

## A DUAL GRAPH CONSTRUCTION FOR HIGHER-RANK GRAPHS, AND $K$ -THEORY FOR FINITE 2-GRAPHS

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ABSTRACT. Given a  $k$ -graph  $\Lambda$  and an element  $p$  of  $\mathbb{N}^k$ , we define the dual  $k$ -graph,  $p\Lambda$ . We show that when  $\Lambda$  is row-finite and has no sources, the  $C^*$ -algebras  $C^*(\Lambda)$  and  $C^*(p\Lambda)$  coincide. We use this isomorphism to apply Robertson and Steger's results to calculate the  $K$ -theory of  $C^*(\Lambda)$  when  $\Lambda$  is finite and strongly connected and satisfies the aperiodicity condition.

### 1. INTRODUCTION

In 1980, Cuntz and Krieger introduced a class of  $C^*$ -algebras, now called Cuntz-Krieger algebras, associated to finite  $\{0, 1\}$ -matrices  $A$  [4]. Enomoto and Watatani then showed that these algebras could be regarded as being associated in a natural way to finite directed graphs by regarding  $A$  as the vertex adjacency matrix of a finite directed graph  $E$  [5]. Generalising this association, Enomoto and Watatani associated  $C^*$ -algebras  $C^*(E)$  to finite graphs  $E$  with no sources<sup>1</sup> ( $E$  has no sources if each vertex of  $E$  is the range of at least one edge). Although not every finite directed graph with no sources has a vertex adjacency matrix with entries in  $\{0, 1\}$ , the vertex adjacency matrix of the dual graph  $\widehat{E}$  formed by regarding the edges of  $E$  as vertices and the paths of length 2 in  $E$  as edges *does* always have entries in  $\{0, 1\}$ , and the Cuntz-Krieger algebras associated to  $E$  and to  $\widehat{E}$  are canonically isomorphic [11]. These results have since been extended to infinite graphs (see for example [10, 9, 3, 7]; see also [2] when  $E$  has sources).

One of the major attractions of graph algebras is their applicability to the classification program for simple purely infinite nuclear  $C^*$ -algebras. Conditions on a graph  $E$  have been identified which guarantee that  $C^*(E)$  is purely infinite, simple, and nuclear, and satisfies the Universal Coefficient Theorem (see, for example, [3]), thus producing a large class of directed graphs whose  $C^*$ -algebras are determined up to isomorphism by their  $K$ -theory [12]. The  $K$ -theory of  $C^*(E)$  for an arbitrary directed graph  $E$  was calculated in [13], and it is shown in [17] that given any two finitely generated abelian groups  $G, H$  such that  $H$  is torsion free, there exists a

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<sup>1</sup>For the sake of consistency with  $k$ -graph notation, we regard directed graphs as 1-graphs, so *no sources* here corresponds to *no sinks* in, for example, [5, 3].

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directed graph  $E$  such that  $C^*(E)$  is simple, purely infinite, nuclear, and satisfies the Universal Coefficient Theorem, with  $K_0(C^*(E)) \cong G$  and  $K_1(C^*(E)) \cong H$ .

In 1999, Robertson and Steger introduced a class of higher-rank Cuntz-Krieger algebras  $\mathcal{A}$ , associated to collections  $M_1, \dots, M_k$  of commuting  $\{0, 1\}$ -matrices satisfying appropriate compatibility conditions [15]. In [16], they went on to calculate the  $K$ -theory of  $\mathcal{A}$ , demonstrating in particular that  $K_1(\mathcal{A})$  need not be torsion free, so that the class of higher-rank Cuntz-Krieger algebras exhausts some  $K$ -invariants which are not achieved by graph algebras. In order to place these higher-rank Cuntz-Krieger algebras in a graph-theoretic setting, and to generalise them as Watatani and Enomoto had generalised the original Cuntz-Krieger algebras, Kumjian and Pask introduced the notion of a higher-rank graph  $\Lambda$ , and defined and investigated the associated higher-rank graph  $C^*$ -algebra  $C^*(\Lambda)$  [8]. Connectivity in a rank- $k$  graph  $\Lambda$  is described in terms of  $k$  commuting vertex adjacency matrices  $\{M_1^\Lambda, \dots, M_k^\Lambda\}$ , called coordinate matrices. Just as in the rank-1 setting, not every  $k$ -graph has coordinate matrices with entries in  $\{0, 1\}$ , but if  $\Lambda$  is a  $k$ -graph whose coordinate matrices are  $\{0, 1\}$ -matrices, then [8, Corollary 3.5(ii)] shows that  $C^*(\Lambda)$  and the  $C^*$ -algebra  $\mathcal{A}$  associated to the coordinate matrices as in [15] are identical.

In this paper we introduce a notion of a dual graph for higher-rank graphs, and show that for a large class of higher-rank graphs  $\Lambda$ , the dual higher-rank graph  $p\Lambda$  and the original higher-rank graph  $\Lambda$  have canonically isomorphic  $C^*$ -algebras for all  $p \in \mathbb{N}^k$  (cf. [1]). We also show that by choosing  $p$  appropriately, we can ensure that  $p\Lambda$  has coordinate matrices with entries in  $\{0, 1\}$ . Using these results, we identify a class of finite rank-2 graphs whose  $C^*$ -algebras are isomorphic to the rank-2 Cuntz-Krieger algebras studied by Robertson and Steger, and we use the results of [16] to show that these  $C^*$ -algebras are purely infinite, simple, unital and nuclear, and to calculate their  $K$ -theory.

The layout of the paper is as follows: in Section 2, we recall the definition of  $k$ -graphs and the associated notation; in Section 3, we introduce the dual graph construction for  $k$ -graphs, and show that this construction preserves the associated  $C^*$ -algebra; and in Section 4, we identify the finite 2-graphs  $\Lambda$  whose  $C^*$ -algebras can be studied using Robertson and Steger's results, and use these results to calculate  $K_*(C^*(\Lambda))$ .

In the final stages of preparation of this paper, the authors became aware of Evans' Ph.D. thesis [6], which appears to obtain more general results regarding  $K$ -theory for 2-graph  $C^*$ -algebras than those established here. The authors thank the referee for helpful comments which we feel have improved the exposition.

## 2. PRELIMINARIES

We regard  $\mathbb{N}^k$  as an additive semigroup with identity 0. Given  $m, n \in \mathbb{N}^k$ , we write  $m \vee n$  for their coordinate-wise maximum and  $m \wedge n$  for their coordinate-wise minimum, and if  $m \leq n$ , then we write  $[m, n]$  for the set  $\{p \in \mathbb{N}^k : m \leq p \leq n\}$ . We denote the canonical generators of  $\mathbb{N}^k$  by  $\{e_1, \dots, e_k\}$ , and for  $n \in \mathbb{N}^k$ , we write  $n_j$  for the  $j^{\text{th}}$  coordinate of  $n$ .

**Definition 2.1.** Let  $k \in \mathbb{N} \setminus \{0\}$ . A  $k$ -graph is a pair  $(\Lambda, d)$  where  $\Lambda$  is a countable category and  $d$  is a functor from  $\Lambda$  to  $\mathbb{N}^k$  which satisfies the *factorisation property*: if  $\lambda \in \text{Mor}(\Lambda)$  and  $d(\lambda) = m + n$ , then there are unique morphisms  $\mu \in d^{-1}(m)$  and  $\nu \in d^{-1}(n)$  such that  $\lambda = \mu\nu$ .

We refer to elements of  $\text{Mor}(\Lambda)$  as *paths* and to elements of  $\text{Obj}(\Lambda)$  as *vertices* and we write  $r$  and  $s$  for the codomain and domain maps. The factorisation property allows us to identify  $\text{Obj}(\Lambda)$  with  $\{\lambda \in \text{Mor}(\Lambda) : d(\lambda) = 0\}$ . So we write  $\lambda \in \Lambda$  in place of  $\lambda \in \text{Mor}(\Lambda)$ , and when  $d(\lambda) = 0$ , we regard  $\lambda$  as a vertex of  $\Lambda$ .

Given  $\lambda \in \Lambda$  and  $E \subset \Lambda$ , we define  $\lambda E = \{\lambda\mu : \mu \in E, r(\mu) = s(\lambda)\}$  and  $E\lambda = \{\mu\lambda : \mu \in E, s(\mu) = r(\lambda)\}$ . In particular if  $d(v) = 0$ , then  $v$  is a vertex of  $\Lambda$  and  $vE = \{\lambda \in E : r(\lambda) = v\}$ ; similarly,  $Ev = \{\lambda \in \Lambda : s(\lambda) = v\}$ . We write  $\Lambda^n$  for the collection  $\{\lambda \in \Lambda : d(\lambda) = n\}$ .

**Definition 2.2.** We say that a  $k$ -graph  $(\Lambda, d)$  is *row-finite* if  $v\Lambda^n$  is finite for all  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ , and that  $\Lambda$  has *no sources* if  $v\Lambda^n$  is nonempty for all  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ . We say that  $\Lambda$  is *strongly connected* if  $v\Lambda w$  is nonempty for all  $v, w \in \Lambda^0$ , and we say that  $\Lambda$  is *finite* if  $\Lambda^0$  and each  $\Lambda^{e_i}$  are finite.

The factorisation property ensures that if  $l \leq m \leq n \in \mathbb{N}^k$  and if  $d(\lambda) = n$ , then there exist unique paths denoted  $\lambda(0, l)$ ,  $\lambda(l, m)$  and  $\lambda(m, n)$  such that  $d(\lambda(0, l)) = l$ ,  $d(\lambda(l, m)) = m - l$ , and  $d(\lambda(m, n)) = n - m$  and such that  $\lambda = \lambda(0, l)\lambda(l, m)\lambda(m, n)$ .

Given  $k \in \mathbb{N} \setminus \{0\}$ , and  $k$ -graphs  $(\Lambda_1, d_1)$  and  $(\Lambda_2, d_2)$ , we call a covariant functor  $x : \Lambda_1 \rightarrow \Lambda_2$  a *graph morphism* if it satisfies  $d_2 \circ x = d_1$ .

**Definition 2.3.** As in [8], given  $k \in \mathbb{N} \setminus \{0\}$ , we write  $\Omega_k$  for the  $k$ -graph given by  $\text{Obj}(\Omega_k) = \mathbb{N}^k$ ,  $\text{Mor}(\Omega_k) = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$ ,  $r(m, n) = m$ ,  $s(m, n) = n$ ,  $(m, n) \circ (n, p) = (m, p)$ , and  $d(m, n) = n - m$ . Given a  $k$ -graph  $\Lambda$ , an *infinite path* of  $\Lambda$  is a graph morphism  $x : \Omega_k \rightarrow \Lambda$ . We denote the collection of all infinite paths of  $\Lambda$  by  $\Lambda^\infty$ . For  $p \in \mathbb{N}^k$ , we write  $\sigma^p : \Lambda^\infty \rightarrow \Lambda^\infty$  for the shift-map determined  $\sigma^p(x)(m, n) = x(m + p, n + p)$ , and we say that  $x \in \Lambda^\infty$  is *aperiodic* if there do not exist  $p, q \in \mathbb{N}^k$  with  $p \neq q$  and  $\sigma^p(x) = \sigma^q(x)$ .

**Definition 2.4.** Let  $(\Lambda, d)$  be a row-finite  $k$ -graph with no sources. A Cuntz-Krieger  $\Lambda$ -family is a collection  $\{t_\lambda : \lambda \in \Lambda\}$  of partial isometries satisfying

- (i)  $\{t_v : v \in \Lambda^0\}$  is a collection of mutually orthogonal projections;
- (ii)  $t_\lambda t_\mu = t_{\lambda\mu}$  whenever  $s(\lambda) = r(\mu)$ ;
- (iii)  $t_\lambda^* t_\lambda = t_{s(\lambda)}$  for all  $\lambda \in \Lambda$ ; and
- (iv)  $t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$  for all  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ .

The *Cuntz-Krieger algebra*  $C^*(\Lambda)$  is the  $C^*$ -algebra generated by a Cuntz-Krieger  $\Lambda$ -family  $\{s_\lambda : \lambda \in \Lambda\}$  which is universal in the sense that for every Cuntz-Krieger  $\Lambda$ -family  $\{t_\lambda : \lambda \in \Lambda\}$  there is a unique homomorphism  $\pi$  of  $C^*(\Lambda)$  satisfying  $\pi(s_\lambda) = t_\lambda$  for all  $\lambda \in \Lambda$ .

### 3. DUAL HIGHER-RANK GRAPHS

In this section we define the higher-rank analog  $p\Lambda$  of the dual graph construction for directed graphs.

**Definition 3.1.** Let  $(\Lambda, d)$  be a  $k$ -graph and let  $p \in \mathbb{N}^k$ . Let  $p\Lambda = \{\lambda \in \Lambda : d(\lambda) \geq p\}$ . Define range and source maps on  $p\Lambda$  by  $r_p(\lambda) = \lambda(0, p)$ , and  $s_p(\lambda) = \lambda(d(\lambda) - p, d(\lambda))$  for all  $\lambda \in p\Lambda$ , and define composition by  $\lambda \circ_p \mu = \lambda\mu(p, d(\mu)) = \lambda(0, d(\lambda) - p)\mu$  whenever  $s_p(\lambda) = r_p(\mu)$ . Finally, define a degree map  $d_p$  on  $p\Lambda$  by  $d_p(\lambda) = d(\lambda) - p$  for all  $\lambda \in p\Lambda$ .

**Proposition 3.2.** *Let  $(\Lambda, d)$  be a  $k$ -graph, and let  $p \in \mathbb{N}^k$ . Then  $(p\Lambda, d_p)$  is a  $k$ -graph.*

*Proof.* It is straightforward to check that  $p\Lambda$  is a category with the indicated operations. If  $\lambda, \mu \in p\Lambda$  and  $s_p(\lambda) = r_p(\mu)$ , then  $\lambda \circ_p \mu = \lambda\mu(p, d(\mu))$  by definition, so  $d_p(\lambda \circ_p \mu) = d(\lambda) + d(\mu) - 2p = d_p(\lambda) + d_p(\mu)$ . Therefore,  $d_p$  is a functor from  $p\Lambda$  to  $\mathbb{N}^k$ .

We need to check that the factorisation property holds for  $p\Lambda$ . Take any  $\lambda \in p\Lambda$  and  $m, n \in \mathbb{N}^k$  with  $m + n = d_p(\lambda)$ , so  $d(\lambda) = m + p + n$ . By the factorisation property for  $\Lambda$  we have  $\lambda = \lambda(0, m)\lambda(m, m + p)\lambda(m + p, m + p + n)$ . But then  $\lambda = (\lambda(0, m)\lambda(m, m + p)) \circ_p (\lambda(m, m + p)\lambda(m + p, m + p + n))$  in  $p\Lambda$ , and  $d_p(\lambda(0, m)\lambda(m, m + p)) = m$  and  $d_p(\lambda(m, m + p)\lambda(m + p, m + p + n)) = n$ . This decomposition is unique by the factorisation property for  $\Lambda$ .  $\square$

*Remark 3.3.* If  $\Lambda$  has no sources, then  $p\Lambda$  has no sources, and if  $\Lambda$  is row-finite, then  $p\Lambda$  is row-finite.

**Proposition 3.4.** *Let  $(\Lambda, d)$  be a  $k$ -graph, and let  $p, q \in \mathbb{N}^k$ . Then  $q(p\Lambda) = (q+p)\Lambda$ .*

*Proof.* By definition, we have  $q(p\Lambda)^n = p\Lambda^{(n+q)} = \Lambda^{(n+q+p)} = (q+p)\Lambda^n$  for all  $n \in \mathbb{N}$ . Hence  $q(p\Lambda)$  and  $(q+p)\Lambda$  have identical elements. For the remainder of the proof, we write  $s_q^{p\Lambda}, r_q^{p\Lambda}, \circ_q^{p\Lambda}$ , and  $d_q^{p\Lambda}$  for the source, range, composition and degree maps of the dual graph  $q(p\Lambda)$ .

Fix  $\lambda \in \Lambda^{n+p+q}$ . We have  $s_{(q+p)}(\lambda) = \lambda(n, n + p + q)$  by definition, while  $s_q^{p\Lambda}(\lambda)$  is the final segment  $\mu$  of  $\lambda$  such that  $d(\mu) - p = d_p(\mu) = q$ ; that is,  $d(\mu) = p + q$ . Hence  $s_{p+q}(\lambda) = s_q^{p\Lambda}(\lambda)$ . Similarly,  $r_{p+q}(\lambda) = \lambda(0, p + q) = r_q^{p\Lambda}(\lambda)$ . Moreover,  $d_{p+q}(\lambda) = d(\lambda) - (p + q) = d_p(\lambda) - q = d_q^{p\Lambda}(\lambda)$ . Since  $\lambda$  was arbitrary, it follows that the range, source, and degree maps for  $(p + q)\Lambda$  and  $q(p\Lambda)$  agree.

This having been established, we have  $r_{p+q}(\lambda) = s_{p+q}(\mu)$  if and only if  $r_q^{p\Lambda}(\lambda) = s_q^{p\Lambda}(\mu)$ , in which case both  $\lambda \circ_{p+q} \mu$  and  $\lambda \circ_q^{p\Lambda} \mu$  are equal to  $\lambda\mu(p + q, d(\mu))$  by definition, completing the proof.  $\square$

**Theorem 3.5.** *Let  $(\Lambda, d)$  be a row-finite  $k$ -graph with no sources, and let  $p \in \mathbb{N}^k$ . Let  $\{s_\lambda : \lambda \in \Lambda\}$  denote the universal generating Cuntz-Krieger  $\Lambda$ -family in  $C^*(\Lambda)$ , and let  $\{t_\lambda : \lambda \in \Lambda\}$  be the universal generating Cuntz-Krieger  $p\Lambda$ -family in  $C^*(p\Lambda)$ . For all  $\lambda \in p\Lambda$ , define  $r_\lambda = s_\lambda s_{s_p(\lambda)}^*$ . There is an isomorphism  $\phi : C^*(p\Lambda) \rightarrow C^*(\Lambda)$  such that  $\phi(t_\lambda) = r_\lambda$  for all  $\lambda \in p\Lambda$ .*

*Proof.* First we show that the family  $\{r_\lambda : \lambda \in p\Lambda\}$  is a Cuntz-Krieger  $p\Lambda$ -family. Since, for any  $\beta \in p\Lambda^0$ , we have  $s_\beta \neq 0$ , it follows that  $r_\beta = s_\beta s_\beta^* \neq 0$  and that it is a projection in  $C^*(\Lambda)$ . Furthermore, for distinct  $\alpha, \beta \in p\Lambda^0$ , we have

$$r_\alpha r_\beta = s_\alpha s_\alpha^* s_\beta s_\beta^* = \delta_{\alpha, \beta} s_\alpha s_\alpha^* = \delta_{\alpha, \beta} r_\alpha.$$

This establishes relation (i).

For relation (ii), let  $\mu, \nu \in p\Lambda$  with  $r_p(\nu) = s_p(\mu)$ , so  $\mu \circ_p \nu = \mu\nu(p, d(\nu))$ . Then,

$$(3.1) \quad r_{\mu \circ_p \nu} = s_{\mu \circ_p \nu} s_{s_p(\mu \circ_p \nu)}^* = s_\mu s_{\nu(p, d(\nu))} s_{s_p(\nu)}^* = s_\mu s_{s_p(\mu)}^* s_{s_p(\mu)} s_{\nu(p, d(\nu))} s_{s_p(\nu)}^*.$$

But  $s_p(\mu) = r_p(\nu) = \nu(0, p)$ , so we can rewrite the right-hand side of (3.1) to obtain  $r_{\mu \circ_p \nu} = s_\mu s_{s_p(\mu)}^* s_\nu s_{s_p(\nu)}^* = r_\mu r_\nu$ . This establishes relation (ii).

Let  $\lambda \in p\Lambda$ , say  $d_p(\lambda) = n$ . Then  $r_\lambda^* r_\lambda = s_{s_p(\lambda)} s_\lambda^* s_\lambda s_{s_p(\lambda)}^* = s_{s_p(\lambda)} s_{s_p(\lambda)}^* = r_{s_p(\lambda)}$  by definition, establishing relation (iii).

Finally, for relation (iv), let  $\beta \in p\Lambda^0$  and let  $n \in \mathbb{N}^k$ . Then

$$r_\beta = s_\beta s_\beta^* = \sum_{\gamma \in s(\beta)\Lambda^n} s_\beta s_\gamma s_\gamma^* s_\beta^* = \sum_{\lambda \in \beta\Lambda^n} s_\lambda s_\lambda^*.$$

Applying the factorisation property and relation (ii) for  $C^*(\Lambda)$  to the right-hand side then gives

$$r_\beta = \sum_{\lambda \in \beta\Lambda^n} s_{\lambda(0,n)} s_{\lambda(n,n+p)} s_{\lambda(n,n+p)}^* s_{\lambda(0,n)}^*,$$

and then since each  $s_{\lambda(n,n+p)} s_{\lambda(n,n+p)}^*$  is a projection, we obtain

$$r_\beta = \sum_{\lambda \in \beta\Lambda^n} (s_{\lambda(0,n)} s_{\lambda(n,n+p)} s_{\lambda(n,n+p)}^*) (s_{\lambda(n,n+p)} s_{\lambda(n,n+p)}^* s_{\lambda(0,n)}^*) = \sum_{\lambda \in \beta(p\Lambda^n)} r_\lambda r_\lambda^*,$$

which establishes relation (iv).

It follows from the universal property of  $C^*(p\Lambda)$  that there exists a homomorphism  $\phi : C^*(p\Lambda) \rightarrow C^*(\Lambda)$  satisfying  $\phi(t_\lambda) = r_\lambda$  for all  $\lambda \in p\Lambda$ . We claim that  $\{r_\lambda : \lambda \in p\Lambda\}$  generates  $C^*(\Lambda)$ . To see this, let  $\sigma \in \Lambda$  with  $d(\sigma) = n$ . An application of relation (iv) for  $C^*(\Lambda)$  gives  $s_\sigma = \sum_{\beta \in s(\sigma)\Lambda^p} s_\sigma s_\beta s_\beta^* = \sum_{\lambda \in \sigma\Lambda^p} s_\lambda s_{s_p(\lambda)}^*$ , and this last is equal to  $\sum_{\lambda \in \sigma\Lambda^p} r_\lambda$  by definition. Thus  $\phi$  maps  $C^*(p\Lambda)$  onto  $C^*(\Lambda)$ .

Now let  $\gamma^\Lambda$  denote the gauge action on  $C^*(\Lambda)$ , and let  $\gamma^{p\Lambda}$  denote the gauge action on  $C^*(p\Lambda)$ . For  $z \in \mathbb{T}^k$  and  $\lambda \in p\Lambda$ , we have

$$\gamma_z^\Lambda(r_\lambda) = \gamma_z^\Lambda(s_\lambda s_{s_p(\lambda)}^*) = z^{d(\lambda)} s_\lambda (z^{d(s_p(\lambda))} s_{s_p(\lambda)}^*)^* = z^{d(\lambda)-p} r_\lambda = \gamma^{p\Lambda}(r_\lambda).$$

Theorem 3.4 of [8] now establishes that  $\phi$  is injective. □

*Remark 3.6.* The hypotheses that  $\Lambda$  be row-finite and have no sources are crucial in Theorem 3.5. To see why, note that for  $v \in \Lambda^0$ , the generator  $s_v$  of  $C^*(\Lambda)$  is recovered in  $C^*(p\Lambda)$  as  $\sum_{\beta \in p\Lambda^0, r(\beta)=v} r_\beta$ . However, even for 1-graphs the Cuntz-Krieger relations only insist that  $p_v = \sum_{r(e)=v} s_e s_e^*$  when  $r^{-1}(v)$  is finite and nonempty.

**Lemma 3.7.** *Let  $(\Lambda, d)$  be a  $k$ -graph, and let  $p \in \mathbb{N}^k$ . For each  $n \in \mathbb{N}^k$  with  $n \leq p$  and  $v, w \in p\Lambda^0$ , there is at most one  $\lambda \in v(p\Lambda^n)w$ .*

*Proof.* Let  $v, w \in p\Lambda^0 = \Lambda^p$  and suppose  $\lambda \in v(p\Lambda^n)w$ . Then  $\lambda \in \Lambda^{n+p}$ ,  $\lambda(0, p) = v$ , and  $\lambda(n, n+p) = w$ . Since  $n \leq p$  we have  $\lambda(0, n) = (\lambda(0, p))(0, n) = v(0, n)$ , so  $\lambda = \lambda(0, n)\lambda(n, n+p) = v(0, n)w$ , and hence is determined by  $v$  and  $w$ . □

*Notation 3.8.* Let  $(\Lambda, d)$  be a  $k$ -graph. We write  $M_i^\Lambda$ ,  $1 \leq i \leq k$ , for the matrices in  $M_{\Lambda^0}(\mathbb{N})$  defined by  $(M_i^\Lambda)_{v,w} = |w\Lambda^{e_i}v|$  for  $w, v \in \Lambda^0$ , and we refer to these matrices as the *coordinate matrices* of  $\Lambda$ .

*Remark 3.9.* In [8, 6]  $(M_i^\Lambda)_{v,w} = |v\Lambda_{e_i}w|$ , so our  $M_i^\Lambda$  is the transpose of theirs. This is for consistency with the matrices in [16, 15]; we will be employing Robertson and Steger’s results to calculate  $K$ -theory in Section 4.

**Corollary 3.10.** *Let  $(\Lambda, d)$  be a  $k$ -graph, and let  $p \in \mathbb{N}^k$  with  $p_i \geq 1$  for  $1 \leq i \leq k$ . Then the coordinate matrices  $M_i^{p\Lambda}$  of  $p\Lambda$  are  $\{0, 1\}$ -matrices.*

#### 4. $K$ -THEORY

In this section we identify a class of 2-graphs whose associated  $C^*$ -algebras are isomorphic to higher rank Cuntz-Krieger algebras in the sense of [16], and use the results of [16] to calculate the  $K$ -theory of the  $C^*$ -algebras of such 2-graphs. To state the main theorem for this section we employ the following notation: given square  $n \times n$  matrices  $M, N$ , we write  $[M \ N]$  for the block  $n \times 2n$  matrix whose first  $n$  columns are those of  $M$  and whose last  $n$  columns are those of  $N$ . We also write  $\mathbf{1}$  for the element  $(1, 1)$  of  $\mathbb{N}^2$ .

**Theorem 4.1.** *Let  $(\Lambda, d)$  be a 2-graph which is finite and strongly connected as in Definition 2.2 and which has an aperiodic infinite path as in Definition 2.3. Then  $C^*(\Lambda)$  is purely infinite, simple, unital and nuclear, and we have*

$$(4.1) \quad \begin{aligned} \text{rank}(K_0(C^*(\Lambda))) &= \text{rank}(K_1(C^*(\Lambda))) \\ &= \text{rank} \left( \text{coker} \begin{bmatrix} I - M_1^{1\Lambda} & I - M_2^{1\Lambda} \end{bmatrix} \right) \\ &\quad + \text{rank} \left( \text{coker} \begin{bmatrix} I - (M_1^{1\Lambda})^t & I - (M_2^{1\Lambda})^t \end{bmatrix} \right); \end{aligned}$$

$$(4.2) \quad \text{tor}(K_0(C^*(\Lambda))) \cong \text{tor} \left( \text{coker} \begin{bmatrix} I - M_1^{1\Lambda} & I - M_2^{1\Lambda} \end{bmatrix} \right); \text{ and}$$

$$(4.3) \quad \text{tor}(K_1(C^*(\Lambda))) \cong \text{tor} \left( \text{coker} \begin{bmatrix} I - (M_1^{1\Lambda})^t & I - (M_2^{1\Lambda})^t \end{bmatrix} \right).$$

The remainder of this section constitutes the proof of Theorem 4.1. We begin by recalling some definitions from [16]. Let  $A$  be a finite set, and let  $M_1, M_2$  be  $A \times A$  matrices with entries in  $\{0, 1\}$ . For  $n \in \mathbb{N}^k$ , let  $W_n = \{w : [0, n] \rightarrow A : M_j(w(l + e_j), w(l)) = 1 \text{ whenever } l, l + e_j \in [0, n]\}$ ; we refer to the elements of  $W_n$  as *allowable words of shape  $n$* , and write  $W$  for the collection  $\bigcup_{n \in \mathbb{N}^2} W_n$  of all *allowable words*. For  $u \in W$ , write  $S(u)$  for the shape of  $u$ ; that is,  $S(u)$  is the unique element of  $\mathbb{N}^2$  such that  $u \in W_{S(u)}$ . We identify  $W_0$  with  $A$ . The matrices  $M_1, M_2$  are said to satisfy (H0)–(H3) if

- (H0) Each  $M_i$  is nonzero;
- (H1a)  $M_1 M_2 = M_2 M_1$ ;
- (H1b)  $M_1 M_2$  is a  $\{0, 1\}$ -matrix;
- (H2) the directed graph with a vertex for each  $a \in A$  and a directed edge  $(a, i, b)$  from  $a$  to  $b$  for each  $a, i, b$  such that  $M_i(b, a) = 1$  is irreducible; and
- (H3) for each  $m \in \mathbb{Z}^2 \setminus \{0\}$ , there exists a word  $w \in W$  and elements  $l_1, l_2$  of  $\mathbb{N}^2$  with  $0 \leq l_1, l_2 \leq S(w)$  such that  $l_2 - l_1 = m$  and  $w(l_1) \neq w(l_2)$ .

*Notation 4.2.* If  $(\Lambda, d)$  is a 2-graph such that the coordinate matrices  $M_1^\Lambda$  and  $M_2^\Lambda$  are  $\{0, 1\}$ -matrices, we write  $W_n^\Lambda$  and  $W^\Lambda$  for the collection of allowable words of shape  $n$  and for the collection of all allowable words, respectively. For  $\lambda \in \Lambda$ , let  $w_\lambda^\Lambda$  be the word in  $W_{d(\lambda)}^\Lambda$  given by  $w_\lambda^\Lambda(m) = s(\lambda(0, m))$  for  $0 \leq m \leq d(\lambda)$ . Since each  $M_i^\Lambda$  is a  $\{0, 1\}$ -matrix, the map  $\lambda \mapsto w_\lambda^\Lambda$  is a bijection between  $\Lambda^n$  and  $W_n^\Lambda$  for all  $n \in \mathbb{N}^2$ .

**Proposition 4.3.** *Let  $(\Lambda, d)$  be a finite 2-graph with no sources, and let  $M_1^{1\Lambda}$  and  $M_2^{1\Lambda}$  be the matrices associated to  $1\Lambda$ . Then*

- (1)  $M_1^{1\Lambda}, M_2^{1\Lambda}$  satisfy (H0), (H1a), and (H1b);
- (2)  $M_1^{1\Lambda}, M_2^{1\Lambda}$  satisfy (H2) if and only if  $\Lambda$  is strongly connected; and
- (3) if  $M_1^{1\Lambda}, M_2^{1\Lambda}$  satisfy (H2), then they satisfy (H3) if and only if  $\Lambda$  has an aperiodic infinite path.

*Proof.* For (1), note that each  $M_i^{1\Lambda}$  is a finite square matrix over  $1\Lambda^0$  by definition, and has entries in  $\{0, 1\}$  by Corollary 3.10. It is easy to see that

$$(M_i^{1\Lambda} M_{3-i}^{1\Lambda})_{v,w} = |\{(\alpha, \beta) \in w(1\Lambda^{e_{3-i}}) \times (1\Lambda^{e_i})v : r(\alpha) = s(\beta)\}| = |w(1\Lambda^1)v|$$

for  $i = 1, 2$ , which establishes (H1a) and, when combined with Lemma 3.7, (H1b).

For (2), notice that  $M_1^{1\Lambda}, M_2^{1\Lambda}$  satisfy (H2) if and only if for every  $v, w \in 1\Lambda^0$  there exist elements  $\alpha_1, \dots, \alpha_k$  in  $1\Lambda^{(1,0)} \cup 1\Lambda^{(0,1)}$  such that  $r(\alpha_1) = v, s(\alpha_k) = w$ , and  $r(\alpha_{i+1}) = s(\alpha_i)$  for  $1 \leq i \leq k - 1$ .

So suppose first that  $M_1^{1\Lambda}, M_2^{1\Lambda}$  satisfy (H2), and let  $v, w \in \Lambda^0$ . Since  $\Lambda$  has no sources, there exist  $\mu, \nu \in \Lambda^1$  with  $r(\mu) = v$  and  $r(\nu) = w$ ; so  $\mu, \nu \in 1\Lambda^0$  by definition, and (H2) ensures that there is a path  $\alpha_1 \dots \alpha_k$  in  $1\Lambda^{(1,0)} \cup 1\Lambda^{(0,1)}$  with  $r_1(\alpha_1) = \mu$  and  $s_1(\alpha_k) = \nu$ . By definition of  $1\Lambda$ , the path  $\alpha_1 \dots \alpha_k$  in  $1\Lambda$  is a path  $\lambda \in \Lambda$  with  $d(\lambda) = d_1(\alpha_1 \dots \alpha_k) + \mathbf{1}$ , and such that  $\lambda(0, \mathbf{1}) = \mu$  and  $\lambda(d(\lambda) - \mathbf{1}, d(\lambda)) = \nu$ . But then  $\lambda(0, d(\lambda) - \mathbf{1}) \in v\Lambda w$ . However,  $v, w \in \Lambda^0$  were arbitrary, so  $\Lambda$  is strongly connected.

Now suppose that  $\Lambda$  is strongly connected, and fix  $\mu, \nu \in 1\Lambda^0$ . Since  $\Lambda$  is strongly connected, there is a path  $\lambda \in s(\mu)\Lambda r(\nu)$ , and then  $\tau := \mu\lambda\nu$  belongs to  $\mu(1\Lambda)\nu$  with  $d_1(\mu\lambda\nu) = d(\lambda) + \mathbf{1}$ . Any factorisation of  $\tau$  into segments from  $1\Lambda^{(1,0)} \cup 1\Lambda^{(0,1)}$  now gives a path in  $1\Lambda^{(1,0)} \cup 1\Lambda^{(0,1)}$  from  $\nu$  to  $\mu$ , so  $M_1^{1\Lambda}, M_2^{1\Lambda}$  satisfy (H2).

Finally, for (3), assume that  $M_1^{1\Lambda}, M_2^{1\Lambda}$  satisfy (H2), so  $\Lambda$  is strongly connected by part (2). For  $x \in \Lambda^\infty$ , define  $\mathbf{1}x \in 1\Lambda^\infty$  by  $(\mathbf{1}x)(m, n) = x(m, n + \mathbf{1})$ . It is easy to see that the map  $x \mapsto \mathbf{1}x$  is a bijection between  $\Lambda^\infty$  and  $1\Lambda^\infty$ .

Claim:  $x \in \Lambda^\infty$  is aperiodic if and only if  $\mathbf{1}x \in 1\Lambda^\infty$  is aperiodic. To see this, let  $m, n \in \mathbb{N}^k$ , and fix  $x \in \Lambda^\infty$ . By definition, we have

$$\begin{aligned} \sigma^m(\mathbf{1}x) = \sigma^n(\mathbf{1}x) &\iff (\mathbf{1}x)(s + m, t + m) = (\mathbf{1}x)(s + n, t + n) \quad \text{for } s \leq t \\ (4.4) \qquad \qquad \qquad &\iff x(s + m, t + m + \mathbf{1}) = x(s + n, t + n + \mathbf{1}) \quad \text{for } s \leq t. \end{aligned}$$

Now if  $x(s + m, t + m + \mathbf{1}) = x(s + n, t + n + \mathbf{1})$  for all  $s \leq t \in \mathbb{N}^2$ , then the uniqueness of factorisations in  $\Lambda$  ensures that  $x(s + m, t + m) = x(s + n, t + n)$  for all  $s \leq t \in \mathbb{N}^2$ . Conversely if  $x(s + m, t + m) = x(s + n, t + n)$  for all  $s \leq t \in \mathbb{N}^2$ , then replacing  $t$  with  $t + \mathbf{1}$  gives  $x(s + m, t + m + \mathbf{1}) = x(s + n, t + n + \mathbf{1})$  for all  $s \leq t \in \mathbb{N}^2$ . Hence (4.4) shows that

$$\begin{aligned} \sigma^m(\mathbf{1}x) = \sigma^n(\mathbf{1}x) &\iff x(s + m, t + m) = x(s + n, t + n) \quad \text{for } s \leq t \in \mathbb{N}^2 \\ &\iff \sigma^m(x) = \sigma^n(x), \end{aligned}$$

establishing the claim. Thus it suffices to show that  $M_i^{1\Lambda}$  satisfy (H3) if and only if  $1\Lambda^\infty$  has an aperiodic element.

Suppose first that there exists an aperiodic path  $x \in 1\Lambda^\infty$ . Fix  $m \in \mathbb{Z}^2$ , and write  $m = m_+ - m_-$  where  $m_+, m_- \in \mathbb{N}^2$ . Since  $|v(1\Lambda^{e_i})w| \in \{0, 1\}$  for all  $v, w \in 1\Lambda^0, i = 1, 2$ , we have that  $x$  is completely determined by its restriction to the objects of  $\Omega_2$ ; that is, by the function from  $\mathbb{N}^2$  to  $1\Lambda^0$  given by  $n \mapsto x(n)$ . Since  $x$  is aperiodic, it follows that  $\sigma^{m_+}(x)(n) \neq \sigma^{m_-}(x)(n)$  for some  $n \in \mathbb{N}^2$ . But then with  $N := n + m_-$ , we have  $x(N + m_+ - m_-) \neq x(N)$ , and  $w := x|_{[0, N+m_+-m_-]} \in W_{N+m_+-m_-}^{1\Lambda}$  satisfies  $w(N) \neq w(N + m)$ . Since  $m \in \mathbb{Z}^2$  was arbitrary, this establishes that  $M_1^{1\Lambda}, M_2^{1\Lambda}$  satisfy (H3).

Now suppose that  $M_1^{1\Lambda}, M_2^{1\Lambda}$  satisfy (H3). For each  $m \in \mathbb{Z}^2 \setminus \{0\}$ , fix  $w_m \in W^{1\Lambda}$  and  $l_m \in \mathbb{N}^2$  such that  $0 \leq l_m, l_m + m \leq S(w_m)$  and  $w_m(l_m) \neq w_m(l_m + m)$ . Let  $\lambda_m$  be the unique path in  $1\Lambda$  such that  $w_m = w_{\lambda_m}^{1\Lambda}$ . We will construct an infinite path  $x$  which contains infinitely many occurrences of each  $\lambda_m$ ; this will ensure that there is no  $m$  for which a sufficiently large shift of  $x$  has period  $m$ , and hence that  $x$  is aperiodic. The details of this construction, and the verification that the resulting  $x$  is aperiodic constitute the remainder of the proof.

Let  $\{m_i : i \in \mathbb{N}\}$  be a listing of  $\mathbb{Z}^2 \setminus \{0\}$ . Fix an arbitrary  $v \in \mathbf{1}\Lambda^0$ , and for each  $i \in \mathbb{N}$ , let  $\alpha_i$  be any element of  $v(\mathbf{1}\Lambda)r(\lambda_{m_i})$ , and let  $\beta_i$  be any element of  $s(\lambda_{m_i})(\mathbf{1}\Lambda)v$  with the property that  $d_{\mathbf{1}}(\alpha_i\lambda_{m_i}\beta_i) \geq \mathbf{1}$ ; this is possible because  $\Lambda$  is strongly connected and has no sources.

For  $i \in \mathbb{N}$ , let  $\rho_i = \alpha_i\lambda_{m_i}\beta_i$ , and let  $\tau_i = \rho_1\rho_2 \dots \rho_i$ . Let  $x$  be the infinite path  $x = \tau_1\tau_2\tau_3 \dots$ . We claim that  $x$  is aperiodic.

To see this, let  $s, t \in \mathbb{N}^2$  be distinct, and let  $I_{s,t}$  be the element of  $\mathbb{N}$  such that  $m_{I_{s,t}} = t - s$ . Let  $J = \max\{s_1, s_2, t_1, t_2\}$ ; since  $d_{\mathbf{1}}(\rho_i) \geq (1, 1)$ , we have that  $i \geq J$  implies  $d_{\mathbf{1}}(\tau_1 \dots \tau_i) \geq s, t$ . Let  $K = \max\{I_{s,t}, J + 1\}$ , and define  $N = d_{\mathbf{1}}(\tau_1 \dots \tau_{K-1}) + d_{\mathbf{1}}(\rho_1 \dots \rho_{I_{s,t}-1}) + d(\alpha_{I_{s,t}}) + l_{t-s} - s$ . We have  $N \geq 0$  by choice of  $K$ , and

$$\begin{aligned} \sigma^s(x)(N) &= x(N + s) \\ &= x(d_{\mathbf{1}}(\tau_1 \dots \tau_{K-1}) + d_{\mathbf{1}}(\rho_1 \dots \rho_{I_{s,t}-1}) + d(\alpha_{I_{s,t}}) + l_{t-s}) \\ &= \lambda_{m_{I_{s,t}}}(l_{t-s}). \end{aligned}$$

A similar calculation shows that  $\sigma^t(x)(N) = \lambda_{m_{I_{s,t}}}(l_{t-s} + (t - s))$ , and hence  $\sigma^s(x)(N) \neq \sigma^t(x)(N)$  by our choice of  $\lambda_{m_{I_{s,t}}}$ . It follows that  $\sigma^s(x) \neq \sigma^t(x)$ , and since  $s, t \in \mathbb{N}^2$  were arbitrary, that  $x$  is aperiodic.  $\square$

*Remark 4.4.* The preceding proof actually shows: (1) that Proposition 4.3 applies to  $M_i^\Lambda$  for any  $\Lambda$  satisfying the property of Lemma 3.7 for  $p = \mathbf{1}$ ; and (2) that  $\mathbf{1}\Lambda$  is strongly-connected (resp. strongly connected and contains an infinite path) if and only if  $\Lambda$  has the same property. Since our motivation is to prove Theorem 4.1, we have compressed this into a single result.

*Notation 4.5.* Let  $\Lambda$  be a finite strongly connected 2-graph with an aperiodic infinite path. We write  $\mathcal{A}^{\mathbf{1}\Lambda}$  for the  $C^*$ -algebra associated to  $M_i^{\mathbf{1}\Lambda}$  as in [16]. That is,  $\mathcal{A}^{\mathbf{1}\Lambda}$  is the universal  $C^*$ -algebra generated by a family  $\{s_{u,v} : u, v \in W^{\mathbf{1}\Lambda}, u(S(u)) = v(S(v))\}$  of partial isometries satisfying

$$(4.5) \quad s_{u,v} = s_{v,u}^* \quad \text{for } u, v \in W^{\mathbf{1}\Lambda};$$

$$(4.6) \quad s_{u,v}s_{v,w} = s_{u,w} \quad \text{for } u, v, w \in W^{\mathbf{1}\Lambda};$$

$$(4.7) \quad s_{u,v} = \sum_{w \in W_{e_j}^{\mathbf{1}\Lambda}, u(S(u))=w(0)} s_{uw}s_{vw}^* \quad \text{for } u, v, \in W^{\mathbf{1}\Lambda}, j \in \{1, 2\}; \text{ and}$$

$$(4.8) \quad s_{a,a}s_{b,b} = 0 \quad \text{for distinct } a, b \in W_0^{\mathbf{1}\Lambda}.$$

**Lemma 4.6.** *Let  $(\Lambda, d)$  be a finite strongly-connected 2-graph which has an aperiodic infinite path. Then  $C^*(\Lambda)$  is isomorphic to  $\mathcal{A}^{\mathbf{1}\Lambda}$ .*

*Proof.* The factorisation property ensures that if  $\Lambda$  is strongly connected and contains an infinite path, then  $\Lambda$  has no sources. By Theorem 3.5, we have that  $C^*(\Lambda)$  is isomorphic to  $C^*(\mathbf{1}\Lambda)$ , so it suffices to show that  $C^*(\mathbf{1}\Lambda)$  is isomorphic to  $\mathcal{A}^{\mathbf{1}\Lambda}$ . It is easy to check using Definition 2.4(i)–(iv), relations (4.5)–(4.8), and the universal properties of  $\mathcal{A}^{\mathbf{1}\Lambda}$  and  $C^*(\mathbf{1}\Lambda)$  that there exists a homomorphism  $\pi : \mathcal{A}^{\mathbf{1}\Lambda} \rightarrow C^*(\mathbf{1}\Lambda)$  satisfying  $\pi(s_{w_\lambda^{\mathbf{1}\Lambda}, w_\mu^{\mathbf{1}\Lambda}}) = s_\lambda s_\mu^*$  for all  $\lambda, \mu \in \mathbf{1}\Lambda$ , and that there exists a homomorphism  $\psi : C^*(\mathbf{1}\Lambda) \rightarrow \mathcal{A}^{\mathbf{1}\Lambda}$  satisfying  $\psi(s_\lambda) = s_{w_\lambda^{\mathbf{1}\Lambda}, w_{s(\lambda)}^{\mathbf{1}\Lambda}}$ . Since these two homomorphisms are mutually inverse, the result follows.  $\square$

*Remark 4.7.* The argument of statement (2) of Proposition 4.3 shows that if  $\Lambda$  has no sources, then for any  $q \geq 1$ , the coordinate matrices of  $q\Lambda$  will satisfy (H2) only if  $\Lambda$  is strongly connected and has no sources. In particular, there exists  $q \in \mathbb{N}^2$  such that  $M_i^{q\Lambda}$  satisfy (H0)–(H3) if and only if  $M_i^{\Lambda}$  satisfy (H0)–(H3).

*Proof of Theorem 4.1.* Theorem 5.9, Proposition 5.11, and Corollary 6.4 of [15], combined with the previous two results, show that  $C^*(\Lambda)$  is simple, purely infinite and nuclear. We have that  $C^*(\Lambda)$  is unital with  $1_{C^*(\Lambda)} = \sum_{v \in \Lambda^0} s_v$ . Proposition 2.14 of [16] establishes (4.1)–(4.3).  $\square$

*Remarks 4.8.* (1) The proof of [16, Proposition 2.14] does not make any use of relations (H2) and (H3). Hence the formulas for  $K_*(C^*(\Lambda))$  in Theorem 4.1 hold when  $\Lambda$  is a finite  $k$ -graph with no sinks or sources, even if it is not strongly connected and does not have an aperiodic infinite path. However, in this case  $C^*(\Lambda)$  is not necessarily simple and purely infinite, and so is not determined up to isomorphism by its  $K$ -theory.

(2) The formulas for  $K_*(C^*(\Lambda))$  given in Theorem 4.1 are in terms of the coordinate matrices  $M_i^{\Lambda}$  of the dual  $k$ -graph. Proposition 5.1 of [6] shows that the same formulas hold if all instances  $M_i^{\Lambda}$  are replaced with  $M_i^{\Lambda}$ , but it is unclear how to show this directly.

## REFERENCES

- [1] T. Bates, *Applications of the gauge-invariant uniqueness theorem for graph algebras*, Bull. Austral. Math. Soc. **65** (2002), 55–67. MR1922607 (2003g:46064)
- [2] T. Bates and D. Pask, *Flow equivalence of graph algebras*, Ergodic Theory Dynam. Systems **24** (2004), 367–382. MR2054048 (2004m:37019)
- [3] T. Bates, D. Pask, I. Raeburn, and W. Szymański, *The  $C^*$ -algebras of row-finite graphs*, New York J. Math. **6** (2000), 307–324. MR1777234 (2001k:46084)
- [4] J. Cuntz and W. Krieger, *A class of  $C^*$ -algebras and topological Markov chains*, Invent. Math. **56** (1980), 251–268. MR0561974 (82f:46073a)
- [5] M. Enomoto and Y. Watatani, *A graph theory for  $C^*$ -algebras*, Math. Japon. **25** (1980), 435–442. MR0594544 (83d:46069a)
- [6] D. G. Evans, *On higher-rank graph  $C^*$ -algebras*, Ph.D. Thesis, Univ. Wales, 2002.
- [7] N. J. Fowler, M. Laca, and I. Raeburn, *The  $C^*$ -algebras of infinite graphs*, Proc. Amer. Math. Soc. **128** (2000), 2319–2327. MR1670363 (2000k:46079)
- [8] A. Kumjian and D. Pask, *Higher rank graph  $C^*$ -algebras*, New York J. Math. **6** (2000) 1–20. MR1745529 (2001b:46102)
- [9] A. Kumjian, D. Pask, and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998), 161–174. MR1626528 (99i:46049)
- [10] A. Kumjian, D. Pask, I. Raeburn, and J. Renault, *Graphs, groupoids and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), 505–541. MR1432596 (98g:46083)
- [11] M.H. Mann, I. Raeburn, and C.E. Sutherland, *Representations of finite groups and Cuntz-Krieger algebras*, Bull. Austral. Math. Soc. **46** (1992), 225–243. MR1183780 (93k:46046)
- [12] N.C. Phillips, *A classification theorem for nuclear purely infinite simple  $C^*$ -algebras*, Documenta Math. **5** (2000), 49–114. MR1745197 (2001d:46086b)
- [13] I. Raeburn and W. Szymański, *Cuntz-Krieger algebras of infinite graphs and matrices*, Trans. Amer. Math. Soc. **356** (2004), 39–59. MR2020023 (2004i:46087)
- [14] I. Raeburn, A. Sims, and T. Yeend, *Higher-rank graphs and their  $C^*$ -algebras*, Proc. Edinb. Math. Soc. **46** (2003), 99–115. MR1961175 (2004f:46068)
- [15] G. Robertson and T. Steger, *Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras*, J. Reine Angew. Math. **513** (1999), 115–144. MR1713322 (2000j:46109)

- [16] G. Robertson and T. Steger, *Asymptotic  $K$ -theory for groups acting on  $\tilde{A}_2$  buildings*, Can. J. Math. **53** (2001), 809–833. MR1848508 (2002f:46141)
- [17] W. Szymański, *The range of  $K$ -invariants for  $C^*$ -algebras of infinite graphs*, Indiana Univ. Math. J. **51** (2002), 239–249. MR1896162 (2003b:46077)

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