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FUNCTIONAL EQUATIONS INVOLVING MEANS AND THEIR GAUSS COMPOSITION

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ABSTRACT. In this paper the equivalence of the two functional equations

$$f(M_1(x,y)) + f(M_2(x,y)) = f(x) + f(y) \qquad (x, y \in I)$$

and

$$2f(M_1 \otimes M_2(x,y)) = f(x) + f(y) \qquad (x, y \in I)$$

is studied, where M_1 and M_2 are two variable strict means on an open real interval I, and $M_1 \otimes M_2$ denotes their Gauss composition. The equivalence of these equations is shown (without assuming further regularity assumptions on the unknown function $f : I \to \mathbb{R}$) for the cases when M_1 and M_2 are the arithmetic and geometric means, respectively, and also in the case when M_1, M_2, and $M_1 \otimes M_2$ are quasi-arithmetic means. If M_1 and M_2 are weighted arithmetic means, then, depending on the algebraic character of the weight, the above equations can be equivalent and also non-equivalent to each other.

1. Introduction

Let $I \subset \mathbb{R}$ be a non-void open interval. A function $M : I^2 \to I$ is called a *strict mean* on I if it is continuous on I^2 and, for all $x, y \in I$ with $x \neq y$,

$$\min\{x, y\} < M(x, y) < \max\{x, y\}.$$

It is obvious that $M(x, x) = x$ for all $x \in I$. Let $M_i : I^2 \to I$ $(i = 1, 2)$ be strict means on I and $x, y \in I$. Consider the sequences (x_n) and (y_n) defined by the Gauss iteration in the following way:

$$\begin{aligned} x_1 &:= x, & y_1 &:= y, \\ x_{n+1} &:= M_1(x_n, y_n), & y_{n+1} &:= M_2(x_n, y_n) \qquad (n \in \mathbb{N}). \end{aligned}$$

It is known (see [2], [6]) that the sequence of the intervals

$$I_n(x,y) = [\min\{x_n, y_n\}, \max\{x_n, y_n\}] \qquad (n \in \mathbb{N})$$

is decreasing, and the intersection of these intervals is a singleton whose unique element is denoted by $M_1 \otimes M_2(x, y)$. Then the function $M_1 \otimes M_2 : I^2 \to I$ so defined is a strict mean on I, and the *invariance equation*

$$M_1 \otimes M_2(M_1(x,y), M_2(x,y)) = M_1 \otimes M_2(x,y) \tag{1.1}$$

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holds for all $x, y \in I$ ([2], [6], [13], [15]). The strict mean $M_1 \otimes M_2$ constructed in this way is called the *Gauss composition* or the *compound mean* of the strict means M_1 and M_2.

In this paper we discuss the following problem. Let M_1 and M_2 be two strict means on I and let $f : I \to \mathbb{R}$ be a function such that functional equation

$$f(M_1(x,y)) + f(M_2(x,y)) = f(x) + f(y) \tag{1.2}$$

holds for all $x, y \in I$. Our main purpose is to find further conditions on the means M_1 and M_2 under which the equation

$$2f(M_1 \otimes M_2(x,y)) = f(x) + f(y) \tag{1.3}$$

is equivalent to (1.2). We emphasize that *we do not assume f to have any further (regularity) properties*. The invariance equation immediately implies that if $f : I \to \mathbb{R}$ satisfies (1.3) for all $x, y \in I$, then f is also a solution of (1.2). Indeed, due to (1.1), the repeated application of (1.3) yields

$$\begin{aligned} f(x) + f(y) &= 2f(M_1 \otimes M_2(x,y)) \\ &= 2f(M_1 \otimes M_2(M_1(x,y)), M_2(x,y)) = f(M_1(x,y)) + f(M_2(x,y)). \end{aligned}$$

On the other hand, if f is a continuous solution of (1.2), then the repeated application of (1.2) and the definition of the Gauss iteration yields that f satisfies (1.3) as well. Therefore the basic problem is to find conditions on the means involved so that arbitrary (not necessarily continuous) solutions of (1.2) also satisfy (1.3).

In Section 2 we consider the case when the two means are the arithmetic and geometric means. It will turn out that f solves (1.2) and (1.3) if and only if it is constant, thus the two equations are equivalent in this case. The case when M_1, M_2, and $M_1 \otimes M_2$ are quasi-arithmetic means is considered in Section 3. In this setting we also obtain an affirmative answer to our problem. In the last section we investigate the case when the two means are weighted arithmetic means. Depending on the algebraic character of the weight as a parameter, the two equations can be equivalent and also non-equivalent to each other.

2. The arithmetic-geometric mean

The arithmetic-geometric mean is the Gauss composition of the arithmetic and geometric means defined by

$$\mathcal{A}(x,y) := \frac{x+y}{2} \qquad \text{and} \qquad \mathcal{G}(x,y) := \sqrt{xy} \qquad (x, y \in \mathbb{R}_+)$$

on $I = \mathbb{R}_+ :=]0, +\infty[$. It is known that

$$\mathcal{A} \otimes \mathcal{G}(x,y) = \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{x^2 \cos^2 t + y^2 \sin^2 t}} \right)^{-1} \qquad (x, y \in \mathbb{R}_+)$$

(cf. [2], [8], [1], [3]). In this case the following theorem holds.

Theorem 2.1. *A function $f : \mathbb{R}_+ \to \mathbb{R}$ is a solution of the functional equation*

$$f(\mathcal{A}(x,y)) + f(\mathcal{G}(x,y)) = f(x) + f(y) \qquad (x, y \in \mathbb{R}_+) \tag{2.1}$$

if and only if f is constant on $\mathbb{R}_+$.

Proof. We apply the following more general result due to Maksa [12]. *If* $(a, b, c) : \mathbb{R}_+ \to \mathbb{R}^3$ *is a solution of the functional equation*

$$a(x+y) + b(xy) = c(x) + c(y) \qquad (x, y \in \mathbb{R}_+), \tag{2.2}$$

then there exist additive functions $A, B : \mathbb{R}_+ \to \mathbb{R}$ *and a logarithmic function* $L : \mathbb{R}_+ \to \mathbb{R}$ *(that is,* $A(x+y) = A(x) + A(y)$, $B(x+y) = B(x) + B(y)$, *and* $L(xy) = L(x) + L(y)$ *for all* $x, y \in \mathbb{R}_+$*) and constants* $k_1, k_2 \in \mathbb{R}$ *such that*

$$\begin{aligned} a(x) &= A(x^2) + B(x) + k_1, \\ b(x) &= L(x) - 2A(x) + k_2, \\ c(x) &= A(x^2) + B(x) + L(x) + (k_1 + k_2)/2 \end{aligned} \tag{2.3}$$

for all $x \in \mathbb{R}_+$.

It is obvious that, if $f : \mathbb{R}_+ \to \mathbb{R}$ is a solution of (2.1), then, with the definitions

$$a(x) := f\left(\frac{x}{2}\right), \qquad b(x) := f(\sqrt{x}), \qquad c(x) := f(x) \qquad (x \in \mathbb{R}_+),$$

the triplet (a, b, c) is a solution of equation (2.2). Therefore it follows from (2.3) that

$$f\left(\frac{x}{2}\right) = A(x^2) + B(x) + k_1, \tag{2.4}$$

$$f(\sqrt{x}) = L(x) - 2A(x) + k_2, \tag{2.5}$$

$$f(x) = A(x^2) + B(x) + L(x) + (k_1 + k_2)/2, \tag{2.6}$$

for all $x \in \mathbb{R}_+$, where $A, B : \mathbb{R}_+ \to \mathbb{R}$ are additive and $L : \mathbb{R}_+ \to \mathbb{R}$ is a logarithmic function, and k_1, k_2 are real constants. We will prove that our statement follows from (2.4) and (2.6) (i.e., it is not necessary to take (2.5) into consideration). Indeed, (2.4) and (2.6) imply, for all $x \in \mathbb{R}_+$, that

$$4A(x^2) + 2B(x) + k_1 = A(x^2) + B(x) + L(x) + (k_1 + k_2)/2,$$

that is,

$$3A(x^2) + B(x) + (k_2 - k_1)/2 = L(x). \tag{2.7}$$

Replacing x by $2x$ in (2.7), we get that

$$12A(x^2) + 2B(x) + (k_1 - k_2)/2 = L(x) + L(2). \tag{2.8}$$

Subtracting (2.7) from (2.8), we deduce the equation

$$9A(x^2) + B(x) = L(2). \tag{2.9}$$

Again replacing x by $2x$ and eliminating $B(x)$ from (2.9) and the equation obtained, we get that

$$18A(x^2) = -L(2).$$

Hence $L(2) = 0$ and $A(x) = 0$ for all $x \in \mathbb{R}_+$. Thus also $B \equiv 0$. Therefore, by (2.4), we obtain that $f(x) = k_1$ for all $x \in \mathbb{R}_+$, that is, f is constant. □

We immediately get the following corollary.

Corollary 2.2. *If* $f : \mathbb{R}_+ \to \mathbb{R}$ *is a solution of the functional equation*

$$2f(\mathcal{A} \otimes \mathcal{G}(x, y)) = f(x) + f(y) \qquad (x, y \in \mathbb{R}_+), \tag{2.10}$$

then f *is a constant function.*

Proof. If f is a solution of (2.10), then it is also a solution for (2.1), thus by virtue of Theorem 2.1, f is constant which is obviously a solution for (2.10). □

The statement of the above corollary can also be expressed in the following way.

Corollary 2.3. *If M_1 is the arithmetic mean and M_2 is the geometric mean on $\mathbb{R}_+$, then the functional equations* (1.2) *and* (1.3) *are equivalent.*

3. Quasi-arithmetic means

Let $I \in \mathbb{R}$ be a non-void open interval. Denote the class of all *continuous and strictly monotonic* functions defined on I by $\mathcal{CM}(I)$. A function $M : I^2 \to I$ is called a *quasi-arithmetic mean* if there exists $\varphi \in \mathcal{CM}(I)$ such that

$$M(x,y) = \varphi^{-1}\Big(\frac{\varphi(x)+\varphi(y)}{2}\Big) =: \mathcal{A}_\varphi(x,y) \tag{3.1}$$

for all $x, y \in I$. The function φ in (3.1) is called the *generating function* of the quasi-arithmetic mean M ([9], [14], [6]). It is obvious that quasi-arithmetic means are strict means on I.

The arithmetic, geometric, and harmonic means are obviously quasi-arithmetic means with the generators $\varphi(x) = x$, $\varphi(x) = \ln(x)$, and $\varphi(x) = 1/x$, respectively.

If $M_1 = \mathcal{A}$ and $M_2 = \mathcal{H}$ (the harmonic mean), then it is immediate to see that their Gauss composition is the geometric mean. In this particular case, the equivalence of the functional equations (1.2) and (1.3) was conjectured by Daróczy [5] and proved by Ebanks [7]. This result is recalled in the following lemma and will be used in the proof of Theorem 3.2 below.

Lemma 3.1. *Let J be an open subinterval of $\mathbb{R}_+$. A function $g : J \to \mathbb{R}$ is a solution of the functional equation*

$$g(\mathcal{A}(x,y)) + g(\mathcal{H}(x,y)) = g(x) + g(y) \qquad (x, y \in J) \tag{3.2}$$

if and only if it is a solution of

$$2g(\mathcal{G}(x,y)) = g(x) + g(y) \qquad (x, y \in J). \tag{3.3}$$

Now we can state the main result of this section.

Theorem 3.2. *If $M_i : I^2 \to I$ $(i = 1, 2, 3)$ are quasi-arithmetic means satisfying*

$$M_3 = M_1 \otimes M_2, \tag{3.4}$$

then the functional equations (1.2) *and* (1.3) *are equivalent.*

Proof. The quasi-arithmetic means satisfying identity (3.4) are derived from the solution of the so-called Matkowski-Sutô problem ([6]). Then there exist a function $\varphi \in \mathcal{CM}(I)$ and a constant $p \in \mathbb{R}$ such that, with the notation

$$\chi_p(x) := \begin{cases} x & \text{if } p = 0, \\ e^{px} & \text{if } p \neq 0 \end{cases} \qquad (x \in \mathbb{R}),$$

(3.5)

$$M_1(x,y) = \mathcal{A}_{\chi_p\circ\varphi}(x,y), \qquad M_2(x,y) = \mathcal{A}_{\chi_{-p}\circ\varphi}(x,y), \qquad M_3(x,y) = \mathcal{A}_{\varphi}(x,y)$$

hold for all $x, y \in I$ (cf. [6, Theorem 4.14]). By substituting the means M_1 and M_2 into equation (1.2), we have that

$$f(\mathcal{A}_{\chi_p\circ\varphi}(x,y)) + f(\mathcal{A}_{\chi_{-p}\circ\varphi}(x,y)) = f(x) + f(y) \tag{3.6}$$

for all $x, y \in I$. If $p = 0$, then (3.6) implies that

$$2f(\mathcal{A}_\varphi(x,y)) = f(x) + f(y) \qquad (x, y \in I).$$

Thus, we obtain (1.2). In the case $p \neq 0$, (3.6) is equivalent to

(3.7)

$$f \circ \varphi^{-1}\left(\frac{1}{p}\log\frac{e^{p\varphi(x)} + e^{p\varphi(y)}}{2}\right) + f \circ \varphi^{-1}\left(-\frac{1}{p}\log\frac{e^{-p\varphi(x)} + e^{-p\varphi(y)}}{2}\right) = f(x) + f(y)$$

for all $x, y \in I$. Now define

$$u := e^{p\varphi(x)}, \qquad v := e^{p\varphi(y)},$$

where $u, v \in e^{p\varphi(I)} =: J \subset \mathbb{R}_+$, and J is a non-void open interval. Furthermore, define

$$g(t) := f \circ \varphi^{-1}\Big(\frac{1}{p}\log t\Big) \qquad (t \in J).$$

Then it follows from (3.7) that, for every $u, v \in J$,

$$g\Big(\frac{u+v}{2}\Big) + g\Big(\frac{2uv}{u+v}\Big) = g(u) + g(v). \tag{3.8}$$

Applying Lemma 3.1, (3.8) implies that

$$2g(\sqrt{uv}) = g(u) + g(v) \qquad (u, v \in J).$$

Therefore,

$$2f \circ \varphi^{-1}\Big(\frac{1}{p}\log\sqrt{e^{p\varphi(x)}e^{p\varphi(y)}}\Big) = f(x) + f(y) \qquad (x, y \in I).$$

Thus, we obtain, for all $x, y \in I$, that

$$2f(\mathcal{A}_\varphi(x,y)) = 2f \circ \varphi^{-1}\Big(\frac{\varphi(x) + \varphi(y)}{2}\Big) = f(x) + f(y).$$

Therefore, by the last equation in (3.5), the functional equation (1.3) holds. □

Remark 1. With the notation $h := f \circ \varphi^{-1}$ and $K := \varphi(I)$, equation (1.3) implies that h satisfies the Jensen functional equation

$$2h\Big(\frac{t+s}{2}\Big) = h(t) + h(s) \qquad (t, s \in K)$$

on the non-void interval $K \subset \mathbb{R}$. This equation has continuous and also discontinuous (particularly non-constant) solutions (cf. [10], [14]). Thus the functional equations (1.2) and (1.3) have a much richer joint solution set in this setting.

4. Weighted arithmetic means

In the following we shall prove that negative answers can also be given to our problem. That is, (1.3) does not follow from equation (1.2), in general. As a basic result we need the following theorem due to Lajkó [11]. (For the terminology, see Székelyhidi [16]).

Theorem 4.1. *Let $0 < p < 1$. If $f : I \to \mathbb{R}$ satisfies the functional equation*

$$f(px + (1-p)y) + f((1-p)x + py) = f(x) + f(y) \qquad (x, y \in I), \tag{4.1}$$

then there exist k-additive, symmetric functions $A_k : \mathbb{R}^k \to \mathbb{R}$ $(k = 0, 1, 2)$ such that

$$A_2(px, (1-p)x) = 0 \qquad (x \in \mathbb{R}) \tag{4.2}$$

and

$$f(x) = A_2(x,x) + A_1(x) + A_0 \tag{4.3}$$

for all $x \in I$.

Lajkó, in his paper [11] mentioned above, studied neither the problem of finding a parameter p for which there exists such a symmetric and biadditive function $A_2 : \mathbb{R}^2 \to \mathbb{R}$ that satisfies (4.2) nor the conditions under which the term $A_2(x,x)$ in (4.3) is not identically zero. It is obvious that if $p \in \mathbb{Q}$ in (4.2), then $A_2(x,x) = 0$ for all $x \in I$. It is also clear that, for $0 < p < 1$,

$$M_1(x,y) := px + (1-p)y, \qquad M_2(x,y) := (1-p)x + py \qquad (x,y \in I) \tag{4.4}$$

are strict means on any non-empty open interval $I \subset \mathbb{R}$, and their Gauss composition is

$$M_1 \otimes M_2(x,y) = \frac{x+y}{2} = \mathcal{A}(x,y). \tag{4.5}$$

Observe that in this setting (1.2) is equivalent to (4.1) and (1.3) reduces to the Jensen functional equation

$$2f\Big(\frac{x+y}{2}\Big) = f(x) + f(y) \qquad (x,y \in I). \tag{4.6}$$

We claim that the following two statements hold:

(I) (4.6) *follows from* (4.1) *if any symmetric and biadditive function* $A_2 : \mathbb{R}^2 \to \mathbb{R}$ *with property* (4.2) *is identically zero.*

(II) (4.6) *does not follow from* (4.1) *if there exists a symmetric and biadditive function* $A_2 : \mathbb{R}^2 \to \mathbb{R}$ *with property* (4.2) *that is not identically zero.*

Statement (I) immediately follows from Theorem 4.1. To prove (II) we need the following lemma.

Lemma 4.2. *Let* $A_2 : \mathbb{R}^2 \to \mathbb{R}$ *be a symmetric biadditive function and let* $I \subset \mathbb{R}$ *be a non-void open interval. Then the function*

$$d(x) := A_2(x,x) \qquad (x \in I)$$

satisfies the functional equation

$$2d\Big(\frac{x+y}{2}\Big) = d(x) + d(y) \qquad (x,y \in I)$$

if and only if A_2 *is identically zero.*

Proof. The "if" part is obvious. To prove the "only if" part, we have

$$2A_2\Big(\frac{x+y}{2}, \frac{x+y}{2}\Big) = A_2(x,x) + A_2(y,y)$$

for all $x, y \in I$. Due to the biadditivity of A_2, this is equivalent to

$$A_2(x-y, x-y) = 0 \qquad (x,y \in I).$$

Thus there exists $\delta > 0$, such that for all $t \in]-\delta, \delta[$, $A_2(t,t) = 0$ holds. If $x \in \mathbb{R}$ and $x \neq 0$ is arbitrary, then there exists a number $r \in \mathbb{Q}$, $r \neq 0$ such that $rx \in]-\delta, \delta[$. Therefore

$$0 = A_2(rx, rx) = r^2 A_2(x,x),$$

thus $A_2(x,x)=0$ for all $x\in\mathbb{R}$. Finally

$$0=A_2(x+y,x+y)=A_2(x,x)+2A_2(x,y)+A_2(y,y)=2A_2(x,y)$$

holds for all $x,y\in\mathbb{R}$, that is, A_2 is identically zero. □

Proof of Statement (II). Let $A_2:\mathbb{R}^2\to\mathbb{R}$ be a function whose existence is stated in (II). It is enough to prove that the function d defined by $d(x):=A_2(x,x)\ \ (x\in I)$ is not a solution for (4.6) since, by virtue of Theorem 4.2, it is a solution of (4.1). However if d is a solution for (4.6), then, by Lemma 4.2, A_2 is identically zero, which is a contradiction. □

Theorem 4.3. *Let $0<p<1$ be either a transcendental number or an algebraic number such that $q:=\frac{p}{2p-1}$ is an algebraic conjugate of p. Then there exists a symmetric, biadditive function $A_2:\mathbb{R}^2\to\mathbb{R}$ with property* (4.2) *which is not identically zero. Consequently* (4.6) *does not follow from* (4.1) *in this case.*

Proof. If p is transcendental, then $q=\frac{p}{2p-1}$ is also transcendental since the algebraic numbers form a field. If p is algebraic, then, by our assumption, q is an algebraic conjugate of p, i.e., it is the root of the defining polynomial of p. In both cases, due to the result of Daróczy [4] (see also Kuczma [10]), there exists an additive function $a:\mathbb{R}\to\mathbb{R}$ which is not *identically zero* such that

$$a(px)=qa(x) \tag{4.7}$$

for all $x\in\mathbb{R}$. Now define the function A_2 by

$$A_2(x,y):=a(x)y+a(y)x \qquad (x,y\in\mathbb{R}).$$

Then $A_2:\mathbb{R}^2\to\mathbb{R}$ is symmetric, biadditive, and *non-identically zero*. Therefore, by (4.7),

$$\begin{aligned} A_2(px,(1-p)x) &= a(px)(1-p)x+a((1-p)x)px \\ &= qa(x)(1-p)x+a(x)px-qa(x)px \\ &= a(x)x(q-qp+p-qp) \\ &= a(x)x\Big(\frac{p}{2p-1}(1-p-p)+p\Big)=0 \end{aligned}$$

for all $x\in\mathbb{R}$. That is, (4.2) holds. Therefore, by Statement (II), (4.6) does not follow from (4.1). □

Summarizing our results obtained so far we have that

(i) (4.6) follows from (4.1) for all rational $0<p<1$.
(ii) (4.6) does not follow from (4.1) for all transcendental $0<p<1$.
(iii) There exist algebraic numbers $0<p<1$ of second degree (for example $p=\sqrt{2}-1$ and then $q=-\sqrt{2}-1$) such that (4.6) does not follow from (4.1) for this p.

In our last result we precisely characterize those algebraic numbers $p\in]0,1[$ of second degree such that (4.6) does not follow from (4.1). In particular, it follows from this result that there are also algebraic numbers of second degree such that (4.6) follows from (4.1).

Theorem 4.4. *Let $p \in]0,1[$ be an algebraic number of second degree whose defining polynomial is x^2+sx+t, where $s,t\in\mathbb{Q}$. Then there exists a non-identically zero, symmetric, and biadditive function $A_2:\mathbb{R}^2\to\mathbb{R}$ with property (4.2) if and only if*

$$s^2-4t>0, \quad \sqrt{s^2-4t}\notin\mathbb{Q} \quad \textit{and} \quad s+2t=0. \tag{4.8}$$

Proof. It is easy to see that, under conditions (4.8), equation $x^2+sx+t=0$ always has a root p in the interval $]0,1[$. Let q be the algebraic conjugate of p. According to the well-known relationships between roots and coefficients and by the last equality in (4.8), we obtain that

$$p+q=-s=2t, \qquad pq=t,$$

whence $p+q-2pq=0$. Therefore $q=\frac{p}{2p-1}$. Applying Theorem 4.3, it follows that there exists a biadditive function $A_2:\mathbb{R}^2\to\mathbb{R}$ with property (4.2). Thus we have proved that condition (4.8) is sufficient.

To prove the necessity of the condition we note that $s^2-4t>0$ and $\sqrt{s^2-4t}\notin\mathbb{Q}$ are necessary for p to be an algebraic number of exactly second degree. Therefore, we only have to prove the necessity of the condition $s+2t=0$.

From identity (4.2), we get

$$0=A_2(p(x-y),(1-p)(x-y))=2A_2(px,py)-A_2(px,y)-A_2(x,py). \tag{4.9}$$

Define $x\in\mathbb{R}$ so that $A_2(x,x)=:\alpha\neq 0$. Furthermore, let $A_2(px,x)=:\beta$ and $A_2(p^2x,x)=:\gamma$. Then

$$\begin{aligned} \gamma+s\beta+t\alpha &= A_2(p^2x,x)+sA_2(px,x)+tA_2(x,x) \\ &= A_2((p^2+sp+t)x,x)=A_2(0,x)=0. \end{aligned} \tag{4.10}$$

By (4.9), we obtain

$$A_2(px,px)=A_2(px,x)=\beta$$

and

$$A_2(p^2x,px)=\frac{1}{2}[A_2(p^2x,x)+A_2(px,px)]=\frac{\gamma+\beta}{2},$$

whence

$$\begin{aligned} \frac{\gamma+\beta}{2}+s\beta+t\beta &= A_2(p^2x,px)+sA_2(px,px)+tA_2(x,px) \\ &= A_2((p^2+sp+t)x,px)=A_2(0,px)=0. \end{aligned} \tag{4.11}$$

Finally, applying (4.9) again, we have that

$$A_2(p^2x,p^2x)=\frac{\beta+\gamma}{2}.$$

Therefore

$$\begin{aligned} \frac{\beta+\gamma}{2}+s\frac{\beta+\gamma}{2}+t\gamma &= A_2(p^2x,p^2x)+sA_2(px,p^2x)+tA_2(x,p^2x) \\ &= A_2((p^2+sp+t)x,p^2x)=A_2(0,p^2x)=0. \end{aligned} \tag{4.12}$$

Equations (4.10), (4.11), and (4.12) form a homogeneous, linear system of equations for the unknowns α, β, and γ, that has non-trivial solution (α,β,γ) since $\alpha\neq 0$.

Therefore the determinant of the system of equations vanishes. That is,

$$\begin{vmatrix} t & s & 1 \\ 0 & s+t+\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1+s}{2} & \frac{1+s}{2}+t \end{vmatrix} = 0.$$

Thus we have that

$$\frac{t}{2}(2t+s)(t+s+1) = 0.$$

On the other hand $t \neq 0$ since $\sqrt{s^2-4t} \notin \mathbb{Q}$. If $t+s+1=0$, then $t=-s-1$, whence

$$\sqrt{s^2-4t} = \sqrt{s^2+4s+4} = \sqrt{(s+2)^2} = |s+2| \in \mathbb{Q},$$

which is also not possible. Therefore, $2t+s=0$, which was to be proved. □

Remark 2. Using the approach followed in the proof of the last theorem, one can prove that, for all algebraic numbers $0 < p < 1$ of cubic order, (4.1) and (4.6) are equivalent functional equations. Unfortunately, we were not able to get similar results for algebraic numbers of order higher than 3. We conjecture that (4.1) and (4.6) can be non-equivalent only either for transcendental or for those second-order algebraic numbers that are described in Theorem 4.3. This problem is left as an open question for the interested reader.

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