

## MULTIPLICITIES OF REPRESENTATIONS IN SPACES OF MODULAR FORMS

BIN YONG HSIE

(Communicated by David E. Rohrlich)

ABSTRACT. This paper shows that for a given irreducible representation  $\rho$  of  $\Gamma/\Gamma_1$ , the two functions  $\dim(M_k(\Gamma_1, \rho))$  and  $\dim(S_k(\Gamma_1, \rho))$  of  $k$  are almost linear functions.

Let  $\Gamma$  and  $\Gamma_1$  be two Fuchsian subgroups of  $\mathrm{SL}_2(\mathbb{Q})$  of the first kind, with  $\Gamma_1$  a normal subgroup of  $\Gamma$  of index  $\mu$ . Let  $\mathbb{H}^*$  denote the extended upper half-plane as in Shimura [4] and put  $Y = \Gamma \backslash \mathbb{H}^*$  and  $X = \Gamma_1 \backslash \mathbb{H}^*$ . Let  $\Omega(X)$  and  $\Omega(Y)$  be the fields of meromorphic functions of  $X$  and  $Y$  respectively. Then  $A_0(\Gamma) \cong \Omega(Y)$  and  $A_0(\Gamma_1) \cong \Omega(X)$  in a natural way, where  $A_k(*)$  denotes the space of meromorphic modular forms for  $*$  of weight  $k$ . We define a representation  $\pi_k$  of  $\Gamma$  on  $A_k(\Gamma_1)$  by the formula

$$\pi_k(\gamma)(f) = f|[\gamma^{-1}]_k \quad (\gamma \in \Gamma, \quad f \in A_k(\Gamma_1)),$$

where the notation  $[*]_k$  is as in [4]. The representation  $\pi_k$  factors through  $\Gamma/\Gamma_1$  to give a representation —also denoted  $\pi_k$ —of the latter group. The space  $M_k(\Gamma_1)$  of holomorphic modular forms of weight  $k$  for  $\Gamma_1$  and the subspace  $S_k(\Gamma_1)$  of cusp forms of weight  $k$  for  $\Gamma_1$  are both stable under  $\pi_k$ , and the resulting representations of  $\Gamma/\Gamma_1$  on  $M_k(\Gamma_1)$  and  $S_k(\Gamma_1)$  will be denoted  $\rho_k$  and  $\sigma_k$  respectively.

Henceforth,  $\rho$  denotes an irreducible complex representation of  $\Gamma/\Gamma_1$ . If  $-I \in \Gamma$ , then the value of  $\rho$  on the coset of  $-I$  in  $\Gamma/\Gamma_1$  is a scalar by Schur's lemma, and we say that  $\rho$  is *even* or *odd* according to whether the scalar is 1 or  $-1$ . If  $-I \in \Gamma_1$ , then  $\rho$  is automatically even.

If  $\pi$  is any finite-dimensional complex representation of  $\Gamma/\Gamma_1$ , then we write  $\langle \rho, \pi \rangle$  for the multiplicity of  $\rho$  in  $\pi$ . For example,  $\langle \rho, \rho_{\mathrm{reg}} \rangle = \dim \rho$ , where  $\rho_{\mathrm{reg}}$  is the regular representation of  $\Gamma/\Gamma_1$ .

**Theorem.** *Fix an irreducible complex representation  $\rho$  of  $\Gamma/\Gamma_1$ , and put*

$$c = \frac{1}{4\pi} \int_{\Gamma \backslash \mathbb{H}} \frac{dx dy}{y^2}.$$

*If  $-I \notin \Gamma$ , then*

$$\lim_{k \rightarrow \infty} \frac{\langle \rho, \rho_k \rangle}{k \langle \rho, \rho_{\mathrm{reg}} \rangle} = \lim_{k \rightarrow \infty} \frac{\langle \rho, \sigma_k \rangle}{k \langle \rho, \rho_{\mathrm{reg}} \rangle} = c.$$

*If  $-I \in \Gamma$ , then the same assertion holds provided  $k$  runs through positive integers of the same parity as  $\rho$ .*

---

Received by the editors June 4, 2004.

2000 *Mathematics Subject Classification.* Primary 11F11.

*Key words and phrases.* Modular form, cusp form.

©2005 American Mathematical Society  
 Reverts to public domain 28 years from publication

*Proof.* We shall prove our assertion only for  $\rho_k$ . The proof for  $\sigma_k$  is similar.

For any positive integer  $k$ , let  $i(k) = (-1)^k$ . If  $-I \in \Gamma_1$ , we assume  $k$  is always even.

Let  $\mathcal{S}$  be the set of the irreducible representations of  $\Gamma/\Gamma_1$  with the same parity as  $k$  if  $-I \in \Gamma$ , and the set of all irreducible representations of  $\Gamma/\Gamma_1$  otherwise.

Let  $p$  be a non-cusp point of  $Y$  which has  $\mu$  different liftings  $p_1, \dots, p_\mu$  on  $X$ , and let  $\tilde{p}_j$  be a lifting of  $p_j$  in  $\mathbb{H}^*$  ( $1 \leq j \leq \mu$ ). By the Riemann-Roch theorem we know that there exists an  $f \in A_{i(k)}(\Gamma_1)$  such that  $f$  has poles only at cusps and  $f(\tilde{p}_1) = 1$ ,  $f(\tilde{p}_j) = 0$  for all  $j = 2, \dots, \mu$ . For any  $\alpha \in \Gamma/\Gamma_1$ , put  $f_\alpha = f|[\alpha]_{i(k)}$ . For a fixed sufficiently large even  $n_0$ , we may choose  $\varphi = \varphi_{n_0} \in S_{n_0}(\Gamma)$  such that  $\varphi f_\alpha$  ( $\alpha \in \Gamma/\Gamma_1$ ) are all cusp forms. Let  $W$  denote the  $\mathbb{C}$ -linear subspace of  $M_{i(k)+n_0}(\Gamma_1)$  spanned by  $\{\varphi f_\alpha | \alpha \in \Gamma/\Gamma_1\}$ . Clearly,  $W$  is stable under  $\pi_{i(k)+n_0}$ . It is easy to see that

$$W \cong \bigoplus_{\rho \in \mathcal{S}} \langle \rho, \rho_{\text{reg}} \rangle \rho$$

since the two-hand sides have the same trace.

When  $k \geq i(k) + n_0$ , the choice of  $f$  ensures that the natural map

$$W \otimes_{\mathbb{C}} M_{k-i(k)-n_0}(\Gamma) \rightarrow M_k(\Gamma_1)$$

is injective. Hence,

$$(1) \quad \frac{\langle \rho, \rho_k \rangle}{k \langle \rho, \rho_{\text{reg}} \rangle} \geq \frac{\dim(M_{k-i(k)-n_0}(\Gamma))}{k}.$$

For  $\rho$  and  $k$  as in the theorem, in [4] Shimura showed that

$$(2) \quad \lim_{k \rightarrow \infty} \frac{\dim(M_{k-i}(\Gamma))}{k} = c$$

and

$$(3) \quad \lim_{k \rightarrow \infty} \frac{\dim(M_k(\Gamma_1))}{k} = c\mu.$$

From (1) and (2) we easily deduce that

$$(4) \quad \liminf_{k \rightarrow \infty} \frac{\langle \rho, \rho_k \rangle}{k} \geq c \langle \rho, \rho_{\text{reg}} \rangle.$$

It is obvious that (3) is equivalent to

$$\lim_{k \rightarrow \infty} \sum_{\rho \in \mathcal{S}} \dim(\rho) \frac{\langle \rho, \rho_k \rangle}{k} = \sum_{\rho \in \mathcal{S}} \dim(\rho) c \langle \rho, \rho_{\text{reg}} \rangle.$$

Combining this equality with (4), we get

$$\lim_{k \rightarrow \infty} \frac{\langle \rho, \rho_k \rangle}{k} = c \langle \rho, \rho_{\text{reg}} \rangle,$$

as desired. □

#### ACKNOWLEDGEMENT

I would like to thank Professor David E. Rohrlich for his kind help in revising this paper.

## REFERENCES

1. O. Forster, *Lectures on Riemann Surfaces*, Springer-Verlag, New York, 1981. MR0648106 (83d:30046)
2. N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer-Verlag, New York, 1984. MR0766911 (86c:11040)
3. J-P. Serre, *Linear Representations of Finite Groups*, Springer-Verlag, New York, 1977. MR0450380 (56:8675)
4. G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press, Princeton, New Jersey, 1971.

LMAM, DEPARTMENT OF MATHEMATICS, PEKING UNIVERSITY, BEIJING 100871, PEOPLE'S REPUBLIC OF CHINA

*E-mail address:* byhsie@math.pku.edu.cn