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A LARGE DEVIATION PRINCIPLE FOR RANDOM UPPER SEMICONTINUOUS FUNCTIONS

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Abstract. We obtain necessary and sufficient conditions in the Large Deviation Principle for random upper semicontinuous functions on a separable Banach space. The main tool is the recent work of Arcones on the LDP for empirical processes.

1. Introduction

In this paper we study the Large Deviation Principle (LDP) for random upper semicontinuous (u.s.c.) functions, giving necessary and sufficient conditions. Our framework includes as particular cases those of random elements and random compact sets in a separable Banach space.

Related results in the literature are scarce. We can mention Cerf's work on sufficient conditions in the LDP for random compact sets [3, Theorem 1], from which we borrow some arguments. More recently, we have obtained a version of Bolthausen's LDP for arrays of random quasiconcave upper semicontinuous functions [19, Theorem 11].

These results ultimately rely on the LDP for random elements of separable Banach spaces. It is interesting to remark that the space of u.s.c. functions, endowed with the metric d_∞ (the strongest one in the literature) is not separable. However, we observe in [19, Theorem 7] that d_∞-Borel random u.s.c. functions, under a technical set-theoretical assumption, take on values almost surely in a separable subspace. Hence, we were able to prove an LDP by methods similar to those in the literature of random sets (based on embedding theorems into separable Banach spaces).

We should also observe that the LDP can be obtained by the same methods without imposing d_∞-Borel measurability or the acceptance of set-theoretical assumptions if we conform ourselves to convergence in weaker separable metrics. Notice that d_∞-Borel measurability is *not* natural in this framework; see [4].

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Another way to obtain limit theorems is the method based on empirical processes. Such an approach allows us to obtain d_∞ convergence but has been used only with a finite-dimensional carrier space [15, 14] or at the cost of further technical hypotheses in the infinite-dimensional case [8]; see Section 3 for details.

We see that all the above-mentioned methods have drawbacks, either in the form of technical requirements or of a weakening of the mode of convergence. By using the empirical process approach, combined with new relative compactness arguments, we will be able to overcome those limitations. Besides, those arguments are especially well suited for the necessary and sufficient conditions in the LDP for empirical processes of Arcones [1], which is our main tool.

The paper is structured as follows. Preliminary notions and results are collected in Section 2. Section 3 is about the support process of a random u.s.c. function (under the additional hypothesis of quasiconcavity) and its relative compactness properties. In Section 4 we derive the LDP for random u.s.c. functions. Finally, some remarks are made in Section 5.

2. Preliminaries

Let $\mathbf{E}$ be a separable Banach space with norm $\|\cdot\|$ and dual space $\mathbf{E}^*$. Denote its closed unit ball by B, and that of $\mathbf{E}^*$ by B^*.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. We denote by $L^p(\mathbf{E})$ the Lebesgue-Bochner spaces $L^p(\mathbf{E}) = \{\xi : \Omega \to \mathbf{E} \mid \xi \text{ measurable}, E\|\xi\|^p < \infty\}$. A random element ξ is Bochner integrable, with integral denoted $E\xi$, if and only if $\xi \in L^1(\mathbf{E})$.

We will say that $\mathbf{E}$ has *type* $p \in [1, 2]$ if

$$E\|\sum_{i=1}^n \xi_i\|^p \leq \sum_{i=1}^n E\|\xi_i\|^p$$

whenever $\xi_i \in L^p(\mathbf{E})$ are independent with $E\xi_i = 0$. Every Banach space has type 1 and every Hilbert space has type 2. $L^p(\mathbf{R})$ spaces, for $p \geq 1$, have type $\min\{p, 2\}$, whereas spaces such as $C[0, 1]$ and $L^\infty(\mathbf{R})$ do not have type p for any $p > 1$.

We will denote by $\mathcal{K}$ the class of non-empty compact subsets of $\mathbf{E}$, endowed with the *Hausdorff metric*

$$d_H(A, C) = \max\{\sup_{x\in A} \inf_{y\in C} \|x - y\|, \sup_{y\in C} \inf_{x\in A} \|x - y\|\}$$

and the operations

$$A + C = \{x + y \mid x \in A, y \in C\},\ \lambda \cdot A = \{\lambda x \mid x \in A\}.$$

$\mathcal{K}_c$ will denote the class of all non-empty subsets of $\mathbf{E}$ which are compact convex. The convex hull of $A \in \mathcal{K}$ will be denoted by $\operatorname{co} A$.

We also set $\|A\| = d_H(A, \{0\})$. For every $A \in \mathcal{K}_c$ we define its *support function* $s(\cdot, A) : B^* \longrightarrow \mathbf{R}$ by $s(x^*, A) = \sup_{x\in A} x^*(x)$.

We will denote by $\mathcal{F}$ the family of non-negative upper semicontinuous functions on $\mathbf{E}$ having relatively compact support and whose maximum value is 1. Observe that every level set $U_\alpha = \{x \mid U(x) \geq \alpha\}$ of $U \in \mathcal{F}$ is in $\mathcal{K}$, for $\alpha \in (0, 1]$. We also denote $U_{\alpha^+} = \{x \mid U(x) > \alpha\}$; the closed support of U will be denoted by U_0.

The subfamily of all U such that U_α is in $\mathcal{K}_c$ for all $\alpha \in [0, 1]$ will be denoted by $\mathcal{F}_c$. It is well known that $U \in \mathcal{F}$ is in $\mathcal{F}_c$ if and only if U is *quasiconcave*, i.e. $U(\lambda x + (1 - \lambda)y) \geq \min\{U(x), U(y)\}$ for all $\lambda \in [0, 1], x, y \in \mathbf{E}$.

For every $U \in \mathcal{F}_c$ we define its *support function* $s(\cdot,\cdot,U) : B^* \times [0,1] \longrightarrow \mathbf{R}$ by $s(x^*,\alpha,U) = \sup_{x\in U_\alpha} x^*(x)$. The indicator function of a set A will be denoted by I_A.

We define the following operations in $\mathcal{F}$:

$$(U+V)(x) = \sup_{x_1+x_2=x} \min\{U(x_1), V(x_2)\},$$

$$(\lambda U)(x) = \begin{cases} U(\lambda^{-1}x), & \lambda \neq 0, \\ I_{\{0\}}(x), & \lambda = 0. \end{cases}$$

These operations extend those of $\mathcal{K}$, in the sense that

$$(\lambda U)_\alpha = \lambda U_\alpha \quad \text{and} \quad (U+V)_\alpha = U_\alpha + V_\alpha$$

for all $\alpha \in [0,1]$.

In order to endow $\mathcal{F}$ with a metric structure, we will use the Puri-Ralescu metric

$$d_\infty(U,V) = \sup_{\alpha\in[0,1]} d_H(U_\alpha, V_\alpha),$$

which is the strongest one considered in the literature. The metric space $(\mathcal{F}, d_\infty)$ is complete but not separable even if $\mathbf{E} = \mathbf{R}$.

Each $A \in \mathcal{F}$ can be identified, preserving all this structure, with its indicator function $I_A \in \mathcal{F}$. For every $U \in \mathcal{F}$, denote by $\operatorname{co} U \in \mathcal{F}_c$ the *quasiconcave envelope* of U. Then $(\operatorname{co} U)_\alpha = \operatorname{co} U_\alpha$ for all $\alpha \in (0,1]$.

An $\mathcal{F}$-valued mapping on a measurable space $(\Omega, \mathcal{A})$ is called *measurable*, a *random u.s.c. function* or a *fuzzy random variable* if $X_\alpha : \omega \mapsto X(\omega)_\alpha$ is a random compact set for each $\alpha \in (0,1]$, equivalently if each X_α is Borel measurable with respect to the Hausdorff metric [17]. For further information on random sets; see e.g. [9, 11].

This definition of measurability is weaker than Borel measurability with respect to the non-separable metric d_∞, and the σ-algebra naturally associated to it is generated by a Polish topology [4].

Denote by $X_0 : \Omega \to \mathcal{K}$ the random compact set such that $X_0(\omega)$ is the closure of the support of $X(\omega)$, for each $\omega \in \Omega$. A random u.s.c. function X is called *integrably bounded* if $E\|X_0\| < \infty$. Then, there exists a unique $EX \in \mathcal{F}$ such that each $(EX)_\alpha$ is the Aumann integral of the random set X_α for all $\alpha \in (0,1]$, namely

$$(EX)_\alpha = \{E\xi \mid \xi \in L^1(\mathbf{E}), \xi \in X_\alpha \text{ almost surely}\}.$$

Whenever (M,ρ) is a pseudometric space, we will denote by $B(x,\varepsilon;\rho)$ the open ρ-ball with center $x \in M$ and radius $\varepsilon > 0$.

3. The support process of a random quasiconcave u.s.c. function

Let $X : \Omega \to \mathcal{F}_c$ be a random quasiconcave u.s.c. function. The *support process* of X is defined to be the process

$$(x^*,\alpha) \in B^* \times [0,1] \mapsto s(x^*,\alpha,X).$$

Observe that each mapping $s(x^*,\alpha,X)$ is indeed a random variable, since each X_α is a random compact set (e.g. [2, Theorem III.15]).

This representation is very appropriate for our problem, because it preserves the metric and algebraic structures:

$$d_\infty(X,Y) = \sup_{(x^*,\alpha)\in B^*\times[0,1]} |s(x^*,\alpha,X) - s(x^*,\alpha,Y)|$$

and

$$s(x^*, \alpha, aX + bY) = as(x^*, \alpha, X) + bs(x^*, \alpha, Y)$$

whenever X, Y are random quasiconcave u.s.c. functions and $a, b \geq 0$.

The idea to use this representation to reformulate the problem seems to be due to Proske [13]. It extends the older idea that limit theorems in Banach spaces can be translated into the language of empirical processes by associating to a random element ξ the process $x^* \in B^* \mapsto x^*(\xi)$. Indeed, if ξ is a random element of $\mathbf{E}$, then $s(x^*, \alpha, I_{\{\xi\}}) = x^*(\xi)$ for all x^*, α. Notice that the same idea appears too in the literature of random sets; e.g. [5].

These techniques have been successfully applied when the carrier space is $\mathbf{R}^d$. Proske and Puri [15, 14] used them in order to prove a strong law of large numbers and a central limit theorem. Li *et al.* [8] recently refined the conditions in the CLT and also presented an extension to the infinite-dimensional setting. Unfortunately, they needed to add a restrictive assumption: that each element of the range of the random u.s.c. function has the property that its support is contained in some multiple of a fixed compact set. These kinds of assumptions already appear in the literature of limit theorems for random sets, see [5, 16].

In view of the results in [19], such an assumption is not needed to obtain LDP results if the random u.s.c. functions are Borel measurable with respect to d_∞. It is interesting to establish that it remains unnecessary when the measurability definition is relaxed. For that purpose, we present some properties of relative compactness related to the support process which differ from those used in former papers.

We introduce some notation. We define

$$L^p(\mathcal{K}_c) = \{\Gamma : \Omega \to \mathcal{K}_c \mid \Gamma \text{ random compact set}, E\|\Gamma\|^p < \infty\}.$$

This space is analogous to the Lebesgue-Bochner spaces, and in fact embeds as a convex cone in $L^p(\mathbf{E}')$ for an appropriate Banach space $\mathbf{E}'$ (see [6]). It is endowed with the metric Δ_p given by

$$\Delta_p(\Gamma, \Gamma') = E[d_H(\Gamma, \Gamma')^p]^{1/p}.$$

Our first compactness result deals with the support process of a random set.

Proposition 3.1. *Let $p \geq 1$. Let $\Gamma \in L^p(\mathcal{K}_c)$. Then, $\{s(x^*, \Gamma)\}_{x^* \in B^*}$ is compact in $L^p(\mathbf{R})$.*

Proof. Since $|s(x^*, \Gamma)| \leq \|\Gamma\| \in L^p(\mathbf{R})$, each $s(x^*, \Gamma)$ is in $L^p(\mathbf{R})$. Let $\hat{\Gamma} : B^* \to L^p(\mathbf{R})$ be the mapping given by $\hat{\Gamma}(x^*) = s(x^*, \Gamma)$. Note that $\{s(x^*, \Gamma)\}_{x^* \in B^*} = \hat{\Gamma}(B^*)$. By the Banach-Alaoglu Theorem, B^* is weak* compact; thus it will suffice to prove that $\hat{\Gamma}$ is weak* continuous. Notice that B^* is weak* metrizable and separable; hence we just have to check that $\hat{\Gamma}$ preserves limits of weak* convergent sequences.

Let $\{x_n^*\}_n \subset B^*$ be weak* convergent to some $x^* \in B^*$. Since the weak* topology coincides in B^* with the topology of uniform convergence on compact sets,

$$|s(x_n^*, \Gamma(\omega)) - s(x^*, \Gamma(\omega))| \leq \sup_{x \in \Gamma(\omega)} |x_n^*(x) - x^*(x)| \to 0$$

for each $\omega \in \Omega$.

Besides, $|s(x_n^*, \Gamma) - s(x^*, \Gamma)| \leq 2\|\Gamma\| \in L^p(\mathbf{R})$. By the dominated convergence theorem, $\|s(x_n^*, \Gamma) - s(x^*, \Gamma)\|_p \to 0$. $\square$

Then, we have another result related to a compactification of the interval $[0,1]$.

Proposition 3.2. *Let $p \geq 1$. Let $X : \Omega \to \mathcal{F}$ be a random u.s.c. function such that $E\|X_0\|^p < \infty$. Then, $\{X_\alpha, X_1, \operatorname{cl} X_{\alpha^+}\}_{\alpha\in[0,1)}$ is compact in $L^p(\mathcal{K}_c)$.*

Proof. Denote $\mathcal{Q} = \{X_\alpha, X_1, \operatorname{cl} X_{\alpha^+}\}_{\alpha\in[0,1)}$. Let $\{\Gamma_n\}_n$ be an arbitrary sequence contained in $\mathcal{Q}$. It suffices to show that it must have a convergent subsequence.

First notice that $\{\Gamma_n\}_n$ has a subsequence of the form $\{X_{\alpha_n}\}_n$ or $\{\operatorname{cl} X_{\alpha_n^+}\}_n$. Both cases are analogous, and we will prove only the latter.

Again by subsequencing we can assume that $\{\alpha_n\}_n$ is monotonic and hence that it converges to some $\alpha \in [0,1]$. Notice also that the following relationships hold for every $\beta \in (0,1)$ and $\varepsilon < 1-\beta$:

$$X_{\beta+\varepsilon} \subset \operatorname{cl} X_{\beta^+} \subset X_\beta, \qquad X_\beta = \bigcap_{\gamma\in(0,\beta)} X_\gamma, \qquad X_{\beta^+} = \bigcup_{\gamma\in(\beta,1)} X_\gamma.$$

Taking that into account, if $\{\alpha_n\}_n$ is increasing, then $\operatorname{cl} X_{\alpha_n^+} \to X_\alpha$ pointwise in the metric d_H (the case $\alpha = 1$ does not pose any problem). If $\{\alpha_n\}_n$ is decreasing, then $\operatorname{cl} X_{\alpha_n^+} \to \operatorname{cl} X_{\alpha^+}$ pointwise in d_H. In any case, we denote by Γ the pointwise limit of the sequence.

Notice that $d_H(\operatorname{cl} X_{\alpha_n^+}, \Gamma)^p \searrow 0$ and $Ed_H(\operatorname{cl} X_{\alpha_1^+}, \Gamma)^p \leq 2E\|X_0\|^p < \infty$. The monotonic convergence theorem yields that $\{\operatorname{cl} X_{\alpha_n^+}\}_n$ Δ_p-converges to an element of $\mathcal{Q}$, as desired. □

Corollary 3.3. *Let $X : \Omega \to \mathcal{F}_c$ be a random quasiconcave u.s.c. function such that $E\|X_0\|^p < \infty$. The space $([0,1], \rho_p^X)$, where ρ_p^X is the pseudometric given by $\rho_p^X(\alpha,\beta) = E[d_H(X_\alpha, X_\beta)^p]^{1/p}$, is totally bounded.*

Remark 3.4. Compare with the approach in [8, Lemmas 1 and 3], where the total boundedness of $[0,1]$ in the pseudometric

$$\tilde{\rho}_2^X(\alpha,\beta) = d_H(E[\|X_0\|X_\alpha], E[\|X_0\|X_\beta])^{1/2}$$

is shown under the condition $E\|X_0\|^2 < \infty$.

Let $Y = \|X_0\|X$. Since $\|Y_0\| = \|X_0\|^2$, under that condition $E\|Y_0\| < \infty$ and by Corollary 3.3, the pseudometric $(\alpha,\beta) \mapsto Ed_H(\|X_0\|X_\alpha, \|X_0\|X_\beta)$ makes $[0,1]$ totally bounded. Now the inequality

$$d_H(E[\|X_0\|X_\alpha], E[\|X_0\|X_\beta]) \leq Ed_H(\|X_0\|X_\alpha, \|X_0\|X_\beta)$$

(see [6, Theorem 4.1]) yields the total boundedness of $([0,1], \tilde{\rho}_2^X)$. That is, our relative compactness result is stronger (and valid also for $p \neq 2$).

Notice, finally, that whether $([0,1], \tilde{\rho}_2^X)$ is totally bounded or not, depends exclusively on EY, a single element of $\mathcal{F}_c$. Hence arguments such as those in [8] or [10, Theorem 2] are not useful in our case and we really need to use compactness arguments in L^p type spaces. □

By combining Proposition 3.1 and Corollary 3.3, we obtain the following. It is directly related to one of the conditions in Arcones's LDP for empirical processes.

Theorem 3.5. *Let $X : \Omega \to \mathcal{F}_c$ be a random u.s.c. function such that $E\|X_0\|^p < \infty$. The space $(B^* \times [0,1], d_p^X)$, where d_p^X is the pseudometric given by*

$$d_p^X((x^*,\alpha),(y^*,\beta)) = E[|s(x^*,\alpha,X) - s(y^*,\beta,X)|^p]^{1/p},$$

is totally bounded.

Proof. Let $\varepsilon > 0$. Take $\{\alpha_i\}_{i=1}^r$ such that

$$\{X_\alpha\}_{\alpha\in[0,1]} \subset \bigcup_{1\le i\le r} B(X_{\alpha_i}, \varepsilon/2; \Delta_p).$$

For each i, take $\{x^*_{i,j}\}_{j=1}^{m(i)}$ such that

$$\{s(x^*, X_{\alpha_i})\}_{x^*\in B^*} \subset \bigcup_{1\le j\le m(i)} B(s(x^*_{i,j}, X_{\alpha_i}), \varepsilon/2; \|\cdot\|_p).$$

For each $(x^*, \alpha) \in B^* \times [0,1]$, there exist then $\alpha_i, x^*_{i,j}$ such that

$$\begin{aligned} d_p^X((x^*,\alpha),(x^*_{i,j},\alpha_i)) &\le d_p^X((x^*,\alpha),(x^*,\alpha_i)) + d_p^X((x^*,\alpha_i),(x^*_{i,j},\alpha_i)) \\ &\le \rho_p^X(\alpha,\alpha_i) + \|s(x^*,X_{\alpha_i}) - s(x^*_{i,j},X_{\alpha_i})\|_p < \varepsilon. \end{aligned}$$

□

4. Large deviation principle

Let $\{Y_n\}_n$ be a sequence of random u.s.c. functions, and let $I : \mathcal{F} \to [0,\infty]$ be a lower semicontinuous function. We will say that $\{Y_n\}_n$ satisfies a *Large Deviation Principle* with good rate function I if

(i) I is lower compact, namely $\{U \in \mathcal{F} \mid I(U) \le \alpha\}$ is compact for each $\alpha \ge 0$,

(ii) for each $\mathcal{H} \subset \mathcal{F}$,

$$\begin{aligned} -\inf I(\operatorname{int}\mathcal{H}) &\le \liminf_n \frac{1}{n}\log P_*(Y_n \in \mathcal{H}) \\ &\le \limsup_n \frac{1}{n}\log P^*(Y_n \in \mathcal{H}) \le -\inf I(\operatorname{cl}\mathcal{H}), \end{aligned}$$

where P_* and P^* respectively denote inner and outer probabilities associated to the probability measure P.

The definition of LDP for stochastic processes with index set T is similar, but replacing $\mathcal{F}$ by the space $\ell_\infty(T)$; see e.g. [1, Definition 1.1].

Let $(S, \mathcal{S}, \nu)$ be a probability space. Let $\Omega = S^{\mathbf{N}}$, $\mathcal{A} = \mathcal{S}^{\mathbf{N}}$, $Q = \nu^{\mathbf{N}}$. Let X_n be the nth coordinate projection from Ω to S; as is well known, $\{X_n\}_n$ is a sequence of i.i.d. measurable mappings from Ω to S. It will be convenient to denote $X = X_1$. From now on, we assume without loss of generality that also i.i.d. random u.s.c. functions arise in this canonical setting.

We also set $h(x) = x\log(x/e) + 1$, if $x \ge 0$, and $h(x) = \infty$ for $x < 0$.

We will obtain the LDP for random u.s.c. functions (under the additional hypothesis of quasiconcavity) as an application of the following theorem to the support process.

Theorem 4.1 ([1, Theorem 2.8])**.** *Let T be an index set and let $\{f(\cdot,t)\}_{t\in T}$ be an image admissible Suslin class of measurable functions from S to $\mathbf{R}$, such that $\sup_{t\in T}|f(X,t)| < \infty$ almost surely. Then, the set of conditions below is necessary and sufficient for the process $t \in T \mapsto n^{-1}\sum_{i=1}^n f(X_i,t)$ to satisfy an LDP in $\ell_\infty(T)$.*

(i) *The space (T,d) is totally bounded, where $d(s,t) = E|f(X,s) - f(X,t)|$.*

(ii) *There exists $\lambda > 0$ such that $E\exp[\lambda F(X)] < \infty$, where $F(X) = \sup_{t\in T}|f(X,t)|$.*

(iii) *For each $\lambda > 0$, there exists $\eta > 0$ for which $E\exp[\lambda F^{(\eta)}(X)] < \infty$, where $F^{(\eta)}(X) = \sup_{d(s,t)\le\eta}|f(X,s) - f(X,t)|$.*

(iv) $\sup_{t\in T} |n^{-1}\sum_{i=1}^n [f(X_i,t) - Ef(X_i,t)]|$ *converges to* 0 *in probability.*

Moreover, the good rate function I *is given by*

$$I(z) = \sup\{I_{t_1,\dots,t_m}(z(t_1),\dots,z(t_m)) \mid t_1,\dots,t_m \in T, m \in \mathbf{N}\},$$

where

$$I_{t_1,\dots,t_m}(u_1,\dots,u_m) = \inf\{Eh(\gamma(X)) \mid E\gamma(X) = 1, \\ E[f(X,t_j)\gamma(X)] = u_j \ \forall\, 1 \le j \le m\}.$$

The hypothesis of quasiconcavity will be removed (if the carrier space has non-trivial type) with the help of the following lemmas.

Lemma 4.2 (see [18, Lemma 10])**.** *Let* $\mathbf{E}$ *be a Banach space of type* $p > 1$*. Then, for every* $U_1,\dots,U_n \in \mathcal{F}$*,*

$$d_\infty(\sum_{i=1}^n U_i, \sum_{i=1}^n \operatorname{co} U_i) \le 2c^{1/p} \cdot (\sum_{i=1}^n \|(U_i)_0\|^p)^{1/p},$$

where c *is the type-*p *constant of* $\mathbf{E}$*.*

Two further lemmas are the following. The first one is elementary, the second one follows closely Cerf's lemma [3, Lemma 2] and so we do not include its proof (in which Lemma 4.2 must replace [16, Theorem 4.1]).

Lemma 4.3. *Let* $\{x_n\}_n, \{y_n\}_n \subset [0,\infty)$ *be such that* $n^{-1}\log x_n \to -\infty$*. Then,* $\limsup_n n^{-1}\log(x_n + y_n) = \limsup_n n^{-1}\log y_n$ *and* $\liminf_n n^{-1}\log(x_n + y_n) = \liminf_n n^{-1}\log y_n$*.*

Proof. This follows from the inequalities $y_n^{1/n} \le (x_n + y_n)^{1/n} \le x_n^{1/n} + y_n^{1/n}$. □

We adopt the following notation. If $\{X_n\}_n$ is a sequence of random u.s.c. functions i.i.d. as X, then we will write $S_n(X)$ for $n^{-1}\sum_{i=1}^n X_i$; and analogously for $S_n(\operatorname{co} X)$, et cetera.

Lemma 4.4. *Let* $\mathbf{E}$ *be a Banach space of type* $p > 1$*. Let* $X : \Omega \to \mathcal{F}_c$ *be a random quasiconcave u.s.c. function such that* $E\exp[\lambda\|X_0\|] < \infty$ *for some* $\lambda > 0$*. Then,*

$$n^{-1}\log P(d_\infty(S_n(X), S_n(\operatorname{co} X)) \ge \varepsilon) \to -\infty$$

for all $\varepsilon > 0$*.*

We present now the LDP for random u.s.c. functions. In a part of the proof, the basic idea of 'deconvexification' using Lemma 4.4 is the same as in Cerf's paper; however, we believe our development of that idea to be clearer and simpler to the reader (e.g. the liminf and limsup parts are dealt with in a symmetric way).

Theorem 4.5. *Let* $\{X_n\}_n$ *be a sequence of integrably bounded* $\mathcal{F}_c$*-valued random u.s.c. functions, i.i.d. as* $X = X_1$*. Then, the following conditions are necessary and sufficient for* $\{S_n(X)\}_n$ *to satisfy an LDP in* $\mathcal{F}_c$*.*

(a) *There exists* $\lambda > 0$ *such that* $E\exp[\lambda\|X_0\|] < \infty$*.*

(b) *For each* $\lambda > 0$*, there exists* $\eta > 0$ *for which* $E\exp[\lambda F^{(\eta)}(X)] < \infty$*, where*

$$F^{(\eta)}(X) = \sup\{|s(t_1,X) - s(t_2,X)| \mid t_1,t_2 \in B^* \times [0,1], E|s(t_1,X) - s(t_2,X)| \le \eta\}.$$

Moreover, the good rate function $I : \mathcal{F}_c \to [0,\infty]$ *is given by*

$$I(U) = \sup\{I_{t_1,\dots,t_m}(s(t_1,U),\dots,s(t_m,U)) \mid t_1,\dots,t_m \in B^* \times [0,1], m \in \mathbf{N}\},$$

where

$$I_{t_1,\dots,t_m}(u_1,\dots,u_m) = \inf\{Eh(\gamma(X)) \mid E\gamma(X) = 1, \\ E[s(t_j,X)\gamma(X)] = u_j \ \forall\, 1 \le j \le m\}.$$

If, additionally, $\mathbf{E}$ *has type* $p > 1$*, then the former holds if* $\{X_n\}_n$ *are* $\mathcal{F}$*-valued random u.s.c. functions; the corresponding good rate function* $J : \mathcal{F} \to [0,\infty]$ *is given by* $J(U) = I(U)$ *if* $U \in \mathcal{F}_c$ *and* $J(U) = \infty$ *if* $U \in \mathcal{F}\backslash\mathcal{F}_c$*.*

Proof. For the first part, we use Theorem 4.1, with the choices $S = \mathcal{F}_c$ (with the σ-algebra given in the preliminaries), $T = B^* \times [0,1]$ and $f(X,t) = s(t,X)$ for $t \in T$.

Notice that each $f(X,t)$ is measurable, as mentioned in Section 3. Besides,

$$\sup_t |f(X,t)| = \sup_{t \in B^* \times [0,1]} |s(t,X)| = d_\infty(X, I_{\{0\}}) = \|X_0\| < \infty$$

almost surely, since $E\|X_0\| < \infty$.

It is also clear that conditions (a) and (b) match conditions (ii) and (iii) in Theorem 4.1. We have to show that conditions (i) and (iv) in that theorem always hold in this setting. Indeed, condition (i) is satisfied by virtue of Theorem 3.5, because $d = d_1^X$. Moreover,

$$\sup_{t\in T} |n^{-1}\sum_{i=1}^n [f(X_i,t) - Ef(X_i,t)]| = \sup_{t \in B^* \times [0,1]} |s(t, n^{-1}\sum_{i=1}^n X_i) - s(t, n^{-1}\sum_{i=1}^n EX_i)| \\ = \sup_{t \in B^* \times [0,1]} |s(t, n^{-1}\sum_{i=1}^n X_i) - s(t, EX)| = d_\infty(n^{-1}\sum_{i=1}^n X_i, EX),$$

which goes to 0 almost surely by the Strong Law of Large Numbers for random u.s.c. functions on a separable Banach space [10, Theorem 3]. Hence, condition (iv) is satisfied too.

The good rate function I is the one given by Theorem 4.1, modulo obvious changes.

As for the second part, for any subset $\mathcal{H} \subset \mathcal{F}$ we define

$$\mathcal{H}^\varepsilon = \bigcup_{U \in \mathcal{H}} B(U,\varepsilon;d_\infty), \qquad \mathcal{H}^{-\varepsilon} = \{U \in \mathcal{H} \mid B(U,\varepsilon;d_\infty) \subset \mathcal{H}\}.$$

Notice that $(\mathcal{H}^{-\varepsilon})^\varepsilon \subset \mathcal{H} \subset \mathcal{H}^\varepsilon$ for an arbitrary $\varepsilon > 0$. Then,

$$P_*(S_n(\operatorname{co} X) \in \mathcal{H}^{-\varepsilon}) \le P_*(S_n(X) \in (\mathcal{H}^{-\varepsilon})^\varepsilon) + P(d_\infty(S_n(X), S_n(\operatorname{co} X)) \ge \varepsilon) \\ \le P_*(S_n(X) \in \mathcal{H}) + P(d_\infty(S_n(X), S_n(\operatorname{co} X)) \ge \varepsilon)$$

and

$$P^*(S_n(X) \in \mathcal{H}) \le P^*(S_n(\operatorname{co} X) \in \mathcal{H}^\varepsilon) + P(d_\infty(S_n(X), S_n(\operatorname{co} X)) \ge \varepsilon).$$

Taking into account Lemmas 4.3 and 4.4, for our purposes we can discard the term $P(d_\infty(S_n(X), S_n(\operatorname{co} X)) \geq \varepsilon)$ and so, from the LDP in the quasiconcave case we obtain the inequalities

$$\begin{aligned} -\inf I(\operatorname{int}(\mathcal{H}^{-\varepsilon} \cap \mathcal{F}_c)) &\leq \liminf_n \frac{1}{n} \log P_*(S_n(X) \in \mathcal{H}) \\ &\leq \limsup_n \frac{1}{n} \log P^*(S_n(X) \in \mathcal{H}) \leq -\inf I(\operatorname{cl}(\mathcal{H}^{\varepsilon} \cap \mathcal{F}_c)), \end{aligned}$$

where I is the rate function in the statement of the theorem. Recall that this is valid for every $\varepsilon > 0$; thus the bounds remain true if we pass to the limit as $\varepsilon \to 0$.

Since I is lower compact, the set function $-\inf I$ is outer semicontinuous on closed sets and inner semicontinuous on open sets, whence

$$\sup_{\varepsilon>0}[-\inf I(\operatorname{int}(\mathcal{H}^{-\varepsilon} \cap \mathcal{F}_c))] = -\inf I(\bigcup_{\varepsilon>0} \operatorname{int}(\mathcal{H}^{-\varepsilon} \cap \mathcal{F}_c)) = -\inf I(\operatorname{int}(\mathcal{H} \cap \mathcal{F}_c))$$

and

$$\inf_{\varepsilon>0}[-\inf I(\operatorname{cl}(\mathcal{H}^{\varepsilon} \cap \mathcal{F}_c))] = -\inf I(\bigcap_{\varepsilon>0} \operatorname{cl}(\mathcal{H}^{\varepsilon} \cap \mathcal{F}_c)) = -\inf I(\operatorname{cl}(\mathcal{H} \cap \mathcal{F}_c)).$$

Notice finally that $-\inf J(\operatorname{int} \mathcal{H}) = -\inf I(\operatorname{int}(\mathcal{H} \cap \mathcal{F}_c))$ and $-\inf J(\operatorname{cl} \mathcal{H}) = -\inf I(\operatorname{cl}(\mathcal{H} \cap \mathcal{F}_c))$. But $\mathcal{F}_c$ is a closed subspace of $\mathcal{F}$, whence J is indeed a lower compact extension of I (and as such is a good rate function).

This proves the sufficiency of (a) and (b) for the LDP in the general case (the assumption that $\mathbf{E}$ has non-trivial type is used when invoking Lemma 4.4). But if, in its turn, the LDP holds for $\mathcal{F}$-valued random u.s.c. functions with rate function J, then in particular it holds in the $\mathcal{F}_c$-valued case with rate function $I = J|_{\mathcal{F}_c}$. Since (a) and (b) are necessary for the quasiconcave LDP, we see that they are also required for the general case. □

5. Concluding remarks

Ogura and Li [12] announce versions of the LDP for random u.s.c. functions, presumably obtained with a different method. Their results have the following weaknesses with respect to ours. First, convergence is weaker than d_∞. Their Corollary 4 provides d_∞-convergence, at the cost of a technical assumption which seems to be so strong as to imply d_∞-separability of the range of X. Second, they only study the finite-dimensional quasiconcave case. Third, they do not give necessary and sufficient conditions (however, it follows from our results that their sufficient condition is also necessary).

On the other side of the scale, in their framework, condition (a) of Theorem 4.5 appears to be sufficient. This is remarkable insofar as $\mathcal{F}_c$ is isomorphic to a cone in an infinite-dimensional space, where the analogous condition is known not to be sufficient in general. There are other instances where the conditions in limit theorems for random u.s.c. functions on $\mathbf{R}^d$ are those of $\mathbf{R}^d$, rather than an infinite-dimensional Banach space. For instance, the central limit theorem in [14] doesn't need the metric entropy condition.

We finally mention that they also announce a Sanov type theorem, which does not have the second limitation above.

References

[1] M. A. Arcones (2003). Large deviations for empirical processes. In: *High dimensional probability III* (J. Hoffmann-Jørgensen, M. B. Marcus, J. A. Wellner, eds.), 205–223. Birkhäuser, Boston. MR2033890 (2005a:60035)

[2] C. Castaing, M. Valadier (1977). *Convex analysis and measurable multifunctions*. Lecture Notes Math. **580**. Springer, Berlin. MR0467310 (57:7169)

[3] R. Cerf (1999). Large deviations for sums of i.i.d. random compact sets. *Proc. Amer. Math. Soc.* **127**, 2431–2436. MR1487361 (99j:60016)

[4] A. Colubi, J. S. Domínguez Menchero, M. López-Díaz, D. Ralescu (2002). A $D_E[0,1]$ representation of random upper semicontinuous functions. *Proc. Amer. Math. Soc.* **130**, 3237–3242. MR1913001 (2003j:49025)

[5] E. Giné, M. G. Hahn, J. Zinn (1983). Limit theorems for random sets: an application of probability in Banach space results. In: *Probability in Banach spaces, IV (Oberwolfach, 1982)*, 112–135. Lecture Notes in Math. **990**, Springer, Berlin. MR0707513 (85d:60019)

[6] F. Hiai, H. Umegaki (1977). Integrals, conditional expectations, and martingales of multivalued functions. *J. Multivar. Anal.* **7**, 149–182. MR0507504 (58:22463)

[7] E. P. Klement, M. L. Puri, D. A. Ralescu (1986). Limit theorems for fuzzy random variables. *Proc. Roy. Soc. London Ser. A* **407**, 171–182. MR0861082 (88b:60092)

[8] S. Li, Y. Ogura, F. N. Proske, M. L. Puri (2003). Central limit theorems for generalized set-valued random variables. *J. Math. Anal. Appl.* **285**, 250–263. MR2000152 (2004j:60050)

[9] G. Matheron (1975). *Random sets and integral geometry*. Wiley, New York. MR0385969 (52:6828)

[10] I. S. Molchanov (1999). On strong laws of large numbers for random upper semicontinuous functions. *J. Math. Anal. Appl.* **235**, 349–355. MR1758687 (2001c:60050)

[11] I. S. Molchanov. *Theory of random sets*. To appear, Springer, Berlin.

[12] Y. Ogura, S. Li On limit theorems for random fuzzy sets including large deviation principles. In: *Soft methodology and random information systems* (M. López-Díaz, M. A. Gil, P. Grzegorzewski, O. Hryniewicz, J. Lawry, editors), 32–44. Springer, Berlin. MR2118078

[13] F. N. Proske (1997). *Grenzwertsätze für Fuzzy-Zufallsvariablen unter dem Gesichspunkt der Wahrscheinlichkeitstheorie auf inseparablen Semigruppen* [*Limit theorems for fuzzy random variables from the viewpoint of probability theory in non-separable semigroups*, in German]. Ph. D. Thesis, Universität Ulm.

[14] F. N. Proske, M. L. Puri (2002). Central limit theorem for Banach space valued fuzzy random variables. *Proc. Amer. Math. Soc.* **130**, 1493–1501. MR1879975 (2003c:28022)

[15] F. N. Proske, M. L. Puri (2003). A strong law of large numbers for generalized random sets from the viewpoint of empirical processes. *Proc. Amer. Math. Soc.* **131**, 2937–2944. MR1974352 (2004j:60071)

[16] M. L. Puri, D. A. Ralescu (1985). Limit theorems for random compact sets in Banach space. *Math. Proc. Cambridge Phil. Soc.* **97**, 151–158. MR0764504 (86c:60024)

[17] M. L. Puri, D. A. Ralescu (1986). Fuzzy random variables. *J. Math. Anal. Appl.* **114**, 409–422. MR0833596 (87f:03159)

[18] P. Terán (2005). An embedding theorem for convex fuzzy sets. *Fuzzy Sets and Systems* **152**, 191–208.

[19] P. Terán (2004). On Borel measurability and large deviations for fuzzy random variables. *Fuzzy Sets and Systems*, to appear.

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