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ON THE PERIOD FUNCTION OF PLANAR SYSTEMS WITH UNKNOWN NORMALIZERS

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ABSTRACT. A necessary and sufficient condition for the period function's monotonicity on a period annulus is given. The approach is based on the theory of normalizers, but is applicable without actually knowing a normalizer. Some applications to polynomial and Hamiltonian systems are presented.

1. Introduction

In this paper we are concerned with plane differential systems,

$$z' = V(z), \qquad z \in \Omega \subset \mathbb{R}^2, \tag{1}$$

with Ω open connected, $V(z) = (V_1(z), V_2(z)) \in C^2(\Omega, \mathbb{R}^2)$, $z = (x, y) \in \Omega$. We denote by $\phi_V(t, z)$ the local flow defined by (1). A connected subset A of Ω is said to be a *period annulus* of (1) if every orbit of V contained in A is a non-trivial cycle of (1). In some cases the inner boundary of A is a single point O, called the *center*, and the largest connected punctured neighbourhood N_O of O covered with non-trivial cycles is called the *central region*. If A is a period annulus, we can define on A the *period function* T by assigning to each point $z \in A$ the minimal period $T(z)$ of the cycle γ_z passing through z. We say that the period function T is *increasing* if outer cycles have larger periods. When T is constant, we say that A is *isochronous*. Let $\delta(s)$ be a curve of class C^1 meeting transversally the cycle γ at the point $s = s_0$. We say that γ is a *critical cycle* if $\left[\frac{d}{ds}T(\delta(s))\right]_{s=s_0} = 0$. It is possible to prove that such a definition does not depend on the particular transversal curve δ chosen.

The monotonicity of the period function is important in approaching several problems related to (1). It is related to boundary value problems, bifurcation or perturbation problems ([1], [2], [14]), delay differential equations [3], thermodynamics ([8], [9]), and linearizability [7].

A recent result presenting a new approach to the monotonicity of T is based on the properties of a suitable class of auxiliary systems, called *normalizers* [5]. Let us denote by $[V, U] = \partial_V U - \partial_U V$ the Lie bracket of V and U. A vector field U,

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transversal to V, is said to be a *non-trivial normalizer* of V on a set $A \subset \Omega$ if $[V,U] \wedge V = 0$ on A. If U is a normalizer of V on A, then there exists a C^1 function μ, defined on A, such that $[V,U] = \mu V$. Let $\phi_U(s,z)$ be the local flow defined by the solutions of

$$z' = U(z). \tag{2}$$

In [5], Theorem 1, it was proved that

$$\partial_U T(z^*) = \left[\frac{d}{ds} T(\phi_U(s,z^*))\right]_{s=0} = \int_0^T \mu(\phi_V(t,z^*))dt. \tag{3}$$

In the same paper a non-trivial normalizer was found for Hamiltonian systems with separable variables

$$x' = -F'(y), \qquad y' = G'(x). \tag{4}$$

The monotonicity of the period function was studied in detail for centers of such systems. Such an approach proves to be very useful and effective when a normalizer is known. In general, it is not known how to find a normalizer of a given system.

In this paper we present an extension of Theorem 1 in [5] that allows us to replace a normalizer with a transversal vector field W. The main result is contained in Theorem 1, which gives an integral formula for the derivative of T along the integral curves of W. Then we apply Theorem 1 to several situations. In Corollary 1, assuming that $[V,W] = \mu V$, $\mu \geq 0$, we show that the period function of a period annulus of

$$z' = V(z) + B(z)W(z)$$

is increasing, even if W is not a normalizer of $V + BW$. In Corollary 2 we do not assume to know a normalizer, and we give a necessary and sufficient condition for T's monotonicity involving an arbitrary transversal vector field. In particular, assuming the cycles to rotate clockwise, we show that T is increasing if and only if there exists W such that $V \wedge W > 0$ and $[V,W] \wedge W \geq 0$. Also, we show that a cycle γ is critical if and only if there exists a transversal W such that $[V,W] \wedge W = 0$ on γ.

Replacing a normalizer with a transversal vector field overcomes the difficulty of finding a normalizer, but usually leads to more complex computations. In spite of this, we show that some previous results can be re-proved with the technique presented here, and we give some new results. For instance, we prove that if the function

$$\Lambda_{PQ} = -(P_y + Q_x)P^2 + 2(P_x - Q_y)PQ + (P_y + Q_x)Q^2$$

has positive sign on an orbit γ, then T is increasing at γ. This extends the classical result giving isochronicity of systems satisfying $P_x = Q_y$, $P_y = -Q_x$. Moreover, we prove that if Λ_{PQ} vanishes on a single orbit γ, then γ is a critical orbit. Other applications to systems with separable variables are given.

2. Results

Let us set $V \wedge U := V_1U_2 - V_2U_1$. If U is a non-trivial normalizer of V on the period annulus A, then $V \wedge U$ vanishes nowhere on A. Moreover, the proportionality function μ satisfies

$$\mu = \frac{[V,U] \cdot V}{|V|^2},$$

so that, if $V, U \in C^k(A)$, then $\mu \in C^k(A)$, $k \geq 2$.

If U is a normalizer of V on A, then the derivative of T at γ_{z^*} along the solution $\phi_U(s, z^*)$ is given by the integral in (3). The value of such an integral depends on the normalizer U, but the sign of $\partial_U T$ is the same for all normalizers crossing γ_{z^*} in the same direction, for instance outwards, as is usually done.

We do not use the words *increasing* and *decreasing* in a strict sense. When dealing with strict mononicity properties, this will be explicitly stated.

In the next lemma we show that on every period annulus A, V has a non-trivial normalizer.

Lemma 1. *Let A be a period annulus of* (1)*. Then V has a non-trivial normalizer $U \in C^2(A, \mathbb{R}^2)$.*

Proof. For $z \in A$, let us consider the new vector field $V(z)T(z)$. If $\gamma_z(t)$ is a $T(z)$-periodic solution to (1), then $\gamma_z(tT(z))$ is a 1-periodic solution to

$$z' = V(z)T(z).$$

The annulus A is an isochronous annulus for such a system, and the construction of Lemma 1 in [10] generates a vector field U commuting with VT. Then,

$$0 = [VT, U] = \partial_{VT} U - \partial_U VT = T[V, U] - (\partial_U T)V.$$

Therefore, $[V, U] = \frac{\partial_U T}{T} V$, proving that U is a normalizer of V. ♣

In the next theorem we consider an arbitrary vector field W transversal to V defined on a period annulus A. By the transversality assumption, there exist functions η, ν, such that $[V, W] = \eta V + \nu W$. If $W \in C^2(A, \mathbb{R}^2)$, then $\eta, \nu \in C^2(A, \mathbb{R}^2)$, since

$$\eta = \frac{[V, W] \wedge W}{V \wedge W}, \qquad \nu = \frac{[V, W] \wedge V}{W \wedge V}. \tag{5}$$

We denote by $\phi_W(r, z)$ the local flow defined by the differential system

$$z' = W(z).$$

We denote by $\partial_W T(z^*)$ the derivative of the period function T with respect to the solution $\phi_W(r, z^*)$ at the point z^*.

Theorem 1. *Let A be a period annulus of* (1) *and $z^* \in A$. Let $W \in C^2(A, \mathbb{R}^2)$ be a vector field transversal to V on A, with $[V, W] = \eta V + \nu W$. Then, for every T-periodic V-cycle $\gamma(t) = \phi_V(t, z^*)$ contained in A, one has*

$$\int_0^T \nu(\gamma(\tau))\, d\tau = 0. \tag{6}$$

Moreover, setting

$$\beta(\gamma(t)) = \beta(\gamma(0)) \exp\left(-\int_0^t \nu(\gamma(\tau))\, d\tau\right), \tag{7}$$

one has

$$\partial_W T(z^*) = \frac{1}{\beta(z^*)} \int_0^T \eta\bigl(\gamma(\tau)\bigr)\beta\bigl(\gamma(\tau)\bigr)\, d\tau. \tag{8}$$

Proof. Let U be the normalizer of V existing by Lemma 1. There exist $\sigma, \beta \in C^2(A, \mathbb{R})$ such that $U = \sigma V + \beta W$, $\beta \neq 0$ by the transversality of U and V. As observed above, the regularity of σ and β comes from the equalities

$$\sigma = \frac{U \wedge W}{V \wedge W}, \qquad \beta = \frac{U \wedge V}{W \wedge V}.$$

One has

$$\mu V = [V, U] = [V, \sigma V + \beta W] = (\partial_V \sigma + \eta\beta)V + (\partial_V \beta + \beta\nu)W.$$

By the transversality of V and W, this implies

$$\partial_V \beta = -\beta\nu, \qquad \mu = \partial_V \sigma + \eta\beta. \tag{9}$$

From the first equation, for every orbit γ of (1) contained in A, one has

$$\beta(\gamma(t)) = \beta(\gamma(0)) exp\left(-\int_0^t \nu(\gamma(\tau))\, d\tau\right), \qquad \beta(\gamma(0)) \neq 0.$$

The condition $\beta(\gamma(0)) \neq 0$ comes from the transversality of V and U. Since $\gamma(T) = \gamma(0)$, one has

$$0 = \beta(\gamma(T)) - \beta(\gamma(0)) = \beta(\gamma(0)) \left(exp\left(-\int_0^T \nu(\gamma(\tau))\, d\tau\right) - 1\right);$$

hence $\int_0^T \nu(\gamma(\tau))\, d\tau = 0$, proving the first statement.

The vector field βW is a normalizer of V, assuming that β satisfies (9). In fact, one has

$$[V, \beta W] = \partial_V \beta W - \partial_{\beta W} V = (\partial_V \beta + \beta\nu)W + \beta\eta V = \beta\eta V.$$

By Theorem 1 in [5], the derivative of T at z^* along a solution of $z' = \beta(z)W(z)$ is

$$\partial_{\beta W} T(z^*) = \int_0^T \eta(\gamma(\tau))\beta(\gamma(\tau))\, d\tau,$$

where $\gamma(0) = z^*$. The statement follows by observing that $\partial_{\beta W} T(z^*) = \beta(z^*)\partial_W T(z^*)$. ♣

Remark 1. The above proof also shows that for every vector field W transversal to V, there exists a function $\beta \neq 0$, such that βW is a normalizer of V.

In general, computing the integral in (8) is not easy, even when W is itself a normalizer of a Hamiltonian system with separable variables [5]. Anyway, β does not change sign on a single orbit, so that if also η has constant sign, then its sign is that of $\partial_W T$.

If V has a normalizer W, with $[V, W] = \mu V$, then $\eta = \mu$. In fact, from the first equality in (5), one has

$$\eta V \wedge W = [V, W] \wedge W = \mu V \wedge W.$$

Remark 2. If $[V, W] = \mu V$, with $\mu \geq 0$, one can apply Theorem 1 to every system of the type

$$z' = V(z) + B(z)W(z) \tag{10}$$

having a period annulus, requesting only that $B \in C^2$. In general, W is not a normalizer of (10). Such a situation will be considered in Corollary 1.

From now on, we assume the cycles of A to rotate clockwise, so that a vector field W satisfying $V \wedge W > 0$ points outwards, that is towards external cycles. One can always reduce to such a situation, by possibly changing V by $-V$. Such a change does not modify the period function.

Corollary 1. *Let γ be a cycle of* (10)*, contained in a period annulus A, γ rotating clockwise. Let $W \in C^2(A, \mathbb{R}^2)$ satisfy $V \wedge W > 0$, $[V, W] = \mu V$, $\mu \geq 0$. Then T is increasing at γ.*

Proof. One has

$$[V + BW, W] \wedge W = [V, W] \wedge W + [BW, W] \wedge W = \mu V \wedge W \geq 0.$$

By formula (8), since β does not change sign on γ, one has $\partial_W T \geq 0$ on γ. ♣

In the next example we give an application of Corollary 1.

Example 1. Since Hamiltonian systems of the form (4) have a normalizer of the form

$$x' = \frac{G(x)}{G'(x)}, \qquad y' = \frac{F(y)}{F'(y)} \tag{11}$$

(see [5]), one can apply Corollary 1 to the systems

$$x' = F'(y) + B(x, y)\frac{G(x)}{G'(x)}, \qquad y' = -G'(x) + B(x, y)\frac{F(y)}{F'(y)}.$$

In order to have a center at the origin, we assume that $G(x) = x^2 + o(x^2)$, $F(y) = y^2 + o(y^2)$, $F(-y) = F(y)$, $G(-x) = G(x)$, $B(-x, y) = -B(x, y)$, $B(x, -y) = -B(x, y)$, $B(x, y) = o(\sqrt{x^2 + y^2})$. For instance, choosing $F(y) = y^2$, $G(x) = 1 - \cos x$, $B(x, y) = xy$, one has the system

$$x' = 2y + \frac{xy(1 - \cos x)}{\sin x}, \qquad y' = -\sin x + \frac{xy^2}{2}, \tag{12}$$

which has a center at O. The system

$$x' = \frac{1 - \cos x}{\sin x}, \qquad y' = \frac{y}{2}$$

is a normalizer of

$$x' = 2y, \qquad y' = -\sin x,$$

with $\mu(x, y) = (1 - \cos x)^2$. By Corollary 1, the system (12) has an increasing period function at O.

In the next corollary, we do not assume W to be a normalizer of any system.

Corollary 2. *Let γ be a cycle of* (1)*, contained in a period annulus A, γ rotating clockwise. Then:*

(i) *T is increasing at γ if and only if there exists a vector field $W \in C^2(A, \mathbb{R}^2)$, such that $V \wedge W > 0$ and $[V, W] \wedge W \geq 0$ on γ; moreover, if $\exists \bar{z} \in \gamma$ such that $([V, W] \wedge W)(\bar{z}) > 0$, then $\partial_W T > 0$ on γ.*
(ii) *γ is a critical orbit if and only if there exists a vector field $W \in C^2(A, \mathbb{R}^2)$, transversal to V, such that $[V, W] \wedge W = 0$ on γ.*

Proof. (i) Assume T to be increasing at γ. Then, for every vector field W, transversal to V on γ and pointing outwards, one has $\partial_W T \geq 0$ at every point of γ. As observed in Lemma 1, A is an isochronous annulus for the vector field VT. Then, VT has a transversal commutator W. Possibly changing W by $-W$, we may assume that W points outwards. Then one has

$$0 = [VT, W] = T[V, W] - (\partial_W T)V,$$

so that

$$T([V, W] \wedge W) = (\partial_W T)(V \wedge W). \tag{13}$$

This shows that $[V, W] \wedge W \geq 0$ on γ.

As for the vice-versa, let us assume that for some transversal W, $[V, W] \wedge W \geq 0$; hence $\eta \geq 0$ on γ. Since β has constant sign, this implies that the integral in (8) is not negative, proving the first part of statement (i).

If, additionally, there exists a point $\bar{z} \in \gamma$ such that $\eta(\bar{z}) > 0$, then the integral in (8) is positive, proving the second part of statement (i).

(ii) Let us first recall that if $\partial_U T(\bar{z}) = 0$, $\bar{z} \in \gamma$, for a transversal U, then $\partial_W T(\bar{z}) = 0$, for every transversal W. Also, if $\partial_W T(\bar{z}) = 0$ for a transversal W at a fixed $\bar{z} \in \gamma$, then $\partial_W T(z) = 0$ at every point $z \in \gamma$. Working as in part (i), from the equality (13) one has that

$$[V, W] \wedge W = 0$$

on all of γ, where W is a commutator of VT.

As for the vice-versa, if for some transversal W, one has $[V, W] \wedge W = 0$ on all of γ, then from the first equality in (5) one has $\eta = 0$ on γ, so that the integral in (8) vanishes. ♣

Remark 3. If case (ii) occurs on all of a period annulus, then every orbit is critical, so that A is an isochronous annulus. This gives a different proof of the main result in [6], [13].

2.1. $W(z) = z$**.** In this subsection we give some examples by choosing $W(z) = z$. In this case the transversality condition is $z \wedge V(z) \neq 0$, which holds if and only if the cycles in A are strictly star-shaped with respect to O. In the following, we write $V(x, y) = (P(x, y), Q(x, y))$. The next corollary's proof is elementary.

Corollary 3. *Assume the orbits of A to be strictly star-shaped with respect to the origin. Then, $W(z) = z$ is transversal to $V(z)$, and one has*

$$\eta(V \wedge W) = [V, W] \wedge W = yP - xQ - y^2 P_y + xy(Q_y - P_x) + x^2 Q_x, \tag{14}$$
$$\nu(W \wedge V) = [V, W] \wedge V = P(xQ_x + yQ_y) - Q(xP_x + yP_y).$$

The next examples show some applications of Corollary 3.

Example 2. We can apply the formula (14) to the following class of systems:

$$x' = A(y) + xB(x, y), \qquad y' = -C(x) - yB(x, y). \tag{15}$$

Assuming O to be a center for (15), one has

$$[V, W] \wedge W = yA(y) - y^2 A'(y) + xC(x) - x^2 C'(x). \tag{16}$$

If $A(y) = y + a_k y^k + o(y^k)$, $C(x) = x + c_h x^h + o(x^h)$, $k, h > 1$, then $[V, W] \wedge W = (1 + k)a_k y^{k+1} + o(y^{k+1}) + (1 + h)c_h x^{h+1} + o(x^{h+1})$. If h, k are even and a_k, c_h have the same sign, then T is monotone at O. If $A(y) = y$, $C(x) = x$, then we have the so-called *uniformly isochronous centers*, studied by several authors (see [4], §8).

Every Liénard equation is equivalent to a system of the form (15) for a suitable choice of $B(x, y)$ and $C(x)$ ([11], Lemma 2). Applying Corollary 2, one can re-prove the results of [11]. Similarly, every second-order equation of the type $x'' + f(x)x'^2 + g(x) = 0$ is equivalent to a system of the form (15) (see [12], Thm. 4). Again, applying Corollary 2, one can re-prove the results of [12].

Example 3. If (1) is a Hamiltonian system, $x' = H_y, y' = -H_x$, choosing $W(x, y) = (x, y)$ gives

$$[V, W] \wedge W = yH_y + xH_x - y^2 H_{yy} - 2xyH_{xy} - x^2 H_{xx}.$$

If H is analytic, we can write $H(z) = \sum_{n=0}^{\infty} H_n(z)$, where $H_n(z)$ is an n-degree homogeneous polynomial. Then we have, from the properties of homogeneous functions:

$$[V, W] \wedge W = \sum_{0}^{\infty}(2n - n^2)H_n.$$

If H has an extremum at the origin O, then O is a center of the Hamiltonian system. The above formula shows that H_2 has no influence on the monotonicity of T, and that the lowest degree term $H_{\bar{n}}$, $\bar{n} > 2$, determines the monotonicity of T at O, if definite in sign. In particular, if $H_{\bar{n}} \geq 0$, then T is decreasing in a neigbourhood of O. If the center is non-degenerate, this can also be proved by computing the first non-zero period constant, as in [2]. On the other hand, if $H_2 \equiv 0$, then the period constants are not defined, because the period function is unbounded at the origin.

Example 4. We can also apply Corollary 2 to a period annulus not contained in a central region. We give an example for a Hamiltonian system with separable variables. In general, systems with separable variables may be better treated by following the approach of [5], provided the normalizer (11) is defined on all of a cycle. This occurs when $G'(x)F'(y)$ does not vanish on a cycle. On the other hand, studying the period function on period annuli surrounding several critical points, it may happen that one of the fractions in (11) diverges. This is the case of $H(x, y) = (x^2 - 1)^2 + y^2$, whose Hamiltonian system

$$x' = 2y, \qquad y' = -4x(x^2 - 1)$$

has two centers at $(-1, 0)$ and $(1, 0)$, a saddle point at the origin, and a period annulus A surrounding the two central regions. The curve $(x^2 - 1)^2 + y^2 = 1$, consisting of a critical point and two homoclinics, is the inner boundary of A. The cycles contained in A are star-shaped with respect to the origin, since $(x, y) \wedge (2y, -4x(x^2 - 1)) = -4x^2(x^2 - 1) - 2y^2 < 0$ out of $(x^2 - 1)^2 + y^2 = 1$. The system

$$x' = \frac{G(x)}{G'(x)} = \frac{x^2 - 1}{4x}, \qquad y' = \frac{F(y)}{F'(y)} = \frac{y}{2}$$

is not defined on the line $x = 0$. In this case one can apply Theorem 1, choosing $W(x, y) = (x, y)$. Then one has $[V, W] \wedge W = -8x^4$, which gives the strict monotonicity of T on A.

2.2. $W = V^\perp$**.** The simplest choice of a globally transversal field consists in taking the orthogonal one, $W = V^\perp = (-Q, P)$, as in the next corollary. Let us set

$$\Lambda_{PQ} = -(P_y + Q_x)P^2 + 2(P_x - Q_y)PQ + (P_y + Q_x)Q^2.$$

Corollary 4. *Let γ be a cycle of* (1)*, contained in a period annulus A, γ rotating clockwise.*

(i) *If $\Lambda_{PQ} \geq 0$ on γ, then T is increasing at γ; if there exists $\bar{z} \in \gamma$ such that $\Lambda_{PQ}(\bar{z}) > 0$ at $\bar{z}$, then $\partial_W T > 0$ at γ.*

(ii) *If $\Lambda_{PQ} = 0$ on γ, then γ is a critical orbit.*

Proof. This is an immediate consequence of Corollary 2, choosing $W = V^\perp$. ♣

If P and Q are conjugate harmonic functions, then $\eta = 0$, so that the period annulus is isochronous. Point (i) gives another proof of a celebrated theorem (see [7], §6, or [4], §6). Every center of such systems is isochronous, which is equivalent to saying that every orbit is critical. Point (ii) extends such a result to single orbits.

Choosing $W = V^\perp$ gives a simple form also for ν's numerator,

$$\nu(W \wedge V) = [V, W] \wedge V = (Q_y - P_x)P^2 - 2(P_y + Q_x)PQ - (Q_y - P_x)Q^2.$$

If one considers Λ_{PQ} as a quadratic form in (P, Q), then Λ_{PQ} cannot be definite, because the coefficients of P^2 and Q^2 have opposite signs. On the other hand, there exist systems whose Λ_{PQ} has constant signs on some period annuli, or vanishes on single orbits. We report here only a class of examples with Λ_{PQ} of constant sign.

Example 5. If (1) is a Hamiltonian system, choosing W as the orthogonal system gives

$$\Lambda_H = [V, W] \wedge W = (H_{xx} - H_{yy})H_y^2 - 4H_{xy}H_xH_y + (H_{yy} - H_{xx})H_x^2.$$

Considering again Hamiltonian systems with separable variables, $H(x, y) = G(x) + F(y)$, one has $\Lambda_H = (G'' - F'')(F'^2 - G'^2)$. If there exists a function $\psi \in C^1(\mathbb{R}, \mathbb{R})$ such that ψ' does not change sign, and $F'' = \psi(F'^2)$, $G'' = \psi(G'^2)$, then T is monotone. In fact, one has

$$\Lambda_H = (F'' - G'')(G'^2 - F^2) = \big(\psi(F'^2) - \psi(G'^2)\big)(G'^2 - F'^2) = -\psi'\big(\xi\big)(G'^2 - F'^2)^2,$$

where ξ is a suitable point in the interval (F'^2, G'^2) or in (G'^2, F'^2). A simple example is given by the system

$$x' = \tan(y), \qquad y' = -\tan(x),$$

which has strictly decreasing period function. In this case one has $\psi(u) = 1 + u$. Similarly for

$$x' = \tanh(y), \qquad y' = -\tanh(x),$$

which has a strictly increasing period function. In this case one has $\psi(u) = 1 - u$.

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