

## FIXED POINT THEOREMS IN ORDERED $L$ -SPACES

ADRIAN PETRUȘEL AND IOAN A. RUS

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ABSTRACT. The purpose of this paper is to present some fixed point results in ordered  $L$ -spaces. Our results generalize and extend a recent result of Ran and Reurings (2004). Some applications to matrix equations are also considered.

### 1. INTRODUCTION

Recently, in a paper of Ran-Reurings (see [9]), the following result was proved:

**Theorem 1.1** (Ran-Reurings [9]). *Let  $X$  be a partially ordered set such that every pair  $x, y \in X$  has a lower and an upper bound. Let  $d$  be a metric on  $X$  such that the metric space  $(X, d)$  is complete. Let  $f : X \rightarrow X$  be a continuous and monotone (i.e., either order-preserving or order-reversing) operator. Suppose that the following two assertions hold:*

- 1) *there exists  $a \in ]0, 1[$  such that  $d(f(x), f(y)) \leq a \cdot d(x, y)$ , for each  $x, y \in X$  with  $x \geq y$ ,*
- 2) *there exists  $x_0 \in X$  such that  $x_0 \leq f(x_0)$  or  $x_0 \geq f(x_0)$ .*

*Then  $f$  has a unique fixed point  $x^* \in X$ , i.e.,  $f(x^*) = x^*$ , and for each  $x \in X$  the sequence  $(f^n(x))_{n \in \mathbb{N}}$  of successive approximations of  $f$  starting from  $x$  converges to  $x^* \in X$ .*

Usually, when a set is endowed with a metric and an order structure, an additional assumption is added, namely the compatibility between the above structures, i.e.,

$$(x_n)_{n \in \mathbb{N}} \rightarrow x, (y_n)_{n \in \mathbb{N}} \rightarrow y \text{ and } x_n \leq y_n, \text{ for each } n \in \mathbb{N} \Rightarrow x \leq y.$$

Following the proof of Ran and Reurings result, we may remark that the conclusion follows from the relations

$$d(f^n(x_1), f^n(x)) \leq a^n \cdot d(x_1, x) \text{ and } d(f^n(x), f^n(x_2)) \leq a^n \cdot d(x, x_2), \quad x_1 \leq x \leq x_2,$$

by letting  $n \rightarrow +\infty$ .

Dropping the hypothesis that each pair of points has an upper and a lower bound, we immediately obtain the following result.

**Theorem 1.2.** *Let  $X$  be a partially ordered set and let  $d$  be a metric on  $X$  such that the metric space  $(X, d)$  is complete and the metric and the ordered structure*

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are compatible. Let  $f : X \rightarrow X$  be a continuous and monotone (i.e., either order-preserving or order-reversing) operator. Suppose that the following two assertions hold:

1) there exists  $a \in ]0, 1[$  such that  $d(f(x), f(y)) \leq a \cdot d(x, y)$ , for each  $x, y \in X$  with  $x \geq y$ ,

2) there exists  $x_0 \in X$  such that  $x_0 \leq f(x_0)$  or  $x_0 \geq f(x_0)$ .

Then  $f$  has at least a fixed point  $x^* \in X$  and for each  $x \in X$  with  $x \geq x_0$  (or  $x \leq x_0$ ) the sequence  $(f^n(x))_{n \in \mathbb{N}}$  of successive approximations of  $f$  starting from  $x$  converges to  $x^* \in X$ .

The purpose of this paper is to generalize and extend Theorem 1.1 and Theorem 1.2 for the case of an ordered L-space.

## 2. NOTATIONS AND BASIC NOTIONS

Let  $f : X \rightarrow X$  be an operator. Then  $f^0 := 1_X$ ,  $f^1 := f$ ,  $\dots$ ,  $f^{n+1} = f \circ f^n$ ,  $n \in \mathbb{N}$ , denote the iterate operators of  $f$ . By  $I(f)$  we will denote the set of all nonempty invariant subsets of  $f$ , i.e.,  $I(f) := \{Y \subset X \mid f(Y) \subseteq Y\}$ .

Also, by  $F_f := \{x \in X \mid x = f(x)\}$  we will denote the fixed point set of the operator  $f$ , while  $A_f(x^*) := \{x \in X \mid f^n(x) \rightarrow x^*, \text{ as } n \rightarrow +\infty\}$  denotes the attractor basin of  $f$  with respect to  $x^* \in X$ .

Let  $X$  be a nonempty set. Denote  $s(X) := \{(x_n)_{n \in \mathbb{N}} \mid x_n \in X, n \in \mathbb{N}\}$ .

Let  $c(X) \subset s(X)$  be a subset of  $s(X)$  and let  $Lim : c(X) \rightarrow X$  be an operator. By definition, the triple  $(X, c(X), Lim)$  is called an L-space (Fréchet [3]) if the following conditions are satisfied:

(i) If  $x_n = x$ ,  $\forall n \in \mathbb{N}$ , then  $(x_n)_{n \in \mathbb{N}} \in c(X)$  and  $Lim(x_n)_{n \in \mathbb{N}} = x$ .

(ii) If  $(x_n)_{n \in \mathbb{N}} \in c(X)$  and  $Lim(x_n)_{n \in \mathbb{N}} = x$ , then for all subsequences,  $(x_{n_i})_{i \in \mathbb{N}}$  of  $(x_n)_{n \in \mathbb{N}}$  we have that  $(x_{n_i})_{i \in \mathbb{N}} \in c(X)$  and  $Lim(x_{n_i})_{i \in \mathbb{N}} = x$ .

By definition an element of  $c(X)$  is a convergent sequence,  $x := Lim(x_n)_{n \in \mathbb{N}}$  is the limit of this sequence and we also write  $x_n \rightarrow x$  as  $n \rightarrow +\infty$ .

In what follows we denote an L-space by  $(X, \rightarrow)$ .

In this setting, an operator  $f : X \rightarrow X$  is called orbitally continuous if  $x \in X$  and  $f^{n(i)}(x) \rightarrow a \in X$ , as  $i \rightarrow +\infty$  imply  $f^{n(i)+1}(x) \rightarrow f(a)$ , as  $i \rightarrow +\infty$ .

Let  $(X, \leq)$  be a partially ordered set. Denote  $X_{\leq} := \{(x, y) \in X \times X \mid x \leq y \text{ or } y \leq x\}$ . Also, if  $x, y \in X$ , with  $x \leq y$ , then by  $[x, y]_{\leq}$  we will denote the ordered segment joining  $x$  and  $y$ , i.e.,  $[x, y]_{\leq} := \{z \in X \mid x \leq z \leq y\}$ . In the same setting, consider  $f : X \rightarrow X$ . Then,  $(LF)_f := \{x \in X \mid x \leq f(x)\}$  is the lower fixed point set of  $f$ , while  $(UF)_f := \{x \in X \mid x \geq f(x)\}$  is the upper fixed point set of  $f$ . Also, if  $f : X \rightarrow X$  and  $g : Y \rightarrow Y$ , then the cartesian product of  $f$  and  $g$  is denoted by  $f \times g$ , and it is defined in the following way:  $f \times g : X \times Y \rightarrow X \times Y$ ,  $(f \times g)(x, y) := (f(x), g(y))$ .

**Definition 2.1.** Let  $X$  be a nonempty set. Then, by definition,  $(X, \rightarrow, \leq)$  is an ordered L-space if and only if:

(i)  $(X, \rightarrow)$  is an L-space,

(ii)  $(X, \leq)$  is a partially ordered set,

(iii)  $(x_n)_{n \in \mathbb{N}} \rightarrow x$ ,  $(y_n)_{n \in \mathbb{N}} \rightarrow y$  and  $x_n \leq y_n$ , for each  $n \in \mathbb{N} \Rightarrow x \leq y$ .

Throughout this paper we suppose that  $(X, \rightarrow, \leq)$  is an ordered L-space satisfying the following additional assertion:

(iv)  $(x_n)_{n \in \mathbb{N}} \rightarrow x$ ,  $(z_n)_{n \in \mathbb{N}} \rightarrow x$  and  $x_n \leq y_n \leq z_n$ , for each  $n \in \mathbb{N} \Rightarrow y_n \rightarrow x$ .

For more considerations on convergence structures see P. P. Zabreiko [13] and De Pascale, Marino and Pietromala [2].

### 3. PICARD AND WEAKLY PICARD OPERATORS

First recall the following important concept.

**Definition 3.1** (I.A. Rus [11]). Let  $(X, \rightarrow)$  be an  $L$ -space. An operator  $f : X \rightarrow X$  is, by definition, a Picard operator (briefly PO) if:

- (i)  $F_f = \{x^*\}$ ,
- (ii)  $(f^n(x))_{n \in \mathbb{N}} \rightarrow x^*$  as  $n \rightarrow \infty$ , for all  $x \in X$ .

In terms of the Picard operators, some classical results in metrical fixed point theory have the following form ([6], [10]).

**Example 3.2** (Contraction principle). Let  $(X, d)$  be a complete metric space and let  $f : X \rightarrow X$  be an  $a$ -contraction, i.e.,  $a \in ]0, 1[$  and  $d(f(x), f(y)) \leq a \cdot d(x, y)$ , for each  $x, y \in X$ . Then  $f$  is a PO.

**Example 3.3** (Nemytzki and Edelstein). Let  $(X, d)$  be a compact metric space and let  $f : X \rightarrow X$  satisfy  $d(f(x), f(y)) < d(x, y)$ , for all  $x, y \in X$  with  $x \neq y$ . Then  $f$  is a PO.

**Example 3.4** (Perov). Let  $(X, d)$  be a complete generalized metric space ( $d(x, y) \in \mathbb{R}_+^n$ ) and  $A \in M_{mm}(\mathbb{R}_+)$ , such that,  $A^n \rightarrow 0$  as  $n \rightarrow \infty$ . If  $f : X \rightarrow X$  is an  $A$ -contraction, i.e.,  $d(f(x), f(y)) \leq Ad(x, y)$ , for all  $x, y \in X$ , then  $f$  is a PO.

**Example 3.5** (Sehgal and Bharucha-Reid, [5]). Let  $(X, F, \min)$  be a complete probabilistic metric space. Let  $f : X \rightarrow X$  be a continuous operator for which there exists  $a \in [0, 1[$  such that  $F_{f(x), f(y)}(a\lambda) \geq F_{x, y}(\lambda)$ , for each  $x, y \in X$  and each  $\lambda > 0$ . Then  $f$  is a PO.

Another important concept is:

**Definition 3.6.** Let  $(X, \rightarrow)$  be an  $L$ -space. By definition,  $f : X \rightarrow X$  is called a weakly Picard operator (briefly WPO) if the sequence  $(f^n(x))_{n \in \mathbb{N}}$  converges for all  $x \in X$  and the limit (which may depend on  $x$ ) is a fixed point of  $f$ .

**Example 3.7.** Let  $(X, d)$  be a complete metric space and let  $f : X \rightarrow X$  be an orbitally continuous operator such that there is  $a \in ]0, 1[$  with the property  $d(f(x), f^2(x)) \leq a \cdot d(x, f(x))$ , for each  $x \in X$ . Then  $f$  is a WPO.

**Example 3.8.** Let  $(X, d)$  be a complete metric space, let  $f : X \rightarrow X$  be an orbitally continuous operator and let  $\varphi : X \rightarrow \mathbb{R}_+$ . We suppose that  $f$  satisfies the Caristi condition with respect to  $\varphi$ , i.e.,  $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$ , for each  $x \in X$ . Then  $f$  is a WPO.

In I. A. Rus [11] the basic theory of Picard and weakly Picard operators is presented. For the multivalued case, see Petruşel [8]. For both settings, see also [12].

## 4. FIXED POINT RESULTS

Our starting result is the following abstract lemma.

**Lemma 4.1.** *Let  $(X, \rightarrow)$  be an  $L$ -space and let  $U$  be a symmetric subset of  $X \times X$  such that  $\Delta(X)$  (the diagonal of  $X \times X$ ) is included in  $U$ . Let  $f : X \rightarrow X$  be an operator. Suppose that:*

- (i) *for each  $x, y \in X$  there exists  $z \in X$  such that  $(x, z) \in U$  and  $(y, z) \in U$ ,*
- (ii) *there exist  $x_0, x^* \in X$  such that  $x_0 \in A_f(x^*)$ ,*
- (iii)  *$(x, y) \in U$  and  $x \in A_f(x^*)$  implies  $y \in A_f(x^*)$ .*

*Then  $A_f(x^*) = X$ . Moreover, if in addition the operator  $f$  is orbitally continuous, then  $F_f = \{x^*\}$ .*

*Proof.* Let  $x \in X$  be arbitrary. If  $(x_0, x) \in U$ , then the conclusion follows from (ii) and (iii). Suppose  $(x_0, x) \notin U$ . Then from (i) there exists  $z \in X$  such that  $(x_0, z) \in U$  and  $(x, z) \in U$ . From (ii) and (iii) we obtain that  $x \in A_f(x^*)$ . The second part of the theorem follows from the orbital continuity of  $f$ .  $\square$

Some consequences of the above abstract result follow by particular choices of  $U$ .

**Theorem 4.2.** *Let  $(X, \rightarrow, \leq)$  be an ordered  $L$ -space and let  $f : X \rightarrow X$  be an operator. We suppose that:*

- (i) *For each  $x, y \in X$  there exist  $m(x, y), M(x, y) \in X$  such that  $x, y \in [m(x, y), M(x, y)]_{\leq}$ .*
- (ii)  *$f : (X, \rightarrow) \rightarrow (X, \rightarrow)$  is orbitally continuous.*
- (iii) *There exist  $x_0, x^* \in X$  such that  $(f^n(x_0))_{n \in \mathbb{N}} \rightarrow x^*$ , as  $n \rightarrow +\infty$ .*
- (iv) *If  $(x, y) \in X_{\leq}$  and  $(f^n(x))_{n \in \mathbb{N}} \rightarrow x^*$ , as  $n \rightarrow +\infty$ , then  $(f^n(y))_{n \in \mathbb{N}} \rightarrow x^*$ , as  $n \rightarrow +\infty$ .*

*Then  $f$  is a PO.*

*Proof. A. A direct proof.* From (ii) and (iii) we have that  $x^* \in F_f$ . Let  $x \in X$  be arbitrary. If  $(x, x_0) \in X_{\leq}$ , then the conclusion follows from (iii) and (iv). If  $(x, x_0) \notin X_{\leq}$ , then by (i) we have that  $x, x_0 \in [m(x, x_0), M(x, x_0)]_{\leq}$ . Since  $x_0 \in [m(x, x_0), M(x, x_0)]_{\leq}$  and taking into account of (iv), it follows that

$$(f^n(m(x, x_0)))_{n \in \mathbb{N}} \rightarrow x^* \text{ and } (f^n(M(x, x_0)))_{n \in \mathbb{N}} \rightarrow x^*, \text{ as } n \rightarrow +\infty.$$

From the fact that  $m(x, x_0) \leq x \leq M(x, x_0)$  and (iv) we obtain  $(f^n(x))_{n \in \mathbb{N}} \rightarrow x^*$ , as  $n \rightarrow +\infty$ . Hence,  $f$  is a PO with respect to  $\rightarrow$ .

**B. A proof based on Lemma 4.1.** Take  $U := X_{\leq}$  and apply Lemma 4.1.  $\square$

From Theorem 4.2. we have the following result.

**Theorem 4.3.** *Let  $(X, d, \leq)$  be an ordered metric space and let  $f : X \rightarrow X$  be an operator. We suppose that:*

- (a) *for each  $x, y \in X$  there exist  $m(x, y), M(x, y) \in X$  such that  $x, y \in [m(x, y), M(x, y)]_{\leq}$ .*
  - (b)  *$X_{\leq} \in I(f \times f)$ .*
  - (c)  *$f : (X, d) \rightarrow (X, d)$  is orbitally continuous.*
  - (d) *The metric  $d$  is complete.*
  - (e) *There exists  $x_0 \in X$  such that  $(x_0, f(x_0)) \in X_{\leq}$ .*
  - (f) *There exists  $\alpha \in ]0, 1[$  such that  $d(f(x), f(y)) \leq \alpha \cdot d(x, y)$ , for each  $x, y \in X_{\leq}$ .*
- Then  $f : (X, d) \rightarrow (X, d)$  is a PO.*

*Proof.* Let  $x_0 \in X$  such that  $(x_0, f(x_0)) \in X_{\leq}$ . From (b), (d) and (f) it follows that there exists  $x^* \in X$  such that, with respect to the metric  $d$ , we have  $(f^n(x_0))_{n \in \mathbb{N}} \rightarrow x^*$ , as  $n \rightarrow +\infty$ . From (c) we get that  $x^* \in F_f$ .

If  $(x, x_0) \in X_{\leq}$ , then, from (c), (f) and (a) we have that  $(f^n(x))_{n \in \mathbb{N}} \rightarrow x^*$  (with respect to  $d$ ), as  $n \rightarrow +\infty$ .

If  $(x, x_0) \notin X_{\leq}$ , then, from (f), (c) and (b), we get the same conclusion.

Our result follows now from Theorem 4.2., namely the case when the  $L$ -structure of the space is that induced by the metric  $d$ . □

*Remark 4.4.* The conclusion of Theorem 4.3 holds if instead the hypothesis (b) we put:

(b')  $f : (X, \leq) \rightarrow (X, \leq)$  is monotone increasing

or

(b'')  $f : (X, \leq) \rightarrow (X, \leq)$  is monotone decreasing.

Of course, it is easy to remark that assertion (b) in Theorem 4.3 is more general. For example, if we consider the ordered  $L$ -space  $(\mathbb{R}^2, \rightarrow, \leq)$ , then  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, f(x, y) := (\varphi(x, y), \varphi(x, y))$  satisfies (ii), for all  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

*Remark 4.5.* Condition (e) from the above theorem is equivalent to:

(e')  $(LF)_f \cup (UF)_f \neq \emptyset$ .

*Remark 4.6.* Theorem 1.1 (see Ran and Reurings [9]) is a particular case of Theorem 4.2 and Theorem 4.3.

Similarly we can establish the following results.

**Theorem 4.7.** *Let  $(X, d, \leq)$  be an ordered generalized metric space (in the sense that  $d(x, y) \in \mathbb{R}_+^n$ ) and let  $f : X \rightarrow X$  be an operator. We suppose that:*

(a) *For each  $x, y \in X$  there exist  $m(x, y), M(x, y) \in X$  such that  $x, y \in [m(x, y), M(x, y)]_{\leq}$ .*

(b)  $X_{\leq} \in I(f \times f)$ .

(c)  $f : (X, d) \rightarrow (X, d)$  is orbitally continuous.

(d) The metric  $d$  is complete.

(e) There exists  $x_0 \in X$  such that  $(x_0, f(x_0)) \in X_{\leq}$ .

(f) *There exists a matrix  $A \in M_{mm}(\mathbb{R}_+)$  with  $A^n \rightarrow 0$  as  $n \rightarrow \infty$ , such that  $d(f(x), f(y)) \leq A \cdot d(x, y)$ , for each  $x, y \in X_{\leq}$ .*

*Then  $f : (X, d) \rightarrow (X, d)$  is a PO.*

**Theorem 4.8.** *Let  $(X, F, \min, \leq)$  be an ordered probabilistic complete metric space and let  $f : X \rightarrow X$  be an operator. We suppose that:*

(a) *For each  $x, y \in X$  there exist  $m(x, y), M(x, y) \in X$  such that  $x, y \in [m(x, y), M(x, y)]_{\leq}$ .*

(b)  $X_{\leq} \in I(f \times f)$ .

(c)  $f : (X, F, \min) \rightarrow (X, F, \min)$  is orbitally continuous.

(d)  $(LF)_f \cup (UF)_f \neq \emptyset$ .

(e) *There exists  $a \in [0, 1[$  such that  $F_{f(x), f(y)}(a\lambda) \geq F_{x, y}(\lambda)$ , for each  $x, y \in X_{\leq}$  and each  $\lambda > 0$ .*

*Then  $f : (X, d) \rightarrow (X, d)$  is a PO.*

*Proof.* From (d) there exists  $x_0 \in (LF)_f \cup (UF)_f$ . Then from (b) and (e) we get that  $(f^n(x_0))_{n \in \mathbb{N}} \rightarrow x^*$ , as  $n \rightarrow +\infty$ , for some  $x^* \in X$ . From (c) we get that  $x^* \in F_f$ .

Further on, if  $(x, x_0) \in X_{\leq}$ , then, from (b) and (e) we have that  $(f^n(x))_{n \in \mathbb{N}} \rightarrow x^*$ , as  $n \rightarrow +\infty$ .

If  $(x, x_0) \notin X_{\leq}$ , then, from (a), (b) and (e) and we get the same conclusion.

Our result now follows from Theorem 4.2. namely the particular case when the L-structure of the space is that induced by the probabilistic metric one.  $\square$

*Remark 4.9.* Theorem 4.7 is an extension of Perov's fixed point theorem, while Theorem 4.8 generalizes Sehgal and Bharucha-Reid's fixed point principle. It is worth noting that other fixed point results can be established by replacing the contraction condition on  $f$  with a generalized contraction assertion (see [6], [1], [10], etc.).

Also, it is not difficult to see that more general fixed point results in probabilistic metric spaces can be proved in this manner. For example, we also have the following generalization of Theorem 4.8 (see Hadžić and Pap [5] for the notions involved).

**Theorem 4.10.** *Let  $(X, F, T, \leq)$  be an ordered complete Menger space, let  $T$  be a  $t$ -norm of  $H$ -type and let  $f : X \rightarrow X$  be an operator. We suppose that:*

(a) *For each  $x, y \in X$  there exist  $m(x, y), M(x, y) \in X$  such that  $x, y \in [m(x, y), M(x, y)]_{\leq}$ .*

(b)  $X_{\leq} \in I(f \times f)$ .

(c)  $f : (X, F, T) \rightarrow (X, F, T)$  is orbitally continuous.

(d)  $(LF)_f \cup (UF)_f \neq \emptyset$ .

(e) *There exists  $a \in [0, 1[$  such that  $F_{f(x), f(y)}(a\lambda) \geq F_{x, y}(\lambda)$ , for each  $x, y \in X_{\leq}$  and each  $\lambda > 0$ .*

*Then  $f : (X, d) \rightarrow (X, d)$  is a PO.*

## 5. APPLICATIONS

Consider first the equation

$$(5.1) \quad X = Q + \sum_{j=1}^m A_j^* \cdot g(X) \cdot A_j,$$

where  $Q$  is a positive definite matrix,  $A_j$  are arbitrary  $n \times n$  matrices and the map  $g : \mathcal{P}(n) \rightarrow \mathcal{P}(n)$ . Denote by  $\mathcal{H}(n)$  the set of all Hermitian matrices and by  $\mathcal{P}(n) \subset \mathcal{H}(n)$  the set of all  $n \times n$  positive definite matrices.  $X \in \mathcal{P}(n)$  will be briefly denoted  $X > 0$ . See [9] for more details.

The next theorem is an extension of Theorem 4.1 and Theorem 4.2 in Ran-Reurings [9], not only with respect to the continuity, but mainly with respect to the monotonicity condition on  $g$ .

**Theorem 5.1.** *Let  $Q \in \mathcal{P}(n)$  and let  $g : \mathcal{P}(n) \rightarrow \mathcal{P}(n)$  be orbitally continuous. Suppose that:*

(a) *there exists  $M > 0$  such that:*

(i)  $\sum_{j=1}^m A_j^* A_j < M \cdot I_n$ ,

(ii)  $\|tr(g(Y) - g(X))\| \leq M^{-1} \cdot \|tr(Y - X)\|$ , for each  $X \leq Y$ .

(b)  $\mathcal{H}(n)_{\leq} \in I(g \times g)$ .

*Then equation (5.1) has a unique solution  $X^*$  in  $\mathcal{P}(n)$  and if  $f(X) := Q + \sum_{j=1}^m A_j^* \cdot g(X) \cdot A_j$ , then for each  $X_0 \in \mathcal{P}(n)$  we have  $f^n(X_0) \rightarrow X^*$ , as  $n \rightarrow +\infty$ .*

*Proof.* It is easy to observe that  $f(Q) \geq Q$ . Next, as in [9] we obtain that there exists  $a \in ]0, 1[$  such that  $\|f(Y) - f(X)\|_1 \leq a \cdot \|Y - X\|_1$  for each  $X \leq Y$ . Since

$\mathcal{H}(n)_{\leq} \in I(g \times g)$  we immediately have  $\mathcal{H}(n)_{\leq} \in I(f \times f)$ . The conclusion follows from Theorem 4.3.  $\square$

Next consider the equation

$$(5.2) \quad X = Q - \sum_{j=1}^m A_j^* \cdot g(X) \cdot A_j.$$

Then we have:

**Theorem 5.2.** *Let  $Q \in \mathcal{P}(n)$  such that  $Q > \sum_{j=1}^m A_j^* \cdot g(X) \cdot A_j$ , for each  $X \in \mathcal{H}(n)$*

*and let  $g : \mathcal{P}(n) \rightarrow \mathcal{P}(n)$  be orbitally continuous. Suppose that:*

(a) *there exists  $M > 0$  such that:*

(i)  $\sum_{j=1}^m A_j^* A_j < M \cdot I_n,$

(ii)  $\|tr(g(Y) - g(X))\| \leq M^{-1} \cdot \|tr(Y - X)\|,$  for each  $X \leq Y.$

(b)  $\mathcal{H}(n)_{\leq} \in I(g \times g).$

*Then equation (5.2) has a unique solution  $X^*$  in  $\mathcal{P}(n)$  and if  $f(X) := Q - \sum_{j=1}^m A_j^* \cdot g(X) \cdot A_j,$  then for each  $X_0 \in \mathcal{P}(n)$  we have  $f^n(X_0) \rightarrow X^*,$  as  $n \rightarrow +\infty.$*

*Proof.* From the hypothesis it follows that  $f(X) > 0$  for each  $X \in \mathcal{H}(n).$  Also  $f(Q) \leq Q.$  Moreover,  $f$  satisfies the assertions  $\|f(Y) - f(X)\|_1 \leq a \cdot \|Y - X\|_1$  for each  $X \leq Y$  and  $\mathcal{H}(n) \in I(f \times f).$  The conclusion follows again by Theorem 4.3.  $\square$

Our final application is for a system of nonlinear equations in  $(\mathbb{R}^m, \rightarrow, \leq).$

**Theorem 5.3.** *Consider the ordered  $L$ -space  $(\mathbb{R}^m, \rightarrow, \leq)$  and  $f : \mathbb{R}^m \rightarrow \mathbb{R}^m.$  Suppose that:*

(i)  $\mathbb{R}_{\leq}^m \in I(f \times f),$

(ii)  $f$  is orbitally continuous,

(iii) there exists  $x_0 \in \mathbb{R}^m$  such that  $(x_0, f(x_0)) \in \mathbb{R}_{\leq}^m,$

(iv) there exists a matrix  $A \in M_{mm}(\mathbb{R}_+),$  such that:

(a)  $A^n \rightarrow 0,$  as  $n \rightarrow \infty,$

(b)

$$\begin{pmatrix} \|f_1(x) - f_1(y)\| \\ \dots \\ \|f_m(x) - f_m(y)\| \end{pmatrix} \leq A \cdot \begin{pmatrix} \|x_1 - y_1\| \\ \dots \\ \|x_m - y_m\| \end{pmatrix},$$

for each  $x, y \in \mathbb{R}^m,$  with  $x \leq y.$

*Then the equation  $x = f(x)$  has a unique solution  $x^* \in \mathbb{R}^m$  and  $f^n(x_0) \rightarrow x^*$  as  $n \rightarrow +\infty,$  for all  $x_0 \in \mathbb{R}^m.$*

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DEPARTMENT OF APPLIED MATHEMATICS, BABEȘ-BOLYAI UNIVERSITY CLUJ-NAPOCA, KOGĂLNICEANU 1, 400084, CLUJ-NAPOCA, ROMANIA  
*E-mail address*: `petrusel@math.ubbcluj.ro`

DEPARTMENT OF APPLIED MATHEMATICS, BABEȘ-BOLYAI UNIVERSITY CLUJ-NAPOCA, KOGĂLNICEANU 1, 400084, CLUJ-NAPOCA, ROMANIA  
*E-mail address*: `iarus@math.ubbcluj.ro`