

## ON VECTORIAL POLYNOMIALS AND COVERINGS IN CHARACTERISTIC 3

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ABSTRACT. For  $K$  a field containing the finite field  $\mathbb{F}_9$  we give explicitly the whole family of Galois extensions of  $K$  with Galois group  $2S_4 * Q_8$  or  $2S_4 * D_8$  and determine the discriminant of such an extension.

### 1. INTRODUCTION

The motivation of this work is the problem of resolution of singularities in positive characteristic, more precisely the ideas presented by S.S. Abhyankar in [4]. Following Abhyankar, loc. cit. Section 18, let  $N_{k,t}^d$  denote a neighborhood of a simple point on a  $d$ -dimensional algebraic variety over an algebraically closed field  $k$  of characteristic  $p$  from which we have deleted a divisor having a  $t$ -fold normal crossing at the simple point and let  $\pi_A^L(N_{k,t}^d)$  be the set of all Galois groups of finite unramified local Galois coverings of  $N_{k,t}^d$ . In his landmark paper [1], Abhyankar, while working on local uniformization of algebraic varieties in a positive characteristic, proved the inclusion  $\pi_A^L(N_{k,t}^d) \subset P_t(p)$ , where  $P_t(p)$  denotes the set of finite groups  $G$  such that the quotient  $G/p(G)$  of  $G$  by the subgroup  $p(G)$  generated by its  $p$ -Sylow subgroups is abelian, generated by  $t$  generators. Later, using so-called projective and vectorial polynomials, he proved (see [2, 4]) that  $\pi_A^L(N_{k,t}^d)$  contains  $\mathrm{PGL}(m, q)$  and  $\mathrm{GL}(m, q)$ , for every integer  $m > 1$  and every power  $q > 1$  of  $p$ . Recently D. Harbater et al. [7] proved that for a group  $G$  to belong to  $\pi_A^L(N_{k,t}^d)$  it is necessary that  $p(G)$  admit an abelian supplement in  $G$  of rank  $\leq t$ . In [4], Abhyankar exhibited some examples due to G. Stroth of groups contained in  $P_t(p)$  but not satisfying the abelian supplement condition. In characteristic 3, and for  $t = 3$ , the Stroth groups are the groups  $2S_4 * H$ , where  $2S_4$  denotes a double cover of the symmetric group  $S_4$ ,  $H$  is either the quaternion group  $Q_8$  or the dihedral group  $D_8$  of order 8 and  $*$  denotes central product. In this paper, for  $K$  a field containing the finite field  $\mathbb{F}_9$  of nine elements, we give explicitly the whole family of Galois extensions of  $K$  with Galois group  $2S_4 * H$ , and determine the discriminant of such an extension. We note that in [5], the first author provided an explicit construction of  $2S_4 * Q_8$ -extensions of fields containing  $\mathbb{F}_9$  using her previous results on Galois embedding problems based on Serre's trace formula, [9]. Here we use a different method of construction combining Abhyankar's embedding criterion [3]

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and Serre's trace formula, and reach a more explicit and simple formula both for  $2S_4 * Q_8$ - and  $2S_4 * D_8$ -extensions as well as an explicit formula for the discriminant of such extensions. The explicit determination of the discriminant of these extensions is a step towards local uniformization for three-dimensional varieties in positive characteristic.

## 2. PRELIMINARIES

Let us first recall the definitions and fix the notation. We denote by  $2S_n$  one of the two double covers of the symmetric group  $S_n$  reducing to the nontrivial double cover  $2A_n$  of the alternating group  $A_n$ , and by  $H$  either the quaternion group  $Q_8$  or the dihedral group  $D_8$ , double covers of the Klein group  $V_4$ . Let  $K$  be a field of characteristic different from 2 and let  $\tilde{L}|K$  be a Galois extension with Galois group the group  $2S_4 * H$ . Then if  $L$  is the field fixed by the center of  $2S_4 * H$ , we have  $\text{Gal}(L|K) \simeq S_4 \times V_4$ , and for  $L_1, L_2$  the fixed subfields of  $L$  by  $V_4$  and  $S_4$ , respectively, we have  $\text{Gal}(L_1|K) \simeq S_4$  and  $\text{Gal}(L_2|K) \simeq V_4$ . Therefore we obtain the whole family of Galois extensions with Galois group  $2S_4 * H$  of a field  $K$  by constructing the whole family of  $2S_4 * H$ -extensions containing a given arbitrary  $S_4$ -extension of the field  $K$ . Let us now be given a polynomial  $f(X) \in K[X]$  of degree 4 with Galois group  $S_4$  and splitting field  $L_1$  over  $K$ . We want to determine when  $L_1$  is embeddable in a Galois extension of  $K$  with Galois group  $2S_4 * H$ . This fact is equivalent to the existence of a Galois extension  $L_2|K$  with Galois group  $V_4$ , disjoint from  $L_1$ , and such that, if  $L$  is the compositum of  $L_1$  and  $L_2$ , the Galois embedding problem

$$(1) \quad 2S_4 * H \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K)$$

is solvable. We recall that a solution to this embedding problem is a quadratic extension  $\tilde{L}$  of the field  $L$ , which is a Galois extension of  $K$  with Galois group  $2S_4 * H$  and such that the restriction epimorphism between the Galois groups  $\text{Gal}(\tilde{L}|K) \rightarrow \text{Gal}(L|K)$  agrees with the given epimorphism  $2S_4 * H \rightarrow S_4 \times V_4$ . If  $\tilde{L} = L(\sqrt{\gamma})$  is a solution, then the general solution is  $L(\sqrt{r\gamma}), r \in K^*$ . Given a Galois extension  $L_1|K$  with Galois group  $S_4$ , in order to obtain all  $2S_4 * H$ -extensions of  $K$  containing  $L_1$ , we have to determine all  $V_4$ -extensions  $L_2$  of  $K$ , disjoint from  $L_1$ , and such that the embedding problem (1) is solvable.

Let us consider the double covers  $2S_4 \rightarrow S_4$  and  $H \rightarrow V_4$  and let  $\varepsilon_1 \in H^2(S_4, \pm 1)$ ,  $\varepsilon_2 \in H^2(V_4, \pm 1)$  denote the corresponding cohomology elements. Let  $\pi_1 : S_4 \times V_4 \rightarrow S_4$  and  $\pi_2 : S_4 \times V_4 \rightarrow V_4$  be the two projections and let  $\pi_1^*, \pi_2^*$  be the induced morphisms between the 2-cohomology groups. Then the element  $\varepsilon = \pi_1^*(\varepsilon_1) \cdot \pi_2^*(\varepsilon_2) \in H^2(S_4 \times V_4, \{\pm 1\})$  corresponds to the double cover  $2S_4 * H$  of  $S_4 \times V_4$ . This implies that the element in  $H^2(G_K, \{\pm 1\})$  giving the obstruction to the solvability of the embedding problem (1) is equal to the product of the elements giving the obstructions to the solvability of the embedding problems  $2S_4 \rightarrow S_4 \simeq \text{Gal}(L_1|K)$  and  $H \rightarrow V_4 \simeq \text{Gal}(L_2|K)$ .

Let us now specify notation by writing  $2^+S_n$  or  $2^-S_n$  depending on whether transpositions in  $S_n$  lift in the double cover to involutions or to elements of order 4. Let  $E = K[X]/(f(X))$ , for  $f(X)$  the polynomial of degree 4 realizing  $L_1$ , let  $Q_E$  denote the trace form of the extension  $E|K$ , i.e.  $Q_E(x) = \text{Tr}_{E|K}(x^2)$ , and let  $d$  be the discriminant of the polynomial  $f(X)$ . Let  $L_2 = K(\sqrt{a}, \sqrt{b})$ . We denote by  $w$  the Hasse-Witt invariant of a quadratic form and by  $(\cdot, \cdot)$  a Hilbert symbol.

By [9] the obstruction to the solvability of the embedding problem  $2^+S_4 \rightarrow S_4 \simeq \text{Gal}(L_1|K)$  is equal to  $w(Q_E) \cdot (\pm 2, d) \in H^2(G_K, \{\pm 1\})$ . By [10], the obstruction to the solvability of  $Q_8 \rightarrow V_4 \simeq \text{Gal}(L_2|K)$  is equal to  $(a, b) \cdot (-1, ab) \in H^2(G_K, \{\pm 1\})$  and by e.g. [6], the obstruction to the solvability of  $D_8 \rightarrow V_4 \simeq \text{Gal}(L_2|K)$  is equal to  $(a, b) \in H^2(G_K, \{\pm 1\})$  (here we assume that the order 4 elements of  $D_8$  are mapped on the nontrivial element in  $\text{Gal}(L_2|K)$  fixing  $\sqrt{ab}$ ).

From now on, we assume that  $K$  is a field of characteristic 3. We write  $f(X) = X^4 + s_2X^2 - s_3X + s_4$ . By computation of the trace form  $Q_E$ , we obtain

$$w(Q_E) = (ds_2, (s_2^2 - s_4)s_2) \cdot (-1, s_2^2 - s_4).$$

If we further assume that  $K$  contains  $\mathbb{F}_9$ , i.e. that  $-1 \in K^2$ , the solvability of the embedding problem (1) is equivalent to

$$(2) \quad (ds_2, (s_2^2 - s_4)s_2) = (a, b),$$

that is, the equality of two Hilbert symbols.

We now recall the isomorphisms  $S_4 \simeq \text{PGL}(2, 3)$  and  $2^+S_4 \simeq \text{GL}(2, 3)$  and state Abhyankar's Embedding Criterion [3] (1.1), and Polynomial Theorem [3] (2.1), (3.7), in our particular case.

**Proposition 1.** *Let  $K$  be a field of characteristic 3, and let  $M|K$  be a Galois extension with Galois group  $\text{PGL}(2, 3)$ . The embedding problem*

$$(3) \quad \text{GL}(2, 3) \rightarrow \text{PGL}(2, 3) \simeq \text{Gal}(M|K)$$

*is solvable  $\Leftrightarrow M|K$  is the splitting field of a projective polynomial  $Y^4 + c_3Y + c_4 \in K[Y]$ . Moreover, if  $|K| \geq 9$ , the splitting field of the vectorial polynomial  $Y(Y^8 + c_3Y^2 + c_4)$  is a solution to the embedding problem (3).*

### 3. MAIN RESULTS

Under the hypothesis  $\text{char} K = 3$ , the two equivalent conditions to the solvability of the Galois embedding problem  $2^+S_4 \rightarrow S_4 \simeq \text{Gal}(M|K)$  obtained by applying Serre's trace formula and Abhyankar's Embedding Criterion can directly be seen to be equivalent. Indeed, let  $M|K$  be a Galois extension with the Galois group  $S_4$  given as the splitting field of a polynomial  $f(X) = X^4 + s_2X^2 - s_3X + s_4 \in K[X]$ , let  $d$  be the discriminant of  $f(X)$ , let  $x$  be a root of  $f(X)$  in  $M$  and let  $E = K(x)$ . Then  $M$  is the splitting field of a polynomial of the form  $Y^4 + c_3Y + c_4 \in K[Y]$  if and only if there exists elements  $a_0, a_1, a_2, a_3 \in K$  such that the irreducible polynomial over  $K$  of the element  $y = a_0 + a_1x + a_2x^2 + a_3x^3$  has such a form. By computation, this is equivalent to the conditions  $a_0 = -a_2s_2$  and  $Q(a_1, a_2, a_3) := s_2a_1^2 + (s_2^2 - s_4)a_2^2 + s_2^3a_3^2 + (s_2^2 + s_4)a_1a_3 + 2s_2s_3a_2a_3 = 0$ . Now the quadratic trace form  $Q_E$ , for  $E = K(x)$ , is equivalent to  $1 + Q$ , for  $Q$  the quadratic form in  $a_1, a_2, a_3$  in the second condition. If we assume  $w(Q_E) = (2, d)$ , then we have  $Q_E \sim \langle 1, 1, 2, 2d \rangle$  (see [9] 3.2) which implies  $Q \sim \langle 1, 2, 2d \rangle$  and this last quadratic form represents 0 over any field  $K$  of characteristic 3. Reciprocally, assume that  $Q$  represents 0 over  $K$ . Diagonalizing  $Q$ , we obtain  $\langle s_2, m, s_2md \rangle$ , with  $m = s_2^2 - s_4$ , and so we have  $s_2b_1^2 + mb_2^2 + s_2mdb_3^2 = 0$ , for some  $b_1, b_2, b_3 \in K$ , which implies  $(-ds_2, -ms_2) = 1$ , and so  $(ds_2, ms_2) = (-1, md) \cdot (-1, -1)$ . Hence we get  $w(Q_E) = (ds_2, ms_2) \cdot (-1, m) = (-1, d) = (2, d)$ .

**Theorem 1.** *Let  $K$  be a field of characteristic 3 containing  $\mathbb{F}_9$ , and let  $f(X) = X^4 + s_2X^2 - s_3X + s_4 \in K[X]$ , with Galois group  $S_4$  and  $L_1$  the splitting field of  $f(X)$*

over  $K$ . Let  $d = s_4^3 + s_2^2 s_4^2 + s_2^4 s_4 - s_2^3 s_3^2$  be the discriminant of the polynomial  $f(X)$ . The family of elements  $a, b$  in  $K$  such that  $(a, b) = (ds_2, ms_2)$ , where  $m = s_2^2 - s_4$ , can be given in terms of an arbitrary invertible matrix  $P = (p_{ij})_{1 \leq i, j, \leq 3} \in \text{GL}(3, K)$  as  $a = dA$ ,  $b = s_2 mF$ , where

$$A = s_2 p_{11}^2 + m p_{21}^2 + d m s_2 p_{31}^2,$$

$$F = d m P_{13}^2 + d s_2 P_{23}^2 + P_{33}^2, \quad \text{with } P_{ij} = \begin{vmatrix} p_{ii} & p_{ij} \\ p_{ji} & p_{jj} \end{vmatrix}.$$

Let  $L_2 = K(\sqrt{a}, \sqrt{b})$  and assume that  $L_2|K$  has Galois group  $V_4$  and  $L_1 \cap L_2 = K$  (i.e. that the elements  $a, b, ab, da, db, dab$  are not squares in  $K$ ). Let  $L = L_1 \cdot L_2$ . For  $x$  a root of the polynomial  $f(X)$ , take  $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ , with

$$a_0 = -s_2 a_2,$$

$$a_1 = d m \sqrt{-1} (n s_2 p_{11} P_{23} - p_{21} P_{33} + m n p_{21} P_{13} - d s_2 p_{31} P_{23}) + m \sqrt{a} (d P_{13} + n P_{33}),$$

$$a_2 = d s_2 \sqrt{-1} (p_{11} P_{33} - s_2^2 s_3 p_{11} P_{23} - m s_2 s_3 p_{21} P_{13} - d m p_{31} P_{13}) - s_2 \sqrt{a} (s_2 s_3 P_{33} + d P_{23}),$$

$$a_3 = d m s_2 \sqrt{-1} (s_2 p_{11} P_{23} + m p_{21} P_{13}) + m s_2 \sqrt{a} P_{33},$$

where  $n = s_2^2 + s_4$ . Then  $L(\sqrt{r y})$ ,  $r \in K^*$ , is the general solution to the embedding problem

$$2^+ S_4 * D_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K).$$

*Proof.* By [8], 3.2, the equality of Hilbert symbols (2) is equivalent to the  $K$ -equivalence of quadratic forms

$$(4) \quad \langle ds_2, ms_2, dm \rangle \sim \langle a, b, ab \rangle.$$

The family of quadratic forms  $K$ -equivalent to  $R := \langle ds_2, ms_2, dm \rangle$  is given by  $P^T R P$ , for  $P$  running over  $\text{GL}(3, K)$ . By diagonalizing  $P^t R P$ , we obtain  $\langle dA, s_2 mF, dA s_2 mF \rangle$ , with  $A$  and  $F$  as in the statement. Let  $a = dA$ ,  $b = s_2 mF$ . Now, we have  $(a, b) = 1 \in H^2(G_{K(\sqrt{a})}, \{\pm 1\})$  and, as  $a \notin K^2$  and  $L_1 \cap K(\sqrt{a}) = K$ , the extension  $L_1(\sqrt{a})|K(\sqrt{a})$  has Galois group  $S_4$ , and the Galois embedding problem  $2^+ S_4 \rightarrow S_4 \simeq \text{Gal}(L_1(\sqrt{a})|K(\sqrt{a}))$  is solvable. By the argument preceding Theorem 1, there exist  $a_1, a_2, a_3 \in K(\sqrt{a})$  such that  $Q(a_1, a_2, a_3) = 0$ , and for the element  $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$  we have that  $\text{Irr}(y, K(\sqrt{a}))$  is a projective polynomial. Also, by Abhyankar's Polynomial Theorem (see Proposition 1), the splitting field of the vectorial polynomial  $Y. \text{Irr}(\sqrt{y}, K(\sqrt{a}))$ , that is, the field  $L_1(\sqrt{a})(\sqrt{y})$ , is a solution to the Galois embedding problem  $2^+ S_4 \rightarrow S_4 \simeq \text{Gal}(L_1(\sqrt{a})|K(\sqrt{a}))$ . Now our aim is to compute explicitly such elements  $a_i$ . Diagonalizing  $Q$ , we obtain  $\langle s_2, m, s_2 m d \rangle$  and from (4) we get that  $\langle s_2, m, s_2 m d \rangle \sim \langle A, s_2 m A F, s_2 m F d \rangle$  and the basis change matrix can be written down explicitly in terms of the matrix  $P$ . Now the vector  $(0, d\sqrt{-1}, \sqrt{a}) \in K(\sqrt{a})^3$  annihilates the quadratic form  $\langle A, s_2 m A F, s_2 m F d \rangle$ , and from it we obtain the values for  $a_1, a_2, a_3 \in K(\sqrt{a})$  such that  $Q(a_1, a_2, a_3) = 0$ .

Now we want to see that  $L(\sqrt{y})|K$  is a Galois extension with Galois group  $2^+ S_4 * D_8$ . By the assumption  $L_1 \cap L_2 = K$ , we have  $\text{Gal}(L(\sqrt{y})|L_2) \simeq 2^+ S_4$ . We now consider the behaviour of  $y$  under the action of  $\text{Gal}(L_2|K)$ . Let  $r, s, t$  be the nontrivial elements of  $\text{Gal}(L_2|K)$  fixing respectively  $\sqrt{ab}, \sqrt{b}, \sqrt{a}$ . By computation we obtain  $y^s y = d^2 h^2 b$ , where  $h = m s_2 p_{31} x^3 + (p_{21} - s_2^2 s_3 p_{31}) x^2 + (m n p_{31} + p_{11}) x + s_2^3 s_3 p_{31} - s_2 p_{21}$ . Now  $y \in K(\sqrt{a})(x)$ , so  $y^t = y$  and  $y^r = y^s$ , so  $L(\sqrt{y})$  is Galois over

$K$ . Now we have  $(dh\sqrt{b})^s = dh\sqrt{b}$  and  $(dh\sqrt{b})^r = -dh\sqrt{b}$ , so  $\text{Gal}(L(\sqrt{y})|L_1) \simeq D_8$ , with  $L(\sqrt{y})|L_1(\sqrt{ab})$  cyclic; hence  $\text{Gal}(L(\sqrt{y})|K) \simeq 2^+ S_4 * D_8$ .  $\square$

*Remark 1.* For the element  $y$  given by Theorem 1, we have  $\text{Irr}(y, K(\sqrt{a})) = Y^4 + c_3 Y + c_4$ , where

$$\begin{aligned} c_3 &= s_3 a_1^3 + m a_1^2 a_2 + s_2 s_3 a_1^2 a_3 + s_2 s_3 a_1 a_2^2 + m s_2 a_1 a_2 a_3 + m s_3 a_1 a_2^2 + s_3^2 a_2^3 \\ &\quad - n s_3 a_2^2 a_3 + (m^2 + s_2 s_3^2) a_2 a_3^2 + s_3^3 a_3^3, \\ c_4 &= \frac{d}{s_2^2} (a_1 a_3 - a_2^2 - s_2 a_3^2)^2. \end{aligned}$$

**Theorem 2.** *Let the fields  $K$  and  $L$  and the elements  $d, a, b$  and  $y$  be as in Theorem 1, let  $\mu = d + (d+1)\sqrt{d}$  and let  $\rho = ab + (ab+1)\sqrt{ab}$ . Then*

1.  $L(\sqrt{r\mu y}), r \in K^*$ , is the general solution to the embedding problem

$$2^- S_4 * D_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K).$$

2.  $L(\sqrt{r\rho y}), r \in K^*$ , is the general solution to the embedding problem

$$2^+ S_4 * Q_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K).$$

3.  $L(\sqrt{r\mu\rho y}), r \in K^*$ , is the general solution to the embedding problem

$$2^- S_4 * Q_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K).$$

*Proof.* For  $\sigma \in S_4 \setminus A_4$ , we have  $\mu^\sigma \mu = -(d-1)^2 d$  and so,  $L(\sqrt{\mu y})|K$  Galois. Now  $(d-1)\sqrt{-d}$  changes sign under the action of  $\sigma$ , so  $\text{Gal}(L(\sqrt{\mu y})|L_2) \simeq 2^- S_4$ , hence  $\text{Gal}(L(\sqrt{\mu y})|K) \simeq 2^- S_4 * D_8$ .

For  $r, s, t \in \text{Gal}(L_2|K)$  fixing  $\sqrt{ab}, \sqrt{b}, \sqrt{a}$ , resp., we have  $\rho^s \rho = \rho^t \rho = -(ab-1)^2 ab$  and  $\rho^r \rho = \rho^2$ , so  $L(\sqrt{\rho y})|K$  Galois. Now  $(ab-1)\sqrt{-ab}$  changes sign under the action of  $s$  and under the action of  $t$ , so  $\text{Gal}(L(\sqrt{\rho y})|L_1) \simeq Q_8$ , hence  $\text{Gal}(L(\sqrt{\rho y})|K) \simeq 2^+ S_4 * Q_8$ .

Combining both arguments, we obtain the third statement in the theorem.  $\square$

**Proposition 2.** *Let the fields  $K$  and  $L$  and the elements  $s_2, s_3, s_4, d, a, b, m, p_{ij}$  and  $y$  be as in Theorem 1;  $\mu, \rho$  as in Theorem 2. We have*

$$\begin{aligned} \text{disc}(L(\sqrt{y})|K) &= d^{104} a^{96} b^{100} D^2, \\ \text{disc}(L(\sqrt{\mu y})|K) &= d^{152} a^{96} b^{100} D^2 (d-1)^{48}, \\ \text{disc}(L(\sqrt{\rho y})|K) &= d^{104} a^{144} b^{148} D^2 (ab-1)^{48}, \\ \text{disc}(L(\sqrt{\mu\rho y})|K) &= d^{152} a^{144} b^{148} D^2 (d-1)^{48} (ab-1)^{48}, \end{aligned}$$

where

$$\begin{aligned} D &= s_4 p_{11}^4 - s_2 s_3 p_{11}^3 p_{21} + m s_2 p_{11}^2 p_{21}^2 - m s_3 p_{11} p_{21}^3 + (m^2 - s_2 s_3^2) p_{21}^4 \\ &\quad + d p_{31} (-p_{11}^3 + m s_2^2 p_{11}^2 p_{31} - s_3 p_{21}^3 + m^2 s_2 p_{31} p_{21}^2) + d^2 p_{31}^3 (s_2 s_3 p_{21} + m p_{11}) + d^3 p_{31}^4. \end{aligned}$$

*Proof.* We have  $\text{disc}(L(\sqrt{y})|K) = \text{disc}(L|K)^2 \cdot N_{L|K}(y)$  and  $\text{disc}(L|K) = (dab)^{48}$ . Now  $N_{L|K}(y) = (N_{L_1(\sqrt{a})|K}(y))^2 = (N_{K(\sqrt{a})|K}(c_4))^2$ , for  $c_4$  the degree 0 coefficient in the irreducible polynomial of  $y$  over  $K(\sqrt{a})$ . By computation we obtain  $c_4 = \frac{d}{s_2^2} (a_1 a_3 - a_2^2 - s_2 a_3^2)^2$  and, by substituting the values of  $a_1, a_2, a_3$  and computing the norm,  $N_{K(\sqrt{a})|K}(c_4) = d^8 b^4 D^2$  for  $D$  as in the statement.

To obtain the other three discriminants, it is now enough to compute  $N_{L|K}(\mu) = N_{K(\sqrt{d})|K}(\mu)^{48} = (d-1)^{48} d^{48}$  and  $N_{L|K}(\rho) = N_{K(\sqrt{ab})|K}(\rho)^{48} = (ab-1)^{48} (ab)^{48}$ .  $\square$

## 4. EXAMPLES

Let  $K = k((Z_1, Z_2, Z_3))$  be the quotient field of the formal power series ring in 3 variables over a field  $k$  containing  $\mathbb{F}_9$ . We consider the polynomial

$$f(X) = X^4 + Z_1X^2 + Z_2X + Z_3 \in K[X],$$

i.e. we are taking  $s_2 = Z_1, s_3 = -Z_2, s_4 = Z_3$ . We can check that the polynomial  $f$  has Galois group  $S_4$  over  $K$  and let  $L_1$  be the splitting field of  $f$  over  $K$ . We consider the extension  $L_2|K$  generated by the elements  $\sqrt{ds_2}, \sqrt{ms_2}, \sqrt{dm}$ , which corresponds to taking the matrix  $P$  in Theorem 1 to be one of the matrices

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We can check that the elements  $ds_2, ms_2, dm, s_2, dms_2, m$  are not squares in  $K$ , and so  $L_2|K$  has Galois group  $V_4$  and is disjoint with  $L_1|K$ . Let  $L = L_1 \cdot L_2$ . We denote by  $y_i$  the element  $y$  given by Theorem 1 for each of the matrices  $P_i$ ,  $i = 1, 2, 3$ . Then we have

$$\text{Gal}(L(\sqrt{y_i})|K) \simeq 2^+S_4 * D_8, \quad i = 1, 2, 3,$$

with  $L(\sqrt{y_1})|L_1(\sqrt{dm})$ ,  $L(\sqrt{y_2})|L_1(\sqrt{ms_2})$ ,  $L(\sqrt{y_3})|L_1(\sqrt{ds_2})$  cyclic. The factors appearing in  $\text{disc}(L(\sqrt{y_1})|K)$  are  $d, s_2, m$  and  $s_4$ , the factors appearing in  $\text{disc}(L(\sqrt{y_2})|K)$  are  $d, s_2, m$  and  $m^2 - s_2s_3^2$ , and the factors appearing in  $\text{disc}(L(\sqrt{y_3})|K)$  are  $d, s_2, m$ . In particular the discriminantal locus remains unchanged when going from  $L$  to  $L(\sqrt{y_3})$ .

We observe that the elements  $d - 1$  and  $ab - 1$ , for each choice of  $a, b$  among  $ds_2, ms_2, dm$ , are invertible elements in the ring  $k[[Z_1, Z_2, Z_3]]$ , and so the discriminant locus will not change if we realize any other of the groups  $2S_4 * H$  over the same field  $K$  by means of Theorem 2.

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