

## STABILITY OF THE FIXED POINT PROPERTY IN HILBERT SPACES

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ABSTRACT. In this paper we prove that if  $X$  is a Banach space whose Banach-Mazur distance to a Hilbert space is less than  $\sqrt{\frac{5+\sqrt{17}}{2}}$ , then  $X$  has the fixed point property for nonexpansive mappings.

### 1. INTRODUCTION

Let  $(X, \|\cdot\|)$  be a Banach space and let  $C$  be a nonempty subset of  $X$ . A mapping  $T : C \rightarrow C$  is said to be *nonexpansive* whenever  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ .

We shall say that a nonempty subset  $C$  of a Banach space has the *fixed point property for nonexpansive mappings* if each nonexpansive mapping  $T : C \rightarrow C$  has a fixed point. A Banach space has the *fixed point property* (*FPP* for short) if each of its nonempty bounded closed convex subsets has the fixed point property for nonexpansive mappings, and the *weak fixed point property* (*wc-FPP*) if each of its nonempty weakly compact convex subsets has the fixed point property for nonexpansive mappings.

In 1971 Zizler [10] showed that every separable Banach space admits an equivalent renorming which is uniformly convex in every direction and consequently has the *wc-FPP* [3]. In 1981 Alspach [2] proved that the separable Banach space  $(L^1[0, 1], \|\cdot\|_1)$  lacks the *wc-FPP*. As a consequence of Zizler's and Alspach's results it was known that, in general, the *wc-FPP* is not preserved under topological isomorphisms.

Despite of this fact, there are results establishing that the *wc-FPP* extends from Banach spaces with nice geometric properties to Banach spaces whose Banach-Mazur distance to them is small enough. We recall that the Banach-Mazur distance between two isomorphic Banach spaces  $X$  and  $Y$  is

$$d(X, Y) = \inf\{\|U\|\|U^{-1}\| : U : X \rightarrow Y \text{ linear isomorphism}\}.$$

Given a Banach space  $X$  with the *wc-FPP* (respectively the *FPP*), the problem of determining a number  $k > 1$  such that any isomorphic Banach space  $Y$  with  $d(X, Y) < k$  has the *wc-FPP* (resp. *FPP*) is known as the stability problem of the *wc-FPP* (resp. *FPP*). A wide survey of the recent developments in the theory of stability of the *FPP* can be found in Chapter 7 of [6].

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In this paper we shall concentrate our attention on the stability problem of the *FPP* in Hilbert spaces. In [7] Pei-Kee Lin proved the following theorem.

**Theorem 1.1.** *Let  $H$  be a Hilbert space. If  $X$  is a Banach space isomorphic to  $H$  such that*

$$d(X, H) < \sqrt{\frac{5 + \sqrt{13}}{2}},$$

*then  $X$  has the *FPP*.*

The aim of this paper is to prove that the conclusion of the above theorem remains true under the weaker hypothesis  $d(X, H) < \sqrt{\frac{5 + \sqrt{17}}{2}}$ .

## 2. PRELIMINARIES

Let  $(X, \|\cdot\|)$  be a Banach space, let  $C$  be a nonempty bounded closed convex subset of  $X$  and let  $T : C \rightarrow C$  be a nonexpansive mapping.

It is known that there is a sequence  $(x_n)$  in  $C$  such that  $\lim_{n \rightarrow \infty} \|x_n - T(x_n)\| = 0$ . Such a sequence is called an *approximate fixed point sequence* (*afps* in short) for  $T$ .

If in addition  $C$  is weakly compact, then the family

$$\mathcal{F} = \{K \subset C : K \text{ nonempty, closed, convex, } T(K) \subset K\}$$

has a minimal element (with respect to set-inclusion). We shall say that a set  $K$  is *minimal for  $T$*  if it is a minimal element of  $\mathcal{F}$ .

A key result in order to establish the *wc - FPP* is the *Goebel-Karlovitz Lemma* [5], [4].

**Lemma 2.1** (Goebel-Karlovitz). *Let  $K$  be a nonempty weakly compact convex subset of a Banach space  $(X, \|\cdot\|)$  which is minimal for a nonexpansive mapping  $T : K \rightarrow K$ . If  $(x_n)$  is an *afps* for  $T$  in  $K$ , then*

$$\lim_{n \rightarrow \infty} \|x - x_n\| = \text{diam}(K)$$

for all  $x \in K$ .

We need to recall some facts about convergence over ultrafilters and the ultrapower of a Banach space. For details we refer to [1], [9] or Chapter 6 of [6].

Throughout this paper  $\mathcal{U}$  will denote a proper nontrivial ultrafilter on  $\mathbb{N}$ .

Recall that, if  $(\Omega, \tau)$  is a Hausdorff topological space and  $(x_n)$  is a sequence in  $\Omega$ , it is said that  $(\tau - \lim_{n, \mathcal{U}})x_n = x_0$  if for every  $\tau$ -neighborhood  $V$  of  $x_0$  we have  $\{n \in \mathbb{N} : x_n \in V\} \in \mathcal{U}$ .

The *ultrapower* of the Banach space  $(X, \|\cdot\|)$ —with respect to  $\mathcal{U}$ —is defined to be the quotient Banach space

$$\tilde{X} = \ell_\infty(X) / \mathcal{N}(X),$$

where

$$\ell_\infty(X) = \{(x_n) \in X^{\mathbb{N}} : \sup_{n \geq 1} \|x_n\| < \infty\}$$

and

$$\mathcal{N}(X) = \{(x_n) \in \ell_\infty(X) : \lim_{n, \mathcal{U}} \|x_n\| = 0\}.$$

The equivalence class of  $(x_n) \in \ell_\infty(X)$  is denoted by  $\widetilde{(x_n)}$ .

If  $\|\cdot\|_{\tilde{X}}$  stands for the quotient norm of  $\tilde{X}$  we have that

$$\|(\widetilde{x_n})\|_{\tilde{X}} = \lim_{n, \mathcal{U}} \|x_n\|.$$

Whenever it does not lead to confusion, we will omit the subindex and still denote by  $\|\cdot\|$  the norm of the ultrapower.

Given a subset  $K$  of  $X$ ,  $\tilde{K}$  will denote the subset of  $\tilde{X}$  defined by

$$\tilde{K} = \{(\widetilde{x_n}) : x_n \in K \text{ for each } n \in \mathbb{N}\},$$

and given a nonexpansive mapping  $T : K \rightarrow K$ , we will consider the associated nonexpansive mapping  $\tilde{T} : \tilde{K} \rightarrow \tilde{K}$  defined by  $\tilde{T}((\widetilde{x_n})) = (\widetilde{T(x_n)})$ .

The following generalization of the Goebel-Karlovitx Lemma, due to P.K. Lin [8], is basic for establishing the *wc-FPP* by using ultrapower methods.

**Lemma 2.2** (Lin). *Let  $K$  be a nonempty weakly compact convex subset of a Banach space  $(X, \|\cdot\|)$  which is minimal for a nonexpansive mapping  $T : K \rightarrow K$ . If  $(\tilde{w}^n)$  is an *afps* for  $\tilde{T}$  in  $\tilde{K}$ , then*

$$\lim_{n \rightarrow \infty} \|\tilde{w}^n - x\| = \text{diam}(K)$$

for all  $x \in K$ .

In particular, if  $W$  is a nonempty closed convex and  $\tilde{T}$ -invariant subset of  $\tilde{K}$ , then

$$\sup\{\|\tilde{w} - x\| : \tilde{w} \in W\} = \text{diam}(K)$$

for all  $x \in K$ .

Let us now consider a Hilbert space  $(H, \|\cdot\|)$ . It is well known that any ultrapower  $\tilde{H}$  of  $H$  is also a Hilbert space.

On the other hand, it is also well known that if  $(x_n)$  is a weakly null sequence in  $H$ , then for each  $x \in H$  we have that

$$(2.1) \quad \limsup_{n \rightarrow \infty} \|x_n - x\|^2 = \limsup_{n \rightarrow \infty} \|x_n\|^2 + \|x\|^2$$

and that the equality is also true if  $\limsup_{n \rightarrow \infty}$  is replaced by  $\liminf_{n \rightarrow \infty}$ .

The following result shows that an analogous identity holds when dealing with weak convergence along ultrafilters.

**Lemma 2.3.** *Let  $(x_n)$  be a bounded sequence in a Hilbert space  $(H, \|\cdot\|)$  with  $(w - \lim_{n, \mathcal{U}})x_n = 0$  and denote  $\tilde{x} = (\widetilde{x_n})$ . Then, for any  $x \in H$ ,*

$$\|\tilde{x} - x\|^2 = \|x\|^2 + \|\tilde{x}\|^2.$$

*Proof.* Let us denote by  $\langle \cdot, \cdot \rangle$  the inner product of  $H$ .

Since, for any  $n \in \mathbb{N}$ ,  $\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - 2\langle x, x_n \rangle$ , taking limits over the ultrafilter, we have that

$$\begin{aligned} \|\tilde{x} - x\|^2 &= \lim_{n, \mathcal{U}} \|x_n - x\|^2 = \lim_{n, \mathcal{U}} \|x_n\|^2 + \|x\|^2 - 2 \lim_{n, \mathcal{U}} \langle x, x_n \rangle \\ &= \|\tilde{x} - x\|^2 + \|x\|^2 - 2 \lim_{n, \mathcal{U}} \langle x, x_n \rangle. \end{aligned}$$

So, the proof is finished if we see that  $\lim_{n, \mathcal{U}} \langle x, x_n \rangle = 0$ , that is, for any  $\varepsilon > 0$   $\{n \in \mathbb{N} : |\langle x, x_n \rangle| < \varepsilon\} \in \mathcal{U}$ .

Let  $\varepsilon > 0$ . Consider the function  $f_x : H \rightarrow \mathbb{R}$  defined by  $f_x(y) := \langle x, y \rangle$ . Since  $f_x \in H^*$ , the set  $V := \{y \in H : |f_x(y) - f_x(0)| < \varepsilon\} = \{y \in H : |\langle x, y \rangle| < \varepsilon\}$

is a neighbourhood of 0 for the weak topology. Consequently, provided that  $(w - \lim_{n,\mathcal{U}})x_n = 0$ , we have that  $\{n \in \mathbb{N} : x_n \in V\} = \{n \in \mathbb{N} : |\langle x, x_n \rangle| < \varepsilon\} \in \mathcal{U}$ , as required.  $\square$

### 3. THE STABILITY RESULT

Let  $(H, \|\cdot\|)$  be a Hilbert space and let  $|\cdot|$  be an equivalent norm in  $H$  satisfying that, for any  $x \in H$ ,

$$\|x\| \leq |x| \leq B\|x\|.$$

The aim is to prove that if  $B < \sqrt{\frac{5+\sqrt{17}}{2}}$ , then  $(H, |\cdot|)$  has the *FPP*. Therefore, by Theorem 1.1, we can assume  $B \geq \sqrt{\frac{5+\sqrt{13}}{2}}$ .

In  $\tilde{H}$ , the ultrapower of  $H$  with respect to  $\mathcal{U}$ , we shall consider the norms associated to  $\|\cdot\|$  and  $|\cdot|$ , that is, the norms respectively defined by  $\|\tilde{x}\| = \lim_{n,\mathcal{U}} \|x_n\|$  and  $|\tilde{x}| = \lim_{n,\mathcal{U}} |x_n|$  for  $\tilde{x} = (\widetilde{x_n}) \in \tilde{H}$ .

In the statement of the following lemmas we shall assume that  $K$  is a nonempty weakly compact convex subset of  $H$ , with  $\text{diam}(K) = 1$  and which is minimal for a  $|\cdot|$ -nonexpansive mapping  $T : K \rightarrow K$ .

Let  $(x_n)$  be an afps for  $T$  in  $K$  and set  $\tilde{x} = (\widetilde{x_n})$ . Let  $r \in (0, 1)$ . Associated to these elements, we define the subset of  $\tilde{K}$

$$\tilde{W}(\tilde{x}, r) = \{\tilde{w} \in \tilde{K} : |\tilde{w} - \tilde{x}| \leq r, D_{|\cdot|}(\tilde{w}) \leq 1 - r\},$$

where  $D_{|\cdot|}(\tilde{w}) = \lim_{n,\mathcal{U}} \lim_{m,\mathcal{U}} |w_n - w_m|$ , if  $\tilde{w} = (\widetilde{w_n})$ .

It is not difficult to check that  $(1 - r)\tilde{x} + rx \in \tilde{W}(\tilde{x}, r)$  for any  $x \in K$ , and that  $\tilde{W}(\tilde{x}, r)$  is a closed convex  $\tilde{T}$ -invariant set.

Let  $\tilde{w} = (\widetilde{w_n}) \in \tilde{W}(\tilde{x}, r)$ . Since  $K$  is weakly compact, we can consider  $w_0 = (w - \lim_{n,\mathcal{U}})w_n$ , and we have

$$(3.1) \quad |\tilde{w} - w_0| \leq D_{|\cdot|}(\tilde{w}) \leq 1 - r.$$

On the other hand, by the Goebel-Karlovitz lemma,  $\lim_{n \rightarrow \infty} |x_n - x_0| = 1$  and therefore

$$|\tilde{w} - w_0 - (\tilde{x} - x_0)| \geq |\tilde{x} - x_0| - |\tilde{w} - w_0| \geq 1 - (1 - r) = r,$$

so

$$(3.2) \quad \|\tilde{w} - w_0 - (\tilde{x} - x_0)\| \geq \frac{r}{B}.$$

We also define the number

$$D(K) = \inf\{\limsup_{n \rightarrow \infty} \|y_n - y\| : (y_n) \text{ is an afps for } T \text{ in } K, y_n \xrightarrow{w} y\}$$

(note that  $D(K) \leq \text{diam}(K) < +\infty$ ).

**Lemma 3.1.** *Let  $(x_n)$  be an afps for  $T$  in  $K$  and denote  $\tilde{x} = (\widetilde{x_n})$ . Let  $r \in (0, 1)$ . For each  $x \in K$  and each  $\varepsilon > 0$  there exists  $\tilde{w} \in \tilde{W}(\tilde{x}, r)$  such that  $|\tilde{w} - x| \geq 1 - \varepsilon$  and  $\|\tilde{w} - x\| \geq D(K) - \varepsilon$ .*

*Proof.* Suppose that there exist  $x \in K$  and  $\varepsilon > 0$  for which, if  $\tilde{w} \in \tilde{W}(\tilde{x}, r)$  satisfies  $|\tilde{w} - x| \geq 1 - \varepsilon$ , then  $\|\tilde{w} - x\| < D(K) - \varepsilon$ .

Since  $\tilde{W}(\tilde{x}, r)$  is closed convex and  $\tilde{T}$ -invariant, it contains an afps for  $\tilde{T}$ ,  $(\tilde{w}^m)$ , that is, a sequence for which, if  $\tilde{w}^m = \widetilde{(w_n^m)}$ , then

$$(3.3) \quad \lim_{m \rightarrow \infty} \lim_{n, \mathcal{U}} \|w_n^m - Tw_n^m\| = 0.$$

From Lin's Lemma,

$$\lim_{m \rightarrow \infty} |\tilde{w}^m - x| = 1,$$

so there exists  $m_0 \in \mathbb{N}$  such that, for all  $m \geq m_0$ ,  $|\tilde{w}^m - x| \geq 1 - \varepsilon$ .

From the previous inequality, according to our assumption, for all  $m \geq m_0$ , we must have

$$(3.4) \quad \lim_{n, \mathcal{U}} \|w_n^m - x\| = \|\tilde{w}^m - x\| < D(K) - \varepsilon.$$

Fix  $k \in \mathbb{N}$ . By (3.3) we can find  $m_k$ , as large as desired. In particular  $m_k \geq m_0$ , for which

$$\lim_{n, \mathcal{U}} \|w_n^{m_k} - T(w_n^{m_k})\| < \frac{1}{k},$$

and so  $A_k = \{n \in \mathbb{N} : \|w_n^{m_k} - T(w_n^{m_k})\| < \frac{1}{k}\} \in \mathcal{U}$ .

On the other hand, by (3.4), the set  $B_k = \{n \in \mathbb{N} : \|w_n^{m_k} - x\| < D(K) - \varepsilon\}$  is also a member of  $\mathcal{U}$ . Therefore,  $A_k \cap B_k$  is an infinite set, and the existence of  $n_k \in A_k \cap B_k$  as large as desired is thus guaranteed.

According to this reasoning, we can construct two strictly increasing sequences of natural numbers,  $(n_k)$  and  $(m_k)$ , for which, defining  $z_k := w_{n_k}^{m_k}$ , we have that, for any  $k \geq 1$ ,

$$\|z_k - Tz_k\| < \frac{1}{k} \text{ and } \|z_k - x\| < D(K) - \varepsilon.$$

Hence, the sequence  $(z_k)$  so defined, is an afps for  $T$  satisfying that, for each  $k \geq 1$ ,

$$\|z_k - x\| < D(K) - \varepsilon.$$

We can assume, by passing to subsequences if necessary, that  $(z_k)$  converges weakly, say  $z_k \xrightarrow{w} z$ .

Then so, by (2.1), we obtain that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \|z_k - z\| &\leq \limsup_{k \rightarrow \infty} \|z_k - x\| \\ &\leq D(K) - \varepsilon < D(K). \end{aligned}$$

But this contradicts the definition of  $D(K)$ , so the proof is complete. □

In the proof of the next result we will use the following generalization of the parallelogram law: Given  $x, y \in H$  and given  $\lambda \in (0, 1)$ , we have the identity

$$\begin{aligned} &\|\lambda x + (1 - \lambda)y\|^2 \\ &= \langle \lambda x + (1 - \lambda)y, \lambda x + (1 - \lambda)y \rangle \\ &= \lambda^2 \|x\|^2 + 2\lambda(1 - \lambda)\langle x, y \rangle + (1 - \lambda)^2 \|y\|^2 \\ &= \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 + \lambda(1 - \lambda)(2\langle x, y \rangle - \langle x, x \rangle - \langle y, y \rangle) \\ &= \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \end{aligned}$$

**Lemma 3.2.** *The inequality*

$$D(K)^2 \geq \frac{B^2 - 1}{B^2(B^2 - 2)}$$

holds.

*Proof.* Consider three parameters  $\lambda \in (0, 1)$ ,  $r \in (0, 1)$  and  $\varepsilon \in (0, r)$ .

By the definition of  $D(K)$ ,  $K$  contains a weakly convergent afps for  $T$ ,  $(x_n)$ , satisfying that, if  $x_0$  is its weak limit, then

$$\limsup_{n \rightarrow \infty} \|x_n - x_0\| \leq D(K) + \varepsilon,$$

so, setting  $\tilde{x} = \widetilde{(x_n)}$ , we have

$$(3.5) \quad \|\tilde{x} - x_0\| \leq D(K) + \varepsilon.$$

Applying Lemma 3.1, we can guarantee the existence of  $\tilde{w} \in \tilde{W}(\tilde{x}, r)$  with

$$(3.6) \quad |\tilde{w} - x_0| \geq 1 - \varepsilon$$

and

$$(3.7) \quad \|\tilde{w} - x_0\| \geq D(K) - \varepsilon.$$

Put  $\tilde{w} = \widetilde{(w_n)}$ . Since  $K$  is weakly compact, we can consider  $w_0 = (w - \lim_{n, \mathcal{U}})w_n$ . From (3.1) and (3.6), we obtain that

$$\begin{aligned} |w_0 - x_0| &= |(\tilde{w} - x_0) - (\tilde{w} - w_0)| \\ &\geq |\tilde{w} - x_0| - |\tilde{w} - w_0| \\ &\geq (1 - \varepsilon) - (1 - r) = r - \varepsilon \end{aligned}$$

and consequently

$$(3.8) \quad \|w_0 - x_0\| \geq \frac{r - \varepsilon}{B}.$$

Since  $\tilde{W}(\tilde{x}, r)$  is convex,  $\lambda\tilde{w} + (1 - \lambda)((1 - r)\tilde{x} + rx_0) \in \tilde{W}(\tilde{x}, r)$ . Moreover,  $(w - \lim_{n, \mathcal{U}})(\lambda w_n + (1 - \lambda)((1 - r)x_n + rx_0)) = \lambda w_0 + (1 - \lambda)x_0$ , so from (3.2),

$$(3.9) \quad \begin{aligned} &\|\lambda(\tilde{w} - w_0) + (1 - \lambda)(1 - r)(\tilde{x} - x_0) - (\tilde{x} - x_0)\| \\ &= \|(\lambda\tilde{w} + (1 - \lambda)((1 - r)\tilde{x} + rx_0)) - (\lambda w_0 + (1 - \lambda)x_0) - (\tilde{x} - x_0)\| \geq \frac{r}{B}. \end{aligned}$$

Now, since

$$(w - \lim_{n, \mathcal{U}})(\lambda(w_n - w_0) + (1 - \lambda)(1 - r)(x_n - x_0) - (x_n - x_0)) = 0,$$

by Lemma 2.3, we have

$$\begin{aligned} &\|\lambda\tilde{w} + (1 - \lambda)((1 - r)\tilde{x} + rx_0) - \tilde{x}\|^2 \\ &= \|\lambda(\tilde{w} - w_0) + (1 - \lambda)(1 - r)(\tilde{x} - x_0) - (\tilde{x} - x_0)\|^2 + \lambda^2\|w_0 - x_0\|^2, \end{aligned}$$

so, from (3.8) and (3.9), we obtain

$$(3.10) \quad \|\lambda\tilde{w} + (1 - \lambda)((1 - r)\tilde{x} + rx_0) - \tilde{x}\|^2 \geq \frac{1}{B^2}(r^2 + \lambda^2(r - \varepsilon)^2).$$

By the generalized parallelogram law, the identities

$$\begin{aligned} & \|\lambda\tilde{w} + (1 - \lambda)((1 - r)\tilde{x} + rx_0) - \tilde{x}\|^2 \\ &= \|\lambda(\tilde{w} - \tilde{x}) + (1 - \lambda)r(x_0 - \tilde{x})\|^2 \\ &= \lambda\|\tilde{w} - \tilde{x}\|^2 + (1 - \lambda)r^2\|x_0 - \tilde{x}\|^2 - \lambda(1 - \lambda)\|\tilde{w} - (1 - r)\tilde{x} - rx_0\|^2 \end{aligned}$$

and

$$\begin{aligned} & \|\tilde{w} - (1 - r)\tilde{x} - rx_0\|^2 \\ &= \|(1 - r)(\tilde{w} - \tilde{x}) + r(\tilde{w} - x_0)\|^2 \\ &= (1 - r)\|\tilde{w} - \tilde{x}\|^2 + r\|\tilde{w} - x_0\|^2 - r(1 - r)\|\tilde{x} - x_0\|^2 \end{aligned}$$

hold, which put together lead to

$$\begin{aligned} & \|\lambda\tilde{w} + (1 - \lambda)((1 - r)\tilde{x} + rx_0) - \tilde{x}\|^2 \\ &= \lambda(1 - (1 - \lambda)(1 - r))\|\tilde{w} - \tilde{x}\|^2 \\ & \quad + (1 - \lambda)r(r + \lambda(1 - r))\|\tilde{x} - x_0\|^2 - \lambda(1 - \lambda)r\|\tilde{w} - x_0\|^2. \end{aligned}$$

Taking into account (3.5), (3.7), (3.10) and the fact that, provided that  $\tilde{w} \in \tilde{W}(\tilde{x}, r)$ , we have  $\|\tilde{w} - \tilde{x}\| \leq |\tilde{w} - \tilde{x}| \leq r$ , we get

$$\begin{aligned} \frac{r^2 + \lambda^2(r - \varepsilon)^2}{B^2} &\leq \lambda(1 - (1 - \lambda)(1 - r))r^2 \\ & \quad + (1 - \lambda)r(r + \lambda(1 - r))(D(K) + \varepsilon)^2 \\ & \quad - \lambda(1 - \lambda)r(D(K) - \varepsilon)^2. \end{aligned}$$

Letting  $\varepsilon$  go to 0, we obtain

$$\frac{r^2(1 + \lambda^2)}{B^2} \leq \lambda(1 - (1 - \lambda)(1 - r))r^2 + (1 - \lambda)^2r^2D(K)^2,$$

from where, simplifying  $r^2$  first and then taking limits as  $r \rightarrow 0$ ,

$$\frac{1 + \lambda^2}{B^2} \leq \lambda^2 + (1 - \lambda)^2D(K)^2,$$

so

$$D(K)^2 \geq \frac{\frac{1 + \lambda^2}{B^2} - \lambda^2}{(1 - \lambda)^2}.$$

For  $\lambda = \frac{1}{B^2 - 1}$  (note that  $\lambda \in (0, 1)$  since we are assuming  $B \geq \sqrt{\frac{5 + \sqrt{13}}{2}} > \sqrt{2}$ ), we finally get

$$D(K)^2 \geq \frac{B^2 - 1}{B^2(B^2 - 2)}.$$

□

**Lemma 3.3.** *Let  $(x_n)$  be an afps for  $T$  in  $K$  such that  $x_n \xrightarrow{w} x_0$  and denote  $\tilde{x} := \widetilde{(x_n)}$ . Then, for any  $r \in (0, 1)$ , the inequality*

$$\sup\{\|\tilde{w} - x_0\|^2 : \tilde{w} \in \tilde{W}(\tilde{x}, r)\} \leq \frac{(1 - r)^2}{2} + r^2\left(1 - \frac{1}{B^2}\right)$$

*holds.*

*Proof.* Let  $\tilde{w} = \widetilde{(w_n)}$  be a member of  $\tilde{W}(\tilde{x}, r)$ . Since  $K$  is weakly compact we can consider  $w_0 = (w - \lim_{n,\mathcal{U}})w_n$ .

Given that  $(w - \lim_{n,\mathcal{U}})(w_n - w_0 - (x_n - x_0)) = 0$ , an application of Lemma 2.3 leads to

$$(3.11) \quad \|\tilde{w} - \tilde{x}\|^2 = \|w_0 - x_0\|^2 + \|\tilde{w} - w_0 - (\tilde{x} - x_0)\|^2.$$

So, taking into account (3.2) and that

$$\|\tilde{w} - \tilde{x}\| \leq |\tilde{w} - \tilde{x}| \leq r,$$

we obtain

$$(3.12) \quad \|w_0 - x_0\|_2^2 \leq r^2 \left(1 - \frac{1}{B^2}\right).$$

Since  $(w - \lim_{n,\mathcal{U}})(w_n - w_0) = 0$ , again applying Lemma 2.3, we have that, for all  $m \geq 1$ ,

$$\|\tilde{w} - w_m\|^2 = \|w_0 - w_m\|^2 + \|\tilde{w} - w_0\|^2,$$

so, taking limits along the ultrafilter, we get

$$\lim_{m,\mathcal{U}} \lim_{n,\mathcal{U}} \|w_n - w_m\|^2 = 2\|\tilde{w} - w_0\|^2$$

and consequently

$$(3.13) \quad \begin{aligned} \|\tilde{w} - w_0\|^2 &= \frac{1}{2} \lim_{n,\mathcal{U}} \lim_{m,\mathcal{U}} \|w_n - w_m\|^2 \\ &\leq \frac{1}{2} D_{|\cdot|}(\tilde{w})^2 \leq \frac{(1-r)^2}{2}. \end{aligned}$$

By Lemma 2.3 once more,

$$\|\tilde{w} - x_0\|^2 = \|w_0 - x_0\|^2 + \|\tilde{w} - w_0\|^2.$$

Inequalities (3.12) and (3.13), along with this last equation, lead to

$$\|\tilde{w} - z_0\|^2 \leq r^2 \left(1 - \frac{1}{B^2}\right) + \frac{(1-r)^2}{2}$$

which completes the proof. □

**Theorem 3.4.** *If  $B < \sqrt{\frac{5+\sqrt{17}}{2}}$ , then  $(H, |\cdot|)$  has the FPP.*

*Proof.* Assume that  $(H, |\cdot|)$  does not have the FPP. Then, by an standard argument, we can suppose that there exists a weakly compact convex subset  $K$  of  $H$ , with  $\text{diam}_{|\cdot|}(K) = 1$ , which is minimal for a  $|\cdot|$ -nonexpansive mapping  $T : K \rightarrow K$  and contains a weakly null afps for  $T$ ,  $(x_n)$ .

From Lemmas 3.1, 3.2 and 3.3, we have that, for any  $r \in (0, 1)$ ,

$$\frac{B^2 - 1}{B^2(B^2 - 2)} \leq D(K)^2 \leq \sup\{\|\tilde{w}\|^2 : \tilde{w} \in \tilde{W}(\tilde{x}, r)\} \leq \frac{(1-r)^2}{2} + r^2 \left(1 - \frac{1}{B^2}\right).$$

In particular, for  $r = \frac{B^2}{3B^2 - 2}$ , it follows that

$$\frac{B^2 - 1}{B^2(B^2 - 2)} \leq \frac{B^2 - 1}{3B^2 - 2},$$

from where

$$B^4 - 5B^2 + 2 \geq 0$$

and in consequence

$$B \geq \sqrt{\frac{5 + \sqrt{17}}{2}},$$

which is the contradiction derived from assuming that  $(H, |\cdot|)$  does not have the *FPP*.  $\square$

**Corollary 3.5.** *Let  $H$  be a Hilbert space. If  $X$  is a Banach space such that*

$$d(X, H) < \sqrt{\frac{5 + \sqrt{17}}{2}},$$

*then  $X$  has the *FPP*.*

*Proof.* Provided that  $d(X, H) < \sqrt{\frac{5 + \sqrt{17}}{2}}$ , we can find  $B > 1$  such that

$$d(X, H) < B < \sqrt{\frac{5 + \sqrt{17}}{2}}.$$

By the definition of the Banach-Mazur distance, there exists a linear isomorphism  $U : X \rightarrow H$  for which

$$\|U\| \|U^{-1}\| < B.$$

The formula

$$|x| = \|U\| \|U^{-1}x\|$$

defines an equivalent norm in  $H$  satisfying

$$\|x\| \leq |x| \leq B\|x\|$$

for all  $x \in H$ .

Since  $B < \sqrt{\frac{5 + \sqrt{17}}{2}}$ , due to the previous results, we know that  $(H, |\cdot|)$  has the *FPP*. From this, it is easy to conclude that  $X$  also enjoys the *FPP*.  $\square$

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