PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 1, Pages 5–8
S 0002-9939(05)08276-6

STANLEY'S THEOREM ON CODIMENSION 3 GORENSTEIN h-VECTORS

FABRIZIO ZANELLO

(Communicated by Bernd Ulrich)

Abstract. In this note we supply an elementary proof of the following well-known theorem of R. Stanley: the h-vectors of Gorenstein algebras of codimension 3 are SI-sequences, i.e. are symmetric and the first difference of their first half is an O-sequence.

We consider standard graded artinian algebras $A = R/I$, where $R = k[x_1, ..., x_r]$, k is any field, the x_i's have degree 1 and I is a homogeneous ideal of R. Recall that the *h-vector* of A is $h(A) = h = (h_0, h_1, ..., h_e)$, where $h_i = \dim_k A_i$ and e is the last index such that $\dim_k A_e > 0$. Since we may suppose that I does not contain non-zero forms of degree 1, $r = h_1$ is defined to be the *codimension* of A.

The *socle* of A is the annihilator of the maximal homogeneous ideal $\overline{m} = (\overline{x_1}, ..., \overline{x_r}) \subset A$, namely $soc(A) = \{a \in A \mid a\overline{m} = 0\}$. Since $soc(A)$ is a homogeneous ideal, we define the *socle-vector* of A as $s(A) = s = (s_0, s_1, ..., s_e)$, where $s_i = \dim_k soc(A)_i$. Note that $s_e = h_e > 0$. The integer e is called the *socle degree* of A (or of $h(A)$). If $s = (0, 0, ..., 0, s_e = 1)$, we say that the algebra A is *Gorenstein*.

The next theorem is a well-known result of Macaulay.

Definition-Remark 1. Let n and i be positive integers. The *i-binomial expansion of n* is

$$n_{(i)} = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + ... + \binom{n_j}{j},$$

where $n_i > n_{i-1} > ... > n_j \geq j \geq 1$.

Under these hypotheses, the i-binomial expansion of n is unique (e.g., see [BH], Lemma 4.2.6).

Furthermore, define

$$n^{\langle i \rangle} = \binom{n_i + 1}{i+1} + \binom{n_{i-1} + 1}{i-1+1} + ... + \binom{n_j + 1}{j+1}.$$

Theorem 2 (Macaulay). *Let $h = (h_i)_{i \geq 0}$ be a sequence of non-negative integers, such that $h_0 = 1$, $h_1 = r$ and $h_i = 0$ for $i > e$. Then h is the h-vector of some standard graded artinian algebra if and only if, for every d, $1 \leq d \leq e-1$,*

$$h_{d+1} \leq h_d^{\langle d \rangle}.$$

Received by the editors June 19, 2004.
2000 *Mathematics Subject Classification.* Primary 13E10; Secondary 13H10.
Key words and phrases. Artinian algebra, Gorenstein algebra, h-vector, SI-sequence.

Proof. See [BH], Theorem 4.2.10. (This theorem holds, with appropriate modifications, for any standard graded algebra, not necessarily artinian.) ☐

Let us now recall a few definitions. We denote by $\lfloor x \rfloor$, as usual, the largest integer less than or equal to x.

Definition 3. i) A sequence of non-negative integers which satisfies the growth condition of Macaulay's theorem is called an *O-sequence*.

ii) A vector of non-negative integers $v = (v_0, v_1, ..., v_d)$ is *differentiable* if its first difference,

$$\Delta v = ((\Delta v)_0 = 1, (\Delta v)_1 = v_1 - v_0, ..., (\Delta v)_d = v_d - v_{d-1}),$$

is an O-sequence. (It is easy to see that if v is differentiable, then v is itself an O-sequence.)

iii) An h-vector $h = (1, h_1, ..., h_e)$ is an *SI-sequence* (named after Stanley and Iarrobino) if it is symmetric with respect to $\frac{e}{2}$ and if its first half, $(1, h_1, ..., h_{\lfloor \frac{e}{2} \rfloor})$, is differentiable.

The study of the possible Gorenstein h-vectors is a central problem in commutative algebra. Initially, Stanley and Iarrobino (independently) conjectured that all Gorenstein h-vectors (of any codimension r) are SI-sequences. (See also Harima's paper [Ha] on the h-vectors of Gorenstein algebras having the weak Lefschetz property.)

The fact that Gorenstein h-vectors are symmetric is well known, and a theorem of Migliore-Nagel ([MN]) and Cho-Iarrobino ([CI]) shows that, in any codimension, an SI-sequence is always a Gorenstein h-vector.

The converse (that, as we just said, was conjectured to be true) instead has been proven false for $r \geq 5$. In particular, not even all Gorenstein h-vectors are *unimodal* (i.e. they do not increase after they start decreasing). The first example of a Gorenstein algebra with a non-unimodal h-vector was given by Stanley (see [St1], Example 4.3) in codimension 13. Later, Bernstein-Iarrobino ([BI]), Boij-Laksov ([BL]) and Boij ([Bo]) exhibited many other non-unimodal Gorenstein h-vectors of codimension 5 or greater.

In codimension 4, we do not know whether or not all Gorenstein h-vectors h are SI-sequences, or even whether they must be all unimodal. There is, however, a remarkable result of Iarrobino and Srinivasan ([IS]) which shows that, if the entry of degree 2 of h is less than or equal to 7, then h must be an SI-sequence.

Instead, in codimension 2, the conjecture that all Gorenstein h-vectors are SI-sequences is correct, as first observed by Macaulay ([Ma]), and is indeed an easy exercise assuming Macaulay's Theorem 2 and the symmetry of Gorenstein h-vectors.

In codimension 3, the conjecture still holds true, as shown by Stanley (see [St1], Theorem 4.2). His proof is based on a deep structure theorem due to Buchsbaum and Eisenbud ([BE], Proposition 3.3). The purpose of the present note is to supply an elementary proof of this important result of Stanley.

Theorem 4. *Let h be a Gorenstein h-vector of codimension 3. Then h is an SI-sequence.*

Before going into the proof we need to recall the following observation of Stanley.

Remark 5 (Stanley). Let $h = (1, h_1, ..., h_e)$ be a Gorenstein h-vector. Then, for any index $j \geq 1$, there exists a Gorenstein h-vector $(a_j = 1, a_{j+1}, ..., a_e)$ such that the vector $(1, h_1, ..., h_{j-1}, h_j - a_j, ..., h_e - a_e)$ is an O-sequence.

Proof. See [St2], bottom of p. 67. □

Proof of Theorem 4. Let $h = (1, h_1 = 3, h_2, ..., h_e)$ be a Gorenstein h-vector of codimension 3. We want to show that its first half is differentiable. By Remark 5 (with $j = 1$), there exists a Gorenstein h-vector $a = (a_1 = 1, a_2, ..., a_{e-1}, a_e)$ such that

$$(1, \Delta_1 = 2, \Delta_2 = h_2 - a_2, ..., \Delta_e = h_e - a_e)$$

is an O-sequence. Note that, by this choice of the indices, a is symmetric with respect to $\frac{e+1}{2}$; in particular, $a_2 = a_{e-1} \leq h_{e-1} = h_1 = 3$.

First we show that h is unimodal. Suppose it is not. We may assume, by induction, that e is the least socle degree for which there exists a non-unimodal Gorenstein h-vector of codimension 3. Hence we have $h_i < h_{i-1}$ for some $i \leq \left\lfloor \frac{e}{2} \right\rfloor$. Since a is unimodal (by the induction hypothesis, if it has codimension 3), we have $\Delta_i < i + 1$ (i.e. Δ_i is not *generic*). But

$$\Delta_{e-(i-1)} = h_{e-(i-1)} - a_{e-(i-1)} = h_{i-1} - a_i > h_i - a_i = \Delta_i,$$

a contradiction, since, by Macaulay's Theorem 2, an O-sequence starting with $(1, 2, ...)$ cannot increase after it is no longer generic. This proves that h is unimodal.

Now we want to show that the first half of h is differentiable. We may suppose, by induction, that all Gorenstein h-vectors of codimension 3 and socle degree lower than e are SI-sequences. The differentiability of h is obvious as long as h is generic (i.e. $h_i = \binom{i+2}{2}$). Hence suppose that h_i is not generic, for some $i \leq \left\lfloor \frac{e}{2} \right\rfloor$. By Macaulay's theorem, we need to show that

$$h_i - h_{i-1} \leq h_{i-1} - h_{i-2}. \tag{1}$$

Let us first consider the case $a_i = \binom{i+1}{2}$. We have

$$\binom{i+1}{2} = a_i = a_{e-(i-1)} \leq h_{e-(i-1)} = h_{i-1} \leq \binom{i+1}{2},$$

and therefore $h_{i-1} = \binom{i+1}{2}$, i.e. h_{i-1} is generic. Thus, (1) becomes $h_i - \binom{i+1}{2} \leq i$, and this is true since $h_i < \binom{i+2}{2}$. This proves the theorem for $a_i = \binom{i+1}{2}$.

Hence, let us assume from now on that $a_i < \binom{i+1}{2}$. Suppose now that Δ_{i-1} is generic (i.e. $\Delta_{i-1} = i$). Therefore (1) becomes

$$a_i + \Delta_i - a_{i-1} - i \leq a_{i-1} + i - a_{i-2} - (i-1),$$

which is true since $\Delta_i - i \leq 1$ and $a_i - a_{i-1} \leq a_{i-1} - a_{i-2}$, because $a_i < \binom{i+1}{2}$ and a is an SI-sequence (by induction, if it has codimension 3). This completes the proof for $\Delta_{i-1} = i$.

Hence let us suppose that $\Delta_{i-1} \leq i - 1$. Therefore $\Delta_{i-1} \geq \Delta_{e-(i-1)} \geq \Delta_{e-(i-2)}$, whence

$$a_{e-(i-1)} - a_{e-(i-2)} \leq h_{e-(i-1)} - h_{e-(i-2)},$$

i.e.

$$a_i - a_{i-1} \leq h_{i-1} - h_{i-2}. \tag{2}$$

Similarly, $\Delta_{i-1} \geq \Delta_i$, i.e.

$$h_i - h_{i-1} \leq a_i - a_{i-1}. \tag{3}$$

Thus, (1) follows from (3) and (2). This proves the theorem. □

References

[BI] D. Bernstein and A. Iarrobino: *A nonunimodal graded Gorenstein Artin algebra in codimension five*, Comm. in Algebra 20 (1992), No. 8, 2323-2336. MR1172667 (93i:13012)

[Bo] M. Boij: *Graded Gorenstein Artin algebras whose Hilbert functions have a large number of valleys*, Comm. in Algebra 23 (1995), No. 1, 97-103. MR1311776 (96h:13040)

[BL] M. Boij and D. Laksov: *Nonunimodality of graded Gorenstein Artin algebras*, Proc. Amer. Math. Soc. 120 (1994), 1083-1092. MR1227512 (94g:13008)

[BH] W. Bruns and J. Herzog: *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, No. 39, revised edition (1998), Cambridge, U.K. MR1251956 (95h:13020)

[BE] D.A. Buchsbaum and D. Eisenbud: *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. 99 (1977), 447-485. MR0453723 (56:11983)

[CI] Y.H. Cho and A. Iarrobino: *Inverse systems of zero-dimensional schemes in* $\mathbf{P}^n$, J. of Algebra, to appear.

[Ha] T. Harima: *Characterization of Hilbert functions of Gorenstein Artin algebras with the weak Stanley property*, Proc. Amer. Math. Soc. 123 (1995), No. 12, 3631-3638. MR1307527 (96b:13014)

[IS] A. Iarrobino and H. Srinivasan: *Some Gorenstein Artin algebras of embedding dimension four, I: components of* $PGOR(H)$ *for* $H = (1, 4, 7, ..., 1)$, J. of Pure and Applied Algebra, to appear.

[Ma] F.H.S. Macaulay: *The Algebraic Theory of Modular Systems*, Cambridge Univ. Press, Cambridge, U.K. (1916). MR1281612 (95i:13001)

[MN] J. Migliore and U. Nagel: *Reduced arithmetically Gorenstein schemes and simplicial polytopes with maximal Betti numbers*, Adv. in Math. 180 (2003), 1-63. MR2019214 (2004k:14082)

[St1] R. Stanley: *Hilbert functions of graded algebras*, Adv. Math. 28 (1978), 57-83. MR0485835 (58:5637)

[St2] R. Stanley: *Combinatorics and Commutative Algebra*, Second Ed., Progress in Mathematics 41 (1996), Birkhäuser, Boston, MA. MR1453579 (98h:05001)

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI GENOVA, GENOVA, ITALY
E-mail address: `zanello@dima.unige.it`