $\omega_\Sigma(n_0, n_1) \cup \omega_\Sigma(m_0, m_1)$ is contained in $\omega_\Sigma(m_2, m_1)$.

Since the linear span of the set $\bigcup_{n_0, n_1, n} \Phi^n_E(\omega_\Sigma(n_0, n_1))$ is dense in $\mathcal{A}_E$, by Proposition 3.8, we show that for each finite set $\omega_\Sigma(n_0, n_1),$

$$ht(\Phi_E, \omega_\Sigma(n_0, n_1)) \leq \max\{h_b(t^*E), h_b(E)\}.$$  

If $s_\alpha s_\beta^* \in \omega(n_0, n_1)$, $r(\alpha) = r(\beta) = v$, then for $l \leq n - 1,$

$$\Phi^n_E(s_\alpha s_\beta^*) = \sum_{|\mu| = l} s_\mu s_\mu^* s_\beta s_\beta^* = \sum_{|\mu| = l} s_\mu s_\mu^* (\sum_{|\nu| = n-i} s_\nu s_\nu^*) s_\beta s_\beta^* = \sum_{|\mu \nu| = n+i+1} s_\mu s_\nu (s_\beta s_\beta^*),$$

because $p_v = \sum_{|\nu| = n-l} s_\nu s_\nu^*$. Hence one sees that

$$\bigcup_{i=0}^{n-1} \Phi^n_E(\omega_\Sigma(n_0, n_1)) \subseteq \left\{ \sum_{|\mu \nu| = n+i+1} s_\mu s_\nu (s_\beta s_\beta^*) s_\alpha s_\beta^* \in \omega(n_0, n_1) \right\}.$$  

Since the set $\{s_\mu s_\mu^* | \mu, \nu \in \bigcup_{i=1}^{n_0} E^{n_1+n}(v_i)\}$ forms a matrix unit, it generates the $C^*$-subalgebra of $\mathcal{A}_E$ which is isomorphic to $M_{k_n}$, where $k_n = \big| \bigcup_{i=1}^{n_0} E^{n_1+n}(v_i) \big|$. Let

$$\rho_n : \text{span}\{s_\alpha s_\beta^* | \alpha, \beta \in \bigcup_{i=1}^{n_0} E^{n_1+n}(v_i)\} \rightarrow M_{k_n}$$

be a $\ast$-isomorphism with the inverse $\rho^{-1}$. Then by Arveson’s extension theorem $\rho$ extends to a contractive cp map $\bar{\rho} : \mathcal{A}_E \rightarrow M_{k_n}$, so that we obtain an element $(\bar{\rho}, \rho^{-1}, M_{k_n}) \in C_{PA}(id, \mathcal{A}_E)$ such that $\|\rho^{-1} \circ \bar{\rho}(x) - x\| = 0$ if

$$x \in \bigcup_{i=0}^{n-1} \Phi^n_E(\omega_\Sigma(n_0, n_1)) \subseteq \text{span}\{s_\alpha s_\beta^* | \alpha, \beta \in \bigcup_{i=1}^{n_0} E^{n_1+n}(v_i)\}.$$
Hence
\[ rcp\left( \bigcup_{i=0}^{n-1} \Phi_E^i(\omega_{\Sigma}(n_0, n_1)), \delta \right) \leq k_n \]
holds for any \( \delta > 0 \). Thus
\[ h_t(\Phi_E, \omega_{\Sigma}(n_0, n_1)) \leq \limsup_{n \to \infty} \frac{1}{n} \log(k_n). \]

On the other hand, the irreducibility of \( E \) implies that there is an \( N \) such that
\[ |E^{n_1+n}(v_i)| \leq |E^{n_1+n+N}(v_1)| \]
for \( 1 \leq i \leq n_0 \). Hence \( k_n = |\bigcup_{i=1}^{n_0} E^{n_1+n}(v_i)| \leq n_0|E^{n_1+n+N}(v_1)| \). Therefore
\[ \limsup_{n \to \infty} \frac{1}{n} \log(k_n) \leq \limsup_{n \to \infty} \frac{1}{n} \log(|E^n(v_1)|), \]
and the assertion then follows from Proposition 3.5(b).

**Example 3.11.** Let \( E := E_{(r_n),(l_n)} \) be Salama’s infinite irreducible graph (see [20]). We assume here that \( l_{n+1} = l_n + 1 \) for each \( n \). There are \( r_k \) edges from the vertex \( k \) to \( k+1 \) and, there is only one path (of length \( l_k - l_{k-1} \)) from the vertex \( v_k \) to \( v_{k+1} \).

\[
\begin{array}{c}
E \\
\begin{array}{cccccccc}
& v_1 & l_1 - l_1 & v_2 & l_3 - l_2 & v_3 & l_4 - l_3 & v_4 & \cdots \\
0 & 1 & 2 & 3 & 4 & \cdots \\
& r_1 & r_2 & r_3 & r_4 & \cdots \\
\end{array}
\end{array}
\]

Note that for each \( n \), \( |E^n(0)\ast| \leq |E^n(0)\ast| \), which then implies by Proposition 3.2 that
\[ h_b(\phi^0) \leq h_b(E). \]

Thus from Theorem 3.10, we have
\[ h_b(\phi^E) \leq h_b(E). \]

In particular, if \( E_p := E_{p,p} \) (\( p > 1 \)) is an irreducible infinite graph of Salama satisfying \( h_1(E_p) = h_b(E_p) = \log p \), by (3) we have
\[ h_t(\phi^E_{E_p} | A_{E_p}) = \log p. \]

**Remark 3.12.** After the paper had been submitted, the authors found a meaningful lower bound for \( h_t(\phi^E_{E_{(r_n),(l_n)}}) \) (see Proposition 3.9) and a better upper bound for \( h_t(\phi^E_{E_{(r_n),(l_n)}}) \) in [12].

**References**


Department of Mathematical Sciences, Seoul National University, Seoul, 151–747 Korea

E-mail address: jajeong@math.snu.ac.kr

Department of Mathematics, Hanshin University, Osan, 447–791 Korea

E-mail address: ghpark@hanshin.ac.kr