

ON THE DETERMINATION OF A MEASURE BY THE ORBITS GENERATED BY ITS LOGARITHMIC POTENTIAL

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(Communicated by Juha M. Heinonen)

ABSTRACT. We say that a logarithmic potential *generates a curve* in the plane if a unit mass traces the curve under the action of the potential. We consider the following problem: *A one-parameter family of plane curves is given. We assume that these curves lie in the complement of a compact set K . Find all measures supported in K whose potentials generate each of the given curves.* We solve this problem when K is the unit circle in three specific cases: (a) when the given curves are straight lines through the origin, (b) when the curves are straight lines through a point on the unit circle, and (c) when the curves are circles centered at the origin. The solution involves the Poisson integral and its boundary behavior.

1. AN INVERSE PROBLEM IN POTENTIAL THEORY

Let σ be a finite Borel measure with support in a compact set K that lies in the plane \mathbb{R}^2 . For notational convenience we identify \mathbb{R}^2 with the complex plane \mathbb{C} . The *logarithmic potential* of σ is the function $V_\sigma : \mathbb{C} \rightarrow (-\infty, \infty]$ defined by

$$(1.1) \quad V_\sigma(z) = \int_K \log \frac{1}{|z - \zeta|} d\sigma(\zeta).$$

It is well known (see e.g. [6, Ch.3]) that V_σ is a function superharmonic in \mathbb{C} and harmonic in the complement of its support (which contains the set $\mathbb{C} \setminus K$).

Let α be an orbit in the plane. By *orbit* here we mean a C^2 function $\alpha(t) = x(t) + iy(t) : I \rightarrow \mathbb{C}$ defined on a closed interval $I \subset \mathbb{R}$. A *curve* is a set of points $x + iy \in \mathbb{C}$ that satisfy an equation of the form $\mathbf{f}(x, y) = q$, where q is a real number and \mathbf{f} is a C^2 real function defined on an open subset of \mathbb{C} . By the usual abuse of language, we will say that an orbit α lies in a set $A \subset \mathbb{C}$ if $\alpha(t) \in A$ for all $t \in I$. Of course every orbit α lies on the curve $\{\alpha(t) : t \in I\}$ which is the *trace* of α . The equation $\mathbf{f}(x, y) = q$ of the trace can be found (in principle) by eliminating t from the equations $x = x(t)$, $y = y(t)$.

We say that the potential V_σ *generates the orbit* α if Newton's second law holds:

$$(1.2) \quad \frac{d^2\alpha}{dt^2}(t) = \frac{d^2x}{dt^2}(t) + i \frac{d^2y}{dt^2}(t) = -\frac{\partial V_\sigma}{\partial x}(\alpha(t)) - i \frac{\partial V_\sigma}{\partial y}(\alpha(t)), \quad t \in I.$$

Received by the editors May 28, 2004 and, in revised form, September 30, 2004.

2000 *Mathematics Subject Classification*. Primary 31A15; Secondary 31A05, 70F99, 70K99.

Key words and phrases. Logarithmic potential, inverse problem, reflection principle, harmonic function, Poisson integral.

Physically this means that a unit point mass placed on a point $\alpha(t_o)$ of α with velocity $\alpha'(t_o)$ traces the orbit α under the attraction of a mass that lies on the cylinder $K \times \mathbb{R}$, and its distribution on every plane parallel to the xy -plane is σ . More details about the physical interpretation of logarithmic potentials can be found in [4, Chapter III]. We say that the potential V_σ *generates the curve* $\mathbf{f}(x, y) = q$ if it generates an orbit lying on this curve.

The problem we wish to study is the following:

Problem 1. Let K be a compact set in \mathbb{C} and let $\mathcal{F} = \{\mathbf{f}(x, y) = p : p \in \mathbb{R}\}$ be a one-parameter family of curves, all lying in $\mathbb{C} \setminus K$. Find all measures σ in K whose logarithmic potentials generate each of the curves of \mathcal{F} .

A related problem, the *inverse problem of dynamics*, has been studied recently. We refer the reader to the paper [2] by G. Bozis for a survey on this problem and its history. A special case of this problem is the following: Suppose that $\mathcal{F} = \{\mathbf{f}(x, y) = p : p \in \mathbb{R}\}$ is a one-parameter family of curves in the plane. Find all C^1 functions V (not necessarily logarithmic potentials) that generate each of the curves of \mathcal{F} . Bozis (see [1], [2] and the references therein) proved that if a C^2 function V is a solution of the above problem, then it satisfies the second-order partial differential equation

$$(1.3) \quad \frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} - \kappa \frac{\partial^2 V}{\partial x \partial y} + \lambda \frac{\partial V}{\partial x} + \mu \frac{\partial V}{\partial y} = 0,$$

where

$$(1.4) \quad \kappa = \frac{1 - \gamma^2}{\gamma}, \quad \lambda = \frac{1}{\gamma \Gamma} \left(\frac{\partial \Gamma}{\partial y} - \gamma \frac{\partial \Gamma}{\partial x} \right), \quad \mu = \lambda \gamma + \frac{3\Gamma}{\gamma},$$

and the functions $\gamma(x, y)$ and $\Gamma(x, y)$ are given by

$$(1.5) \quad \gamma = \frac{\partial \mathbf{f}}{\partial y} \left(\frac{\partial \mathbf{f}}{\partial x} \right)^{-1} \quad \text{and} \quad \Gamma = \gamma \frac{\partial \gamma}{\partial x} - \frac{\partial \gamma}{\partial y}.$$

In fact (1.3) holds when the given family is not a family of straight lines. If the given family consists of straight lines (i.e. when $\Gamma(x, y) = 0$), then every function V that generates them must satisfy the first-order partial differential equation

$$(1.6) \quad \frac{\partial V}{\partial x} + \gamma \frac{\partial V}{\partial y} = 0.$$

Returning to Problem 1, we see that if the measure σ is one of its solutions, then its potential must satisfy (1.3) or (1.6). Using (1.1) and differentiation under the integral sign (provided, of course, that such an operation can be justified), we arrive at an integral equation. This equation must be satisfied by every measure σ supported on K and generating each of the curves of the family \mathcal{F} .

In this paper we study Problem 1 under the additional assumption that σ is supported in the unit circle \mathbb{T} , that is, $K = \mathbb{T}$. Under this assumption, we solve the problem in three cases:

Case (a): The given family consists of linear segments on lines passing through the origin. This case is studied in Section 4. We find that the measure σ must be uniformly distributed on \mathbb{T} .

Case (b): The given family consists of linear segments on lines passing through the point $1 \in \mathbb{T}$. This case is studied in Section 5. We find that if the observed segments are inside \mathbb{T} , then σ must be the sum of a uniform distribution on \mathbb{T} and

a Dirac measure on 1, while if the observed segments are outside \mathbb{T} , then σ must be a Dirac measure on 1.

Case (c): The given family consists of circular arcs lying on circles centered at the origin. This case is studied in Section 6. We find that σ must be uniformly distributed on \mathbb{T} .

The proof of the above results involves various properties of harmonic functions. A reflection principle (Section 2) is used to prove in Section 3 that if a logarithmic potential generates a linear segment, then it must be locally symmetric with respect to this segment. We also use the Poisson integral and its boundary behavior.

2. A REFLECTION PRINCIPLE FOR HARMONIC FUNCTIONS

In this section we prove a form of the reflection principle for harmonic functions (Theorem 2 below). This result is certainly known, but we prove it for the sake of completeness. We will need some new pieces of notation. If $z = x + iy \in \mathbb{C}$, then by \hat{z} we denote the reflection of z in the real axis: $\hat{z} = x - iy$; if $A \subset \mathbb{C}$, then $\hat{A} = \{\hat{z} : z \in A\}$. Below we will use some results related to the Dirichlet problem; for these results we refer to [6].

Theorem 1. *Let G be a bounded plane domain, regular for the Dirichlet problem and symmetric with respect to the real axis. Let f be a bounded, continuous, real function on ∂G with $f(\zeta) = f(\hat{\zeta})$ for all $\zeta \in \partial G$. Let u be the harmonic function in G with $u = f$ on ∂G . Then*

$$(2.1) \quad u(z) = u(\hat{z}), \quad \forall z \in G,$$

and

$$(2.2) \quad \frac{\partial u}{\partial y}(x) = 0, \quad \forall x \in G \cap \mathbb{R}.$$

Proof. Let $v(z) = (u(z) + u(\hat{z}))/2$. Then $v(z) = v(\hat{z})$, $z \in G$, and

$$\lim_{z \rightarrow \zeta} v(z) = f(\zeta), \quad \forall \zeta \in \partial G.$$

Therefore $u = v$ and (2.1) is proved. Also, because of (2.1),

$$\frac{\partial u}{\partial y}(x) = \lim_{h \rightarrow 0} \frac{u(x + ih) - u(x - ih)}{2h} = 0, \quad \forall x \in G \cap \mathbb{R},$$

and (2.2) is proved. □

Theorem 2. *Let D be a bounded plane domain, regular for the Dirichlet problem. Assume that D lies in the upper half-plane and that the boundary of D contains an open subset τ of \mathbb{R} . Let u be a function harmonic in D and continuous on the closure of D . If $\frac{\partial u}{\partial y}(x) = 0$ for all $x \in \tau$, then the function*

$$u_1(z) = \begin{cases} u(z), & z \in D \cup \tau, \\ u(\hat{z}), & z \in \hat{D}, \end{cases}$$

is harmonic in $D \cup \tau \cup \hat{D}$.

Proof. Let v be the harmonic function in $G := D \cup \tau \cup \hat{D}$ with boundary values

$$v(z) = \begin{cases} u(\zeta), & \zeta \in (\partial G \cap \{z : \text{Im}z \geq 0\}), \\ u(\hat{\zeta}), & \zeta \in \partial G \cap \{z : \text{Im}z < 0\}. \end{cases}$$

Then, by Theorem 1, $\frac{\partial v}{\partial y} = 0$ on τ . Therefore the function $h = u - v$ is the solution of the Dirichlet-Neumann problem in D with boundary conditions $h = 0$ on $\partial D \setminus \tau$, $\frac{\partial h}{\partial y} = 0$ on τ . By the uniqueness of the solution (see [5, Chapter 2, Theorem 9]), we have $v(z) = u(z)$, $\forall z \in D$. By the same argument, $v(z) = u(\hat{z})$, $\forall z \in \hat{D}$. Both v and u_1 are continuous in G . Hence $v = u_1$ in G and the theorem is proved. \square

3. POTENTIALS GENERATING A STRAIGHT LINE

In this section we prove the following theorem.

Theorem 3. *Let σ be a finite Borel measure with compact support $K \subset \mathbb{C}$. Suppose that the logarithmic potential V_σ of σ generates an orbit α that lies in a straight line c . Then V_σ is locally symmetric with respect to c in the sense that $V_\sigma(z) = V_\sigma(\hat{z})$ for all z in a neighborhood of the trace of α (\hat{z} is the reflection of z in c).*

Proof. Without loss of generality we assume that $c = \mathbb{R}$. Then the orbit α has a formula of the form $\alpha(t) = x(t) \in \mathbb{R}$, $t \in [a, b]$. We fix $t_o \in [a, b]$ and set $x_o = x(t_o)$. Let G be a disk of center x_o and radius small enough so that $G \cap \mathbb{R} \subset \{x(t) : t \in [a, b]\}$ and $G \subset \mathbb{C} \setminus K$. Since V_σ generates the orbit α , we have

$$\frac{\partial V_\sigma}{\partial y}(x(t)) = 0, \quad t \in [a, b].$$

Hence $\frac{\partial V_\sigma}{\partial y}(x) = 0$, $\forall x \in \mathbb{R} \cap G$. We consider the function u defined as follows:

$$u(z) = \begin{cases} V_\sigma(z), & z \in G \cap \{z : \text{Im}z \geq 0\}, \\ V_\sigma(\hat{z}), & z \in G \cap \{z : \text{Im}z < 0\}. \end{cases}$$

By Theorem 2, u is harmonic in G . Since u coincides with V_σ in an open subset of G , we have $u = V_\sigma$ in G (by the identity principle for harmonic functions; see [6]). Hence $V_\sigma(z) = V_\sigma(\hat{z})$ for all $z \in G$. By a compactness argument, this equality holds for all z in a neighborhood of the trace of α . \square

4. STRAIGHT LINES PASSING THROUGH THE ORIGIN

From now on we consider a finite Borel measure σ whose support is a subset of the unit circle \mathbb{T} , and we suppose that it generates each curve of a one-parameter family of curves. Our aim will be to determine the measure σ .

In this section the observed orbits lie on straight lines through the origin. More precisely, let D be an open disk in $\mathbb{C} \setminus \mathbb{T}$, let $\{c_p : p \in J\}$ be the family of all straight lines that pass through the origin and intersect D , and let $s_p = c_p \cap D$ (the parameter p takes values in a suitable open interval J). If a measure on \mathbb{T} generates the linear segments s_p , then it must be uniformly distributed on \mathbb{T} :

Theorem 4. *Suppose that a finite Borel measure σ supported in \mathbb{T} generates each linear segment s_p in the disk D . Then for every Borel set $A \subset \mathbb{T}$, we have $\sigma(A) = C \cdot \Lambda(A)$, where $\Lambda(A)$ is the Lebesgue measure (length) of A and C is a constant.*

Proof. By Theorem 3, V_σ is locally symmetric with respect to each line c_p . This implies that V_σ is radial in D . Since V_σ is also harmonic in $\mathbb{C} \setminus \mathbb{T}$, we have

$$(4.1) \quad V_\sigma(z) = a \log \frac{1}{|z|} + b, \quad \forall z \in D,$$

where a, b are real constants. Here we distinguish two cases.

Case 1: $D \subset \mathbb{C} \setminus \mathbb{D}$.

By the identity principle for harmonic functions, (4.1) holds for all $z \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$. We write $z = re^{i\theta} \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$, $\zeta = e^{i\varphi} \in \mathbb{T}$ and we compute the derivative $\frac{\partial V_\sigma}{\partial \theta}$:

$$(4.2) \quad \begin{aligned} \frac{\partial V_\sigma}{\partial \theta}(re^{i\theta}) &= \frac{\partial}{\partial \theta} \int_0^{2\pi} \log \frac{1}{|re^{i\theta} - e^{i\varphi}|} d\sigma(\varphi) \\ &= \int_0^{2\pi} \frac{\partial}{\partial \theta} \log \frac{1}{|re^{i\theta} - e^{i\varphi}|} d\sigma(\varphi). \end{aligned}$$

The differentiation under the integral sign is justified by a standard application of Lebesgue’s dominated convergence theorem; see e.g. [3, Theorem (2.27)]. A simple computation gives

$$(4.3) \quad \frac{\partial}{\partial \theta} \log \frac{1}{|re^{i\theta} - e^{i\varphi}|} = \frac{r \sin(\varphi - \theta)}{1 + r^2 - 2r \cos(\varphi - \theta)}.$$

The equations (4.2) and (4.3) yield

$$(4.4) \quad \begin{aligned} \frac{\partial V_\sigma}{\partial \theta}(re^{i\theta}) &= \int_0^{2\pi} \frac{r \sin(\varphi - \theta)}{1 + r^2 - 2r \cos(\varphi - \theta)} d\sigma(\varphi) \\ &= \int_0^{2\pi} \frac{\frac{1}{r} \sin(\varphi - \theta)}{1 + \frac{1}{r^2} - 2\frac{1}{r} \cos(\varphi - \theta)} d\sigma(\varphi). \end{aligned}$$

The function

$$Q(\rho e^{i\theta}) := \frac{2\rho \sin \theta}{1 + \rho^2 - 2\rho \cos \theta}, \quad 0 \leq \rho < 1, \theta \in [0, 2\pi],$$

is the *conjugate Poisson kernel*. Therefore the function $\frac{\partial V_\sigma}{\partial \theta}$ is harmonic conjugate to the function

$$u(re^{i\theta}) := \pi P_{1/r}[\sigma](\theta), \quad r > 1, \theta \in [0, 2\pi],$$

where

$$P_{1/r}[\sigma](\theta) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - \frac{1}{r^2}}{1 + \frac{1}{r^2} - 2\frac{1}{r} \cos(\varphi - \theta)} d\sigma(\varphi)$$

is the Poisson integral of the measure σ .

It follows from (4.1) that $\frac{\partial V_\sigma}{\partial \theta} = 0$ in $\mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$. Hence $P_{1/r}[\sigma](\theta)$ is a real constant C for $e^{i\theta}/r \in \mathbb{D}$. By the theory of the boundary behavior of Poisson integrals (see [7, Theorem 11.30]), we infer that $\sigma = C \cdot \Lambda$ on \mathbb{T} .

Case 2: $D \subset \mathbb{D}$.

By the identity principle for harmonic functions, (4.1) holds for all $z \in \mathbb{D} \setminus \{0\}$. But V_σ is harmonic at the origin; hence $a = 0$ and we conclude that $V_\sigma = b$, a constant, in \mathbb{D} . Now, as in Case 1, we use the equality, $\frac{\partial V_\sigma}{\partial \theta} = 0$, the Poisson integral and its conjugate, and conclude that σ is a constant multiple of the Lebesgue measure on \mathbb{T} . □

Remark 1. Another proof of the theorem above can be found by using the integral equation mentioned in the first section. The integral equation leads, after some calculations, to the equality

$$\int_0^{2\pi} \frac{r \sin(\varphi - \theta)}{1 + r^2 - 2r \cos(\varphi - \theta)} d\sigma(\varphi) = 0,$$

and from now on this new proof is the same as the proof above.

Remark 2. It is interesting that σ is determined by the observation of the line segments it generates in an *arbitrarily* small disk. Also this disk can be arbitrarily far away from the support of the measure σ .

Remark 3. We found that the logarithmic potential which generates the line segments s_p is constant inside \mathbb{T} and radial outside \mathbb{T} . We observe now that the *orbits* on s_p cannot be arbitrary. For example there is no potential (of a measure supported on \mathbb{T}) that can generate orbits that lie on s_p inside \mathbb{T} , and the velocity is not constant for each one of them.

Remark 4. From the proof of Theorem 4 we infer that if the logarithmic potential of a measure supported in \mathbb{T} is constant in \mathbb{D} , then the measure is necessarily a constant multiple of the equilibrium measure of \mathbb{T} .

5. STRAIGHT LINES PASSING THROUGH THE POINT 1

In this section we consider a finite Borel measure σ supported in the unit circle \mathbb{T} . Let D be an open disk in $\mathbb{C} \setminus \mathbb{T}$, let $\{s_p : p \in J\}$ be the family of all straight lines that pass through the point 1 and intersect D , and let $c_p = s_p \cap D$ (the parameter p takes values in a suitable open interval J).

Theorem 5. *Suppose that a finite Borel measure σ supported in \mathbb{T} generates each linear segment c_p in the disk D . If $D \subset \mathbb{D}$, then $\sigma = C_1 \cdot \Lambda + C_2 \cdot \delta_1$, where C_1, C_2 are constants and δ_1 is the Dirac measure on 1. If $D \subset \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$, then $\sigma = C_3 \cdot \delta_1$, where C_3 is a constant.*

Proof. Suppose first that $D \subset \mathbb{D}$. By Theorem 3, V_σ is locally symmetric with respect to each line s_p . This implies that $V_\sigma(z)$ in D depends only on the distance between z and 1. Since V_σ is also harmonic in \mathbb{D} , we have

$$(5.1) \quad V_\sigma(z) = C_2 \log \frac{1}{|1-z|} + b, \quad \forall z \in D,$$

where C_2, b are real constants. By the identity principle, (5.1) holds for all $z \in \mathbb{D}$.

Now consider the signed measure $\tau = \sigma - C_2 \delta_1$. For $z \in \mathbb{D}$, we have

$$(5.2) \quad V_\tau(z) = V_\sigma(z) - C_2 V_{\delta_1}(z) = b.$$

By the same arguments as in the proof of Theorem 4 (involving the Poisson integral), we conclude that τ has the form $C_1 \cdot \Lambda$. Hence $\sigma = C_1 \cdot \Lambda + C_2 \cdot \delta_1$.

Next suppose that $D \subset \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$. Then, as above, we find that

$$(5.3) \quad V_\sigma(z) = C_3 \log \frac{1}{|1-z|} + C_4, \quad \forall z \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T}),$$

where C_3, C_4 are real constants. Consider the signed measure $\mu = \sigma - C_3 \delta_1$. For $z \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$, we have

$$(5.4) \quad V_\mu(z) = V_\sigma(z) - C_3 V_{\delta_1}(z) = C_4.$$

Again by the argument involving the Poisson integral, we conclude that μ has the form $C_5 \cdot \Lambda$, where C_5 is a constant. Hence $\sigma = C_5 \cdot \Lambda + C_3 \cdot \delta_1$. Therefore

$$(5.5) \quad V_\sigma(z) = 2\pi C_5 \log \frac{1}{|z|} + C_3 \log \frac{1}{|1-z|} + C_6, \quad z \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T}).$$

Comparing (5.5) with (5.3) we conclude that $C_5 = 0$. Hence $\sigma = C_3 \cdot \delta_1$. \square

6. CIRCLES CENTERED AT THE ORIGIN

In this section we again consider a finite Borel measure σ supported on the unit circle \mathbb{T} . Also let $\{s_p\}$ be the one-parameter family of circles centered at the origin. Let D be a disk in $\mathbb{C} \setminus \mathbb{T}$ and let $c_p = s_p \cap D$ (the parameter p takes values in a suitable open interval J). If σ generates the circular arcs c_p , then σ is necessarily a constant multiple of the Lebesgue measure on \mathbb{T} :

Theorem 6. *Suppose that a finite Borel measure σ supported on \mathbb{T} generates each circular arc c_p in the disk D . Then $\sigma = C \cdot \Lambda$, where C is a constant.*

Proof. We will assume that $D \subset \mathbb{D}$; the case $D \subset \mathbb{C} \setminus (\mathbb{T} \cup \mathbb{D})$ can be treated in an analogous way. We also assume without loss of generality that $\overline{D} \cap \mathbb{R} = \emptyset$ and $\overline{D} \cap \mathbb{T} = \emptyset$; otherwise we consider a smaller disk D .

The equation of the circles s_p is $\mathbf{f}(x, y) := x^2 + y^2 = p; p \in J$. For this \mathbf{f} , we compute the functions $\gamma, \Gamma, \kappa, \lambda, \mu$ using (1.5) and (1.4), and we find

$$(6.1) \quad \gamma = \frac{y}{x}, \Gamma = -\frac{y^2}{x^3} - \frac{1}{x},$$

$$(6.2) \quad \kappa = \frac{x^2 - y^2}{xy}, \lambda = \frac{3}{x}, \mu = -\frac{3}{y}, \quad (x, y) \in D.$$

So, Bozis' equation (1.3) becomes

$$(6.3) \quad \frac{\partial^2 V_\sigma}{\partial x^2} - \frac{\partial^2 V_\sigma}{\partial y^2} - \frac{x^2 - y^2}{xy} \frac{\partial^2 V_\sigma}{\partial x \partial y} + \frac{3}{x} \frac{\partial V_\sigma}{\partial x} - \frac{3}{y} \frac{\partial V_\sigma}{\partial y} = 0, \quad (x, y) \in D.$$

In this equation we substitute V_σ from (1.1), differentiate under the integral sign (an operation justified by [3, Theorem (2.27)]), write $x + iy = re^{i\theta} \in D, \zeta = e^{i\varphi} \in \mathbb{T}$, identify σ with a measure on $[0, 2\pi]$, and (6.3), after some calculations, becomes

$$(6.4) \quad \int_0^{2\pi} \frac{[r^2 + 3 - 4r \cos(\theta - \varphi)] \sin(\theta - \varphi)}{r \sin 2\theta [r^2 + 1 - 2r \cos(\theta - \varphi)]^2} d\sigma(\varphi) = 0, \quad re^{i\theta} \in D.$$

We set

$$B(r, \theta, \varphi) := \frac{r \sin(\theta - \varphi)}{r^2 + 1 - 2r \cos(\theta - \varphi)}$$

and

$$A(r, \theta) = \int_0^{2\pi} B(r, \theta, \varphi) d\sigma(\varphi).$$

After some calculations (6.4) becomes

$$(6.5) \quad \frac{1}{r^2 \sin 2\theta} \int_0^{2\pi} \left[2B(r, \theta, \varphi) + r \frac{\partial B}{\partial r}(r, \theta, \varphi) \right] d\sigma(\varphi) = 0, \quad re^{i\theta} \in D.$$

Performing a differentiation under the integral sign (justified again by [3, Theorem (2.27)]), we see that (6.5) is equivalent to

$$(6.6) \quad 2A(r, \theta) + r \frac{\partial A}{\partial r}(r, \theta) = 0, \quad re^{i\theta} \in D.$$

For each fixed θ , (6.6) is an ordinary linear differential equation in the variable r which has the solutions

$$(6.7) \quad A(r, \theta) = \frac{K(\theta)}{r^2},$$

where $K(\theta)$ is an arbitrary function of θ . But A is harmonic function of $re^{i\theta}$ in D (it is the conjugate Poisson integral of σ). Using the Laplacian in polar coordinates we find that

$$(6.8) \quad A(r, \theta) = \frac{c_1 \cos \theta + c_2 \sin \theta}{r^2}, \quad re^{i\theta} \in D,$$

where c_1, c_2 are constants. By the identity principle this equality holds for all $re^{i\theta} \in \mathbb{D} \setminus \{0\}$. Since A is also harmonic at 0, we have $c_1 = c_2 = 0$ and therefore $A = 0$ in \mathbb{D} . We continue as in the proof of Theorem 4: The Poisson integral of σ must be equal to a real constant and therefore σ must be a constant multiple of the Lebesgue measure on \mathbb{T} . \square

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