NON-COMMUTATIVE METRIC TOPOLOGY
ON MATRIX STATE SPACE

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Abstract. We present an operator space version of Rieffel’s theorem on the agreement of the metric topology, on a subset of the Banach space dual of a normed space, from a seminorm with the weak*-topology. As an application we obtain a necessary and sufficient condition for the matrix metric from an unbounded Fredholm module to give the BW-topology on the matrix state space of the C*-algebra. Motivated by recent results we formulate a non-commutative Lipschitz seminorm on a matrix order unit space and characterize those matrix Lipschitz seminorms whose matrix metric topology coincides with the BW-topology on the matrix state space.

1. Introduction

One basic idea in modern analysis is that C*-algebras are non-commutative C(K) spaces [7, 12]. The basic idea of non-commutative functional analysis is to study operator spaces as a generalization of Banach spaces [6, 18, 9]. Let A be a unital C*-algebra. Connes has shown that an appropriate way to specify a Riemannian metric in this non-commutative situation is by means of a spectral triple [5]. This consists of a representation of A on a Hilbert space H, together with the generalized Dirac operator D on H satisfying certain conditions. The Lipschitz seminorm, L, is defined on the set L(A) of Lipschitz elements of A by the operator norm L(a) = ||[D, a]||. From this, Connes defines a metric ρ on the state space of A by

\[ ρ(μ, ν) = \sup\{|μ(a) - ν(a)| : a ∈ L(A), L(a) ≤ 1\} \]

A natural question is when the metric topology from the metric ρ agrees with the underlying weak*-topology on the state space of A.

In [10], Rieffel defined a metric on a subset of the Banach space dual in a very rudimentary Banach space setting and gave necessary and sufficient conditions under which the topology from the metric agrees with the weak*-topology. This problem has also been considered by Pavlović [15]. For an operator with domain contained in a unital C*-algebra, range contained in a normed space and kernel C1, she defined a metric on the state space of the C*-algebra from the operator and then obtained the necessary and sufficient conditions for the metric to give the weak*-topology. The theory of operator spaces naturally leads to the problem of
whether we may replace the Banach space by an operator space. The main purpose of this paper is to extend these results to the category of operator spaces.

The paper is organized as follows. We begin in section 2 with the explanation of terminology and notation. In section 3 we present the operator space version of Rieffel’s theorem and in section 4 we use it to the matrix metrics from Dirac operators. Finally in section 5 we define a non-commutative version of Lipschitz seminorms on matrix order unit spaces and then use the results of the previous sections to determine when the metric topology from a matrix Lipschitz seminorm agrees with the BW-topology on the matrix state space.

2. TERMINOLOGY AND NOTATION

All vector spaces are assumed to be complex throughout this paper. Given a vector space $V$, let $M_{m,n}(V)$ denote the matrix space of all $m$ by $n$ matrices $v = [v_{ij}]$ with $v_{ij} \in V$, and we set $M_n(V) = M_{n,n}(V)$. If $V = \mathbb{C}$, we write $M_{m,n} = M_{m,n}(\mathbb{C})$ and $M_n = M_{n,n}(\mathbb{C})$, which means that we may identify $M_{m,n}(V)$ with the tensor product $M_{m,n} \otimes V$. We identify $M_{m,n}$ with the normed space $B(\mathbb{C}^n, \mathbb{C}^m)$. We use the standard matrix multiplication and *-operation for compatible scalar matrices. For other terminology and notation, we will follow [3] and [10].

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3. MATRIX METRICS ON MATRIX STATES

Let $V$ be an operator space with the matrix norm $\| \cdot \| = (\| \cdot \|_n)$, and let $W$ be a subspace of $V$. As in [16], we will not assume that $W$ is closed. Suppose that $L = (L_n)$ is a matrix seminorm on $W$ and $\varphi^{(0)}(L)$ is a bounded linear functional on the subspace $K_1 = \{ a \in W : L_1(a) = 0 \}$ with $\| \varphi^{(0)} \| = 1$. Set $\varphi^{(0)} = \varphi^{(0)}_1 \oplus \cdots \oplus \varphi^{(0)}_n$.

$n \in \mathbb{N}$. Then $(\varphi^{(0)}_n)$ is a sequence of completely bounded linear mappings $\varphi^{(0)}_n : K_1 \mapsto M_n$, with $\| \varphi^{(0)}_n \|_{cb} = 1$. Here $\| \varphi^{(0)}_n \|_{cb}$ is the completely bounded norm of $\varphi^{(0)}_n$. In the applications, $K_1$ will always be closed in $V$. Thus we assume that $K_1$ is closed in $V$. Denote $K_n = \{ a \in M_n(W) : L_n(a) = 0 \}$ for $n \in \mathbb{N}$, and let $K = (K_n)$. The inequality $L_1(v_{ij}) \leq L_n([v_{ij}]) \leq \sum_{i,j=1}^n L_1(v_{ij})$ holds for $[v_{ij}] \in M_n(W)$ and $n \in \mathbb{N}$ [10]. So $K_n = M_n(K_1)$ for $n \in \mathbb{N}$.
Denote $GCS_n(V) = \{ \varphi : \varphi \text{ is a completely bounded linear mapping from } V \text{ to } M_n, \varphi = \varphi_n \}$ on $K_1$, and $\|\varphi\|_{cb} = 1$. We write $CB_n(V) = \{ \varphi : \varphi \text{ is a completely bounded linear mapping from } V \text{ to } M_n \}$, and $CB_n^1(V) = \{ \varphi \in CB_n(V) : \|\varphi\|_{cb} \leq 1 \}$, for $n \in \mathbb{N}$. Then $GCS_n(V)$ is a convex subset of $CB_n(V)$. Define $S : M_n(V^*) \to CB_n(V)$ by $(S(p_{ij}))(a) = [p_{ij}(a)]$, where $[p_{ij}] \in M_n(V^*)$ and $a \in V$. Then $S$ is a linear isomorphism from $M_n(V^*)$ onto $CB_n(V)$. As a linear mapping from operator space $M_n(V^*)$ onto operator space $CB_n(V)$, $S$ is a complete isometry \[ \|\varphi\|_{cb} = \|S(\varphi)\|_{cb} \]. On $GCS(V)$, we can define a sequence $D_L = (D_{L_n})$ of metrics by

$$D_{L_n}(\varphi, \psi) = \sup\{\|\langle \varphi, a \rangle - \langle \psi, a \rangle\| : a \in M_n(W), L_n(a) \leq 1, r \in \mathbb{N}\},$$

for $\varphi, \psi \in GCS_n(V)$. (These may take the value $+\infty$.) We call $D_L$ the matrix metric from $L$. We will refer to the topology on $GCS(V)$ defined by $D_L$, as the $D_L$-topology or the matrix metric topology. The natural topology on $GCS(V)$ is the $BW$-topology; that is, the topologies on each $GCS_n(V)$ are induced by the $BW$-topology. We say that $W$ separates the points of $GCS(V)$ if given $\varphi, \psi \in GCS_n(V)$ and $\varphi \neq \psi$ for $n \in \mathbb{N}$ there is an $a \in W$ such that $\varphi(a) \neq \psi(a)$.

**Proposition 3.1.** For $\varphi, \psi \in GCS_n(V)$,

$$D_{L_n}(\varphi, \psi) = \sup\{\|\langle \varphi, a \rangle - \langle \psi, a \rangle\| : a \in M_n(W), L_n(a) \leq 1\}.$$  

**Proof.** Given $\varphi, \psi \in GCS_n(V)$, let $K_{L_n}(\varphi, \psi) = \sup\{\|\langle \varphi, a \rangle - \langle \psi, a \rangle\| : a \in M_n(W), L_n(a) \leq 1\}$. Clearly, $K_{L_n}(\varphi, \psi) \leq D_{L_n}(\varphi, \psi)$. For any $a \in M_n(W)$, we have that $\|\langle \varphi, a \rangle - \langle \psi, a \rangle\| \leq K_{L_n}(\varphi, \psi) L_n(a)$. Since $L$ is a matrix gauge on $W$, we have that $\|\langle \varphi, a \rangle - \langle \psi, a \rangle\| \leq K_{L_n}(\varphi, \psi) L_n(a)$ for all $a \in M_r(W)$ with $r \in \mathbb{N}$ arbitrary by Lemma 5.2 in \[ \|\varphi - \psi\| \leq K_{L_n}(\varphi, \psi) \]. So $D_{L_n}(\varphi, \psi) = \sup\{\|\langle \varphi, a \rangle - \langle \psi, a \rangle\| : a \in M_r(W), L_r(a) \leq 1, r \in \mathbb{N}\} \leq K_{L_n}(\varphi, \psi)$. Therefore, $K_{L_n} = D_{L_n}$. \qed

**Proposition 3.2.** $GCS(V)$ is $BW$-compact.

**Proof.** This follows immediately from the observation that $GCS_n(V) = \bigcap_{k \in K_1} \{ \varphi \in CB_n^1(V) : \varphi(k) = \varphi_n(k) \}$ and the fact that $CB_n^1(V)$ is $BW$-compact \[ \|\varphi - \psi\| \leq K_{L_n}(\varphi, \psi) \]. \qed

**Proposition 3.3.** If $W$ separates the points of $GCS(V)$, then the $D_L$-topology on $GCS(V)$ is finer than the $BW$-topology.

**Proof.** Let $\{\varphi_k\}$ be a sequence in $GCS_n(V)$ which converges to $\varphi \in GCS_n(V)$ for the metric $D_{L_n}$. Then for any $r \in \mathbb{N}$ and $a \in M_r(W)$ with $L_r(a) \leq 1$, we have $\lim_{k \to \infty} \|\langle \varphi_k, a \rangle - \langle \varphi, a \rangle\| = 0$. So, for every $r \in \mathbb{N}$ and $a \in M_r(W)$, $\{\langle \varphi_k, a \rangle\}$ converges to $\langle \varphi, a \rangle$ in the norm topology, and hence in the weak operator topology.

Denote $\phi(a, \xi, \eta)(\psi) = \langle \psi, a, \xi, \eta \rangle$ for $a \in M_r(W), \psi \in GCS_n(V), \xi, \eta \in \mathbb{C}^n$ and $r, n \in \mathbb{N}$. The paragraph above shows that $\phi(a, \xi, \eta)(\varphi_k)$ converges to $\phi(a, \xi, \eta)(\varphi)$ for any $a \in M_r(W), \xi, \eta \in \mathbb{C}^n$ and $r \in \mathbb{N}$. Let $Q_{r,n} = \{\phi(a, \xi, \eta) : a \in M_r(W), \xi, \eta \in \mathbb{C}^n\}$ and $Q_n = \bigcup_{r=1}^{\infty} Q_{r,n}$. Then $Q_n$ is a linear space of $BW$-continuous functions on $GCS_n(V)$. Since $W$ separates the points of $GCS(V)$, $Q_n$ separates the points of $GCS_n(V)$. For $a \in K_1$ and $\xi, \eta \in \mathbb{C}^n$, we have that $\phi(a, \xi, \eta)(\varphi) = \langle \psi, a, \xi, \eta \rangle = \langle \psi_n, a \rangle \xi, \eta \rangle$, where $\varphi \in GCS_n(V)$. So $Q_n$ contains the constant functions. $GCS_n(V)$ $BW$-compact implies that $Q_n$ determines the $BW$-topology of $GCS_n(V)$. Therefore, $\{\varphi_k\}$ converges to $\varphi$ in the $BW$-topology. \qed
For $a \in M_n(W)$, define $\|a\|_{\tilde{n}}^2 = \sup \{ \| \langle \varphi, a \rangle \| : \varphi \in GCS_n(V) \}$. Then we obtain a sequence $\| \cdot \|^2 = (\| \cdot \|_n^2)$ of seminorms, and $\|a\|_{\tilde{n}}^2 \leq \|a\|_n$ for $a \in M_n(W)$.

**Proposition 3.4.** If $\| \cdot \|^2$ is a matrix seminorm on $W$, then

$$\|a\|_{\tilde{n}}^2 = \sup \{ \| \langle \varphi, a \rangle \| : \varphi \in GCS_n(V), r \in \mathbb{N} \}.$$ 

**Proof.** Denote $\|a\|_{\tilde{n}}^2 = \sup \{ \| \langle \varphi, a \rangle \| : \varphi \in GCS_n(V), r \in \mathbb{N} \}$. Clearly, $\|a\|_{\tilde{n}}^2 \leq \|a\|_n^2$. Given $\varphi \in GCS_n(V)$, if $r < n$, we choose a $\varphi_1 \in GCS_{n-r}(V)$ and then obtain that

$$\| \langle \varphi, a \rangle \| \leq \max \{ \| \langle \varphi, a \rangle \|, \| \langle \varphi_1, a \rangle \| \} = \| \langle \varphi \oplus \varphi_1, a \rangle \| \leq \|a\|_{\tilde{n}}^2. \quad \text{When } r > n, \text{ we have that } \| \langle \varphi, a \rangle \| = \| \langle \varphi, a \oplus 0_{n-r} \rangle \| \leq \|a\|_{\tilde{n}}^2, \text{ since } \| \cdot \|^2 \text{ is a matrix seminorm. Hence } \|a\|_{\tilde{n}}^2 \leq \|a\|_n^2, \text{ and this, with the previous inequality, gives that } \|a\|_{\tilde{n}}^2 = \sup \{ \| \langle \varphi, a \rangle \| : \varphi \in GCS_n(V), r \in \mathbb{N} \}. \qed$$

Let $\tilde{W} = W/K_1$ and $\tilde{V} = V/K_1$. Then $L$ drops to a matrix norm on $\tilde{W}$ [10], which we denote by $L = (L_\circ)$. But on $\tilde{W}$ and $\tilde{V}$ we also have the quotient matrix norm from $\| \cdot \|$ on $V$ [9], which we denote by $\| \cdot \|_\circ$ = $(\| \cdot \|^\circ)$. The image in $M_n(\tilde{W})$ of $a \in M_n(W)$ will be denoted by $\tilde{a}$. For each $n \in \mathbb{N}$, we let $L_n = \{ a \in M_n(W) : L_n(a) \leq t \}$ and $B_n = \{ a \in M_n(W) : L_n(a) \leq 1, \|a\|_n \leq t \}$ for $t > 0$, and set $L^t = (L^t_n)$ and $B^t = (B^t_n)$. We say that the image of a graded set $E = (E_n)$ $(E_n \subseteq M_n(W))$ in $\tilde{W}$ is totally bounded for $\| \cdot \|_\circ$ if each $E_n$ is totally bounded in $M_n(\tilde{V})$ for $\| \cdot \|_\circ$.

**Definition 3.5.** Let $V$ be a vector space, and let $\| \cdot \| = (\| \cdot \|_n)$ and $\| \cdot \|^2 = (\| \cdot \|^2_n)$ be two matrix norms on $V$. We say that $\| \cdot \|$ and $\| \cdot \|^2$ are equivalent if for any $n \in \mathbb{N}$, $\| \cdot \|_n$ is equivalent to $\| \cdot \|^2_n$.

**Proposition 3.6.** Suppose that $\| \cdot \| = (\| \cdot \|_n)$ and $\| \cdot \|^2 = (\| \cdot \|^2_n)$ are two matrix norms on the vector space $V$. If $\| \cdot \|_1$ and $\| \cdot \|^2_1$ are equivalent, then $\| \cdot \|$ and $\| \cdot \|^2$ are equivalent.

**Proof.** Since $\| \cdot \|_1$ and $\| \cdot \|^2_1$ are equivalent, there are positive constants $\alpha$ and $\beta$ such that $\alpha \|x\|_1^2 \leq \|x\|_1^\circ \leq \beta \|x\|_1^2$ for $x \in V$. Because $\| \cdot \|_1$ and $\| \cdot \|^2_1$ are matrix norms, we have $\|x_{ij}\|_1 \leq \|x_{ij}\|_n \leq \sum_{j=1}^n \|x_{ij}\|_1$ and $\|x_{ij}\|_1^2 \leq \|x_{ij}\|_n^2 \leq \sum_{j=1}^n \|x_{ij}\|_1^2$ for $x_{ij} \in M_n(V)$ [18]. Hence $\alpha^{-\frac{1}{2}}\|x_{ij}\|_n^2 \leq \|x_{ij}\|_n \leq n^2 \beta \|x_{ij}\|_1^2$. So $\| \cdot \|_n$ and $\| \cdot \|^2_n$ are equivalent. Therefore, $\| \cdot \|$ and $\| \cdot \|^2$ are equivalent. \qed

**Theorem 3.7.** Let $V$ be an operator space with the matrix norm $\| \cdot \| = (\| \cdot \|_n)$, and let $W$ be a subspace of $V$, which is not necessarily closed. Suppose that $L = (L_n)$ is a matrix seminorm on $W$ with closed (in $V$) kernels $K_n = \{ a \in M_n(W) : L_n(a) = 0 \} (n \in \mathbb{N})$ and $(\varphi^{(0)}_n)$ is a sequence of completely bounded linear mappings $\varphi^{(0)}_n : K_1 \rightarrow M_n$, with $\varphi^{(0)}_n = \varphi^{(0)}_1 + \cdots + \varphi^{(0)}_n$ and $\|\varphi^{(0)}_n\|_{cb} = 1$. Denote $GCS_n(V) = \{ \varphi : \varphi$ is a completely bounded linear mapping from $V$ to $M_n, \varphi = \varphi^{(0)}_n$ on $K_1$, and $\|\varphi\|_{cb} = 1 \}$ and $\|a\|_n^2 = \sup \{ \| \langle \varphi, a \rangle \| : \varphi \in GCS_n(V), a \in M_n(W) \}$. Set $GCS(V) = (GCS_n(V))$ and $\| \cdot \|^2 = (\| \cdot \|_n^2)$. We assume that $W$ separates the points of $GCS(V)$. 

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(1) Suppose that $\| \cdot \|_2$ is a matrix norm on $W$ and that it is equivalent to the matrix norm $\| \cdot \|$. If the $D_L$-topology on $GCS(V)$ agrees with the BW-topology, then the image of $L^1$ in $\hat{W}$ is totally bounded for $\| \cdot \|_\infty$.

(2) If the image of $L^1$ in $\hat{W}$ is totally bounded for $\| \cdot \|_\infty$, then the $D_L$-topology on $GCS(V)$ agrees with the BW-topology.

Proof. (1) If $\| \cdot \|_2$ is equivalent to $\| \cdot \|$, then $\| \cdot \|_2$ is equivalent to $\| \cdot \|_1$. When $n = 1$, $GCS_1(V)$ is just the space $S$ in [1]. Since the $D_{L_1}$-topology agrees with the BW-topology on $GCS_1(V)$, the image of $L^1$ in $\hat{W}$ is totally bounded for $\| \cdot \|_\infty$ by Theorem 1.8 in [10] and Proposition 1.1. So given $\epsilon > 0$, there exist elements $a_1, \ldots, a_k \in L^1$ such that $\tilde{L}^1_1 \subseteq \bigcup_{j=1}^k \{ \tilde{a} \in \hat{W} : \| \tilde{a} - \tilde{a}_j \|_\infty < \epsilon \}$. For a matrix gauge $\mathcal{G} = (g_n)$ on a vector space $Y$ and $v = [v_{ij}] \in M_n(Y)$, we have the constraint $g_1(\psi_{ij}) \leq g_n(v) \leq \sum_{i,j=1}^n g_1(\psi_{ij})$. From the first inequality, we get that $L^1_n \subseteq M_n(L^1_1)$.

Denote $X_n = \{ a \in M_n(W) : a = [a_{ij}], a_{ij} \in \{a_1, \ldots, a_k\} \}$. Then $X_n$ is a finite set for every $n \in \mathbb{N}$. Now for $b = [b_{ij}] \in L^1_n$, there exists an $a = [a_{ij}] \in X_n$ such that $\| b_{ij} - a_{ij} \|_\infty < \epsilon$. So $\| \tilde{b} - \tilde{a} \|_\infty \leq \sum_{i,j=1}^n \| b_{ij} - a_{ij} \|_\infty < n^2 \epsilon$. Therefore, the image of $L^1_n$ in $M_n(\hat{W})$ is totally bounded for $\| \cdot \|_\infty$. Consequently, the image of $L^1$ in $\hat{W}$ is totally bounded for $\| \cdot \|_\infty$.

(2) Suppose that the image of $L^1$ in $\hat{W}$ is totally bounded for $\| \cdot \|_\infty$. From the proof of (1), we see that the image of $L^1$ in $\hat{W}$ is totally bounded for $\| \cdot \|_\infty$. For any $n \in \mathbb{N}$, let $\varphi \in GCS_n(V)$ and $\epsilon > 0$ be given, and let $B(\varphi, \epsilon)$ be the $D_{L_n}$-ball of radius $\epsilon$ at $\varphi$. By the total boundedness of $L^1_n$, we can find elements $a_1, a_2, \ldots, a_k \in L^1_n$ such that the $\| \cdot \|_\infty$-balls of radius $\frac{\epsilon}{n}$ at the $\tilde{a}_j$'s cover $L^1_n$.

Assume that $\varphi = [\varphi_{pq}]$ and $a_j = [a_{ij}^{(j)}]$ for $j \in \{1, 2, \ldots, k\}$. Let $Q = \{ \psi = [\psi_{ij}] \in GCS_n(V) : \| \psi_{pq}(a_{ij}^{(j)}) - \varphi_{pq}(a_{ij}^{(j)}) \| < \frac{\epsilon}{n}, s, t, p, q \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, k\} \}$. Then $Q$ is BW-open and $\varphi \in Q$. Set $N = \{ \psi \in GCS_n(V) : \| \| \varphi - \psi \|, a_j \|_\infty \| < \frac{\epsilon}{n}, 1 \leq j \leq k \}$. Then $Q \subseteq N$ and so $N$ is a BW-neighborhood of $\varphi$. For any $a \in L^1_n$, there are $a \in \{1, 2, \ldots, k\}$ and a $b \in K_n$ such that $\| a - (a_j + b) \|_\infty < \frac{\epsilon}{n}$. For any $\psi \in N$, we have that $\| \| \varphi - \psi, a_j \|_\infty \| \leq \frac{\epsilon}{n} \| a - (a_j + b) \|_\infty + \| \| \varphi - \psi, a_j \|_\infty \| + \| \varphi_{cb} \| (a_j + b - a) \|, a \|_\infty < \frac{\epsilon}{n}$. By Lemma 5.2 in [10], we obtain $\| \| \varphi - \psi, a \|_\infty \| \leq \frac{\epsilon}{n}$ for all $a \in L^1_n$ with $r \in \mathbb{N}$ arbitrary. Thus $D_{L_n}(\varphi, \psi) \leq \frac{\epsilon}{n}$. Consequently $N \subseteq B(\varphi, \epsilon)$. Since $W$ separates the points of $GCS(V)$, by Proposition 3.8 the $D_{L_n}$-topology on $GCS_n(V)$ is finer than the BW-topology. Therefore, the $D_{L_n}$-topology on $GCS_n(V)$ agrees with the BW-topology. We conclude that the $D_L$-topology on $GCS(V)$ agrees with the BW-topology. □

For a sequence $\{r_n\}$ of positive constants, we set $CB^n_{r_n}(V) = \{ f \in M_n((V)^*) : \| f \|_{cb} \leq r_n \}$, and this is just the subset of $M_n(V^*)$ consisting of those $f \in M_n(V^*)$ such that $\| f \|_{cb} \leq r_n$ and $f(k) = 0$ for $k \in K_1$. If $CB^n_{r_n}(V) \subseteq (GCS_n(V) - GCS_n(V)) + i(GCS_n(V) - GCS_n(V))$ for each $n \in \mathbb{N}$, we say that $V$ has the matrix $\{r_n\}$-decomposable property about $K$; if $r_n = r$ for all $n \in \mathbb{N}$, we say that $V$ has the matrix $r$-decomposable property about $K$.

Proposition 3.8. If $V$ has the matrix $\{r_n\}$-decomposable property about $K$ and $D_L$ is bounded, that is, each $D_{L_n}$ is bounded, then there is a sequence $\{d_n\}$ of positive constants such that $\| \tilde{a} \|_\infty \leq d_n$ for all $a \in L^1_n$ and $n \in \mathbb{N}$. 
Proof. Assume that each $D_{L_n}$ is bounded by $c_n$. Then for $a \in M_n(W)$ with $L_n(a) \leq 1$, we have that
\[
\|\tilde{a}\|_n^\sim = \sup \{ \| \langle f, \tilde{a} \rangle \| : f \in CB_n(\tilde{V}), \| f \|_{cb} \leq 1 \} 
\leq r_n^{-1} \sup \{ \| \langle (\varphi_1 \varphi_2 + i(\varphi_3 \varphi_4), \tilde{a} \rangle \| : \varphi_i \in GCS_n(V), i = 1, 2, 3, 4 \}
\leq 2r_n^{-1} \sup \{ \| \langle \varphi_1 \varphi_2, \tilde{a} \rangle \| : \varphi_1, \varphi_2 \in GCS_n(V) \}
= 2r_n^{-1} \sup \{ \| \langle \varphi_1, a \rangle - \langle \varphi_2, a \rangle \| : \varphi_1, \varphi_2 \in GCS_n(V) \}
\leq 2r_n^{-1} \sup \{ D_{L_n}(\varphi_1, \varphi_2) : \varphi_1, \varphi_2 \in GCS_n(V) \} \leq 2c_n r_n^{-1}
\]
by Theorem 4.2 in [18] and Lemma 2.1 in [8]. Letting $d_n = 2c_n r_n^{-1}$, we have that $\|\tilde{a}\|_n^\sim \leq d_n$ for all $a \in L_n^1$. \hfill \Box

Here, we give another additional premise such that for the $D_{\mathcal{L}}$-topology on $GCS(V)$ to agree with the BW-topology it is necessary that the image of $L^1$ in $W$ is totally bounded for $\| \cdot \|^\sim$.

**Theorem 3.9.** With the notation of Theorem 3.7, suppose that $V$ has the matrix property $\{r_n\}$-decomposable property about $K = (K_n)$. If the $D_{\mathcal{L}}$-topology on $GCS(V)$ agrees with the BW-topology, then the image of $L^1$ in $W$ is totally bounded for $\| \cdot \|^\sim$.

Proof. If each $D_{L_n}$-topology agrees with the BW-topology on $GCS_n(V)$, then each $D_{L_n}$ must be bounded since $GCS_n(V)$ is BW-compact. By Proposition 3.8, there exists a sequence $\{d_n\}$ of positive constants such that $\|\tilde{a}\|_n^\sim \leq d_n$ for $a \in L_n^1$ and $n \in \mathbb{N}$. Choose $k_n > d_n$. Then $\|\tilde{a}\|_n^\sim < k_n$ if $a \in L_n^1$. So the image of $B_{k_n}^n$ in $M_n(\tilde{W})$ is the same as the image of $L_n^1$. For $r, n \in \mathbb{N}$, $a \in L_n^1$ and $\varphi, \psi \in GCS_n(V)$, we have that $\|\langle \varphi, a \rangle - \langle \psi, a \rangle \| \leq D_{L_n}(\varphi, \psi)$. Let $\epsilon > 0$. For each $\varphi \in GCS_n(V)$, let $U_\varphi$ be an open neighborhood of $\varphi$ such that $\|\langle \varphi, a \rangle - \langle \psi, a \rangle \| < \frac{\epsilon}{2}$ for $a \in L_n^1(n \in \mathbb{N})$, and $\psi \in U_\varphi$. Now, $\{U_\varphi : \varphi \in GCS_n(V)\}$ is an open covering of $GCS_n(V)$. Since $GCS_n(V)$ is BW-compact, there are points $\varphi_1, \ldots, \varphi_k$ in $GCS_n(V)$ such that $GCS_n(V) = \bigcup_{j=1}^k U_{\varphi_j}$. Assume that $\varphi_j = [\varphi_{st}^j], j = 1, \ldots, k$, and denote $D = \{ a : \tilde{a} = (\varphi_{11}^j(a_{11}), \varphi_{11}^j(a_{12}), \ldots, \varphi_{11}^j(a_{nm}), \varphi_{12}^j(a_{11}), \varphi_{12}^j(a_{12}), \ldots, \varphi_{12}^j(a_{nm}), \ldots, \varphi_{nn}^j(a_{nn}), \varphi_{nn}^j(a_{nn}), \varphi_{nn}^j(a_{nn}), \varphi_{nn}^j(a_{nn}), \varphi_{nn}^j(a_{nn}), \varphi_{nn}^j(a_{nn}), \varphi_{nn}^j(a_{nn}), \varphi_{nn}^j(a_{nn}), \varphi_{nn}^j(a_{nn}), a = [a_{ij}] \in B_n^k \}$. For $a \in B_n^k$, we have
\[
\| \tilde{a} \| = \sqrt{\sum_{j=1}^k \sum_{s,t=1}^n \sum_{r,l=1}^n |\varphi_{st}^j(a_{rl})|^2} 
= \sum_{j=1}^k \sum_{s,t=1}^n \sum_{r,l=1}^n |e_\varphi^j(\varphi_j, a)| e_\varphi^j|^2 
\leq \sum_{j=1}^k \sum_{s,t=1}^n \sum_{r,l=1}^n \| \varphi_j \|_{cb}^2 \| a \|_n^2 
\leq \sum_{j=1}^k \sum_{s,t=1}^n \sum_{r,l=1}^n \| \varphi_j \|_{cb}^2 \| a \|_n^2 \leq n^2 k_n \sqrt{k},
\]
where $e_\varphi^j$ is the $nr \times 1$ column matrix $[0, \ldots, 0, 1_{n+j}, 0, \ldots, 0]^\ast$. So $D$ is totally bounded, and hence there are elements $a_1, a_2, \ldots, a_m \in B_n^k$ such that $D \subseteq \bigcup_{j=1}^m \{ a : \| \tilde{a} - \tilde{a}_j \| < \frac{\epsilon}{3n^2}, a \in B_n^k \}$. Now for $a \in B_n^k$, there exists a $p \in \{1, 2, \ldots, m\}$ such that $\| \tilde{a} - \tilde{a}_p \| < \frac{\epsilon}{4}$. For any $\varphi \in GCS_n(V)$, there is a $j \in \{1, 2, \ldots, k\}$ such that $\varphi \in U_{\varphi_j}$. So by Cauchy’s inequality, we have
\[
\| \langle a - a_p, \varphi \rangle \| \leq \| \langle a - a_p, \varphi_j \rangle \| + \| \langle a - a_p, \varphi - \varphi_j \rangle \| 
\leq n^2 \| \tilde{a} - \tilde{a}_p \| + \| \langle a, \varphi - \varphi_j \rangle \| + \| \langle a_p, \varphi - \varphi_j \rangle \| < \epsilon.
\]
By Lemma 2.1 in [8], we have that
\[ \| \tilde{a} - \tilde{a}_p \|_{N} = \sup \{ \| \langle \tilde{a} - \tilde{a}_p, f \rangle \| : f \in M_n(\tilde{V}), \| f \|_{cb} \leq 1 \} \]
\[ = r^{-1}_n \sup \{ \| \langle \tilde{a} - \tilde{a}_p, (\varphi_1 - \varphi_2) + i(\varphi_3 - \varphi_4) \rangle \| : \varphi_i \in GCS_n(V), i = 1, 2, 3, 4 \} \]
\[ \leq 2r^{-1}_n \sup \{ \| \langle \tilde{a} - \tilde{a}_p, \varphi_1 - \varphi_2 \rangle \| : \varphi_1, \varphi_2 \in GCS_n(V) \} \]
\[ = 2r^{-1}_n \sup \{ \| \langle a - a_p, \varphi_1 - \varphi_2 \rangle \| : \varphi_1, \varphi_2 \in GCS_n(V) \} \]
\[ \leq 4r^{-1}_n \sup \{ \| \langle a - a_p, \varphi \rangle \| : \varphi \in GCS_n(V) \} \]
\[ \leq 4r^{-1}_n \epsilon. \]

Therefore, the image of \( B_{n}^{k} \) in \( M_n(\tilde{V}) \) is totally bounded for \( \| \cdot \|_{N} \); then so is \( L_{1}^{k} \).

From the arbitrariness of \( n \), the image of \( L_{1} \) in \( \tilde{W} \) is totally bounded for \( \| \cdot \|_{*} \). \( \square \)

4. Matrix metrics from Dirac operators

Let \( \mathcal{A} \) be a unital \( C^{*} \)-algebra. An unbounded Fredholm module \((\mathcal{H}, D)\) over \( \mathcal{A} \) is [4][9]: a Hilbert space \( \mathcal{H} \) which is a left \( \mathcal{A} \)-module, that is, a Hilbert space \( \mathcal{H} \) and a \( \ast \)-representation of \( \mathcal{A} \) on \( \mathcal{H} \); an unbounded, self-adjoint operator \( D \) (the generalized Dirac operator) on \( \mathcal{H} \) such that \( \mathcal{B} = \{ a \in \mathcal{A} : [D, a] \) densely defined and extends to a bounded operator on \( \mathcal{H} \) \} is norm dense in \( \mathcal{A} \); \( (1 + D^2)^{-1} \) is a compact operator. Given a triple \((\mathcal{A}, \mathcal{H}, D)\), where \((\mathcal{H}, D)\) is an unbounded Fredholm module over a unital \( C^{*} \)-algebra \( \mathcal{A} \), then \( \mathcal{B}(\mathcal{H}) \) is a Banach \( \mathcal{A} \)-module, and the mapping \( T : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \), defined by \( Ta = [D, a] \), is a densely defined derivation from \( \mathcal{A} \) into \( \mathcal{B}(\mathcal{H}) \). For \( n \in \mathbb{N} \), we define \( L_{n}(a) = \| T_{n}(a) \|, a \in M_{n}(\mathcal{B}), \) and denote \( \mathcal{L} = (L_{n}) \). For the unital \( C^{*} \)-algebra \( \mathcal{A} \), the matrix state space of \( \mathcal{A} \) is the collection \( \mathcal{CS}(\mathcal{A}) = (CS_{n}(\mathcal{A})) \) of matrix states \( CS_{n}(\mathcal{A}) = \{ \varphi : \varphi \) is a completely positive linear mapping from \( \mathcal{A} \) to \( M_{n}, \varphi(1) = 1 \} \), where \( 1 \) is the identity of \( \mathcal{A} \). This space is important in the study of \( C^{*} \)-algebras [19].

Proposition 4.1. \( \mathcal{L} \) is a matrix seminorm on \( \mathcal{B} \), and \( M_{n}(\mathcal{C}) \subseteq K_{n} = \{ a \in M_{n}(\mathcal{B}) : L_{n}(a) = 0 \} \).

Proof. It is clear that \( L_{n} \) is a seminorm on \( M_{n}(\mathcal{B}) \) for each \( n \in \mathbb{N} \). For \( a = [a_{ij}] \in M_{m}(\mathcal{B}), b = [b_{st}] \in M_{n}(\mathcal{B}), \alpha = [\alpha_{kl}] \in M_{m,m} \) and \( \beta = [\beta_{uv}] \in M_{m,n}, \) we have that
\[ L_{m+n}(a \oplus b) = \| T_{n}(a \oplus b) \| = \| T_{n}(a) \| + \| T_{n}(b) \| \]
\[ = \max \{\| [D, a_{ij}] \|, \| [D, b_{st}] \| \} = \max \{\| [D, a_{ij}] \|, \| [D, b_{st}] \| \} \]
\[ = \max \{\| T_{n}(a) \|, \| T_{n}(b) \| \}, \quad \forall a, b \in M_{n}(\mathcal{B}), \]
and
\[ L_{n}(aa \beta) = \| T_{n}(aa \beta) \| \]
\[ = \| T_{n}(\sum_{i,j=1}^{m} \alpha_{si}a_{ij}\beta_{j1}) \| = \| \sum_{i,j=1}^{m} \alpha_{si}T(a_{ij})\beta_{j1} \| \]
\[ = \| \alpha T_{m}(a) \| = \| \alpha \| \| [D, a_{ij}] \| \| \beta \| \leq \| \alpha \| \| [D, a_{ij}] \| \| \beta \| \]
\[ = \| \alpha \| \| T_{m}(a) \| \| \beta \| = \| \alpha \| \| L_{m}(a) \| \| \beta \|. \]

So \( \mathcal{L} \) is a matrix seminorm on \( \mathcal{B} \). For any \([a_{ij}] \in M_{1}(\mathcal{C}) \), we have that \( L_{n}([a_{ij}]) = \| T_{n}([a_{ij}]) \| = \| T([a_{ij}]) \| = \| [0] \| = 0 \), that is, \( M_{n}(\mathcal{C}) \subseteq K_{n}. \)

\( \square \)

Proposition 4.2. Let \( P_{n} = M_{n}(\mathcal{C}) \) and \( \mathcal{P} = (P_{n}) \). Then \( \mathcal{A} \) has the matrix 2-decomposable property \( \mathcal{P} \).
Proof. Given \( f \in CB_{n}^{2} (\mathfrak{A}) \), suppose that \( f \) is self-adjoint, that is, \( f^{*} = f \). By Satz 4.5 in [21] or Proposition 1.3, Remark 1.5 and Theorem 1.6 in [11] and the injectivity of \( M_{\mathfrak{A}} \), there are completely positive linear mappings \( \psi_{1} \) and \( \psi_{2} \) from \( \mathfrak{A} \) to \( M_{n} \) such that \( f = \psi_{1} - \psi_{2} \) and \( \| \psi_{1} + \psi_{2} \| = \| \psi_{1} - \psi_{2} \|_{cb} = \| f \|_{cb} \). But \( 0_{n} = f(1) = \psi_{1}(1) - \psi_{2}(1) \). So \( \psi_{1}(1) = \psi_{2}(1) \). Therefore, \( \| \psi_{1} \|_{cb} = \| \psi_{1}(1) \| = \| \psi_{2}(1) \| = \| \psi_{2} \|_{cb} = \| \psi_{2} \| \) by Proposition 3.5 in [14]. Since \( \| f \|_{cb} \leq 2 \), we have that \( \| \psi_{1}(1) + \psi_{2}(1) \| = \| \psi_{1} + \psi_{2} \| \leq 2 \). So \( 0_{n} = \psi_{1}(1) = \psi_{2}(1) \leq 1_{n} \). If \( \psi_{1}(1) \leq 1_{n} \) but \( \psi_{1}(1) \neq 1_{n} \), choosing a \( \phi \in CS_{1} (\mathfrak{A}) \) and setting \( \varphi_{1}(a) = \psi_{1}(a) + (1_{n} - \psi_{1}(1))\phi(a) \) and \( \varphi_{2}(a) = \psi_{2}(a) + (1_{n} - \psi_{2}(1))\phi(a) \) for \( a \in \mathfrak{A} \), then we have that \( \varphi_{1}(1) = \varphi_{2}(1) = 1_{n} \) and \( \varphi_{1} - \varphi_{2} = \psi_{1} - \psi_{2} \). For \( k \in \mathbb{N} \) and \( [a_{ij}] \in M_{k}(\mathfrak{A}) \), we have that

\[
\begin{align*}
(\varphi_{1})_{k} ([a_{ij}]) &= [\varphi_{1}(a_{ij})] = [\psi_{1}(a_{ij}) + (1_{n} - \psi_{1}(1))\phi(a_{ij})] \\
&= [\psi_{1}(a_{ij})] + [(1_{n} - \psi_{1}(1))\phi(a_{ij})] \\
&= (\psi_{1})_{k} ([a_{ij}]) + (\phi(a_{ij})) \otimes (1_{n} - \psi_{1}(1)) \\
&= \varphi(a_{ij}) + \phi(a_{ij}) \otimes (1_{n} - \psi_{1}(1)).
\end{align*}
\]

From this observation and the complete positivity of \( \psi_{1} \) and \( \phi \), we see that \( \varphi_{1} \in CS_{n}(\mathfrak{A}) \). Similarly, we have that \( \varphi_{2} \in CS_{n}(\mathfrak{A}) \). If \( f \) is not self-adjoint, we set \( f_{1} = \frac{1}{2} (f + f^{*}) \), \( f_{2} = \frac{1}{2} (f - f^{*}) \). Then \( f_{1} \) and \( f_{2} \) are self-adjoint completely bounded linear mappings. If \( \| f \|_{cb} \leq 2 \), then \( \| f_{1} \|_{cb} \leq 2 \) and \( \| f_{2} \|_{cb} \leq 2 \). So the result above implies that there exist \( \varphi_{1}, \varphi_{2}, \varphi_{3}, \varphi_{4} \in CS_{n}(\mathfrak{A}) \) such that \( f = (\varphi_{1} - \varphi_{2}) + i(\varphi_{3} - \varphi_{4}) \). Accordingly, \( \mathfrak{A} \) has the matrix 2-decomposable property about \( \mathcal{P} \).

It is apparent that \( C_{1} \) is a closed subspace of \( \mathfrak{A} \) and \( B/\langle C_{1} \rangle \) is an operator space with the quotient matrix norm \( \| \cdot \|_{\sim} = (\| \cdot \|_{n})^{\infty} \) from \( \| \cdot \| = (\| \cdot \|_{n}) \) on \( B \) [13].

**Theorem 4.3.** Let \( (\mathcal{H}, D) \) be an unbounded Fredholm module over a unital C*-algebra, and let the matrix metric \( D_{\mathcal{L}} \) on \( CS(\mathfrak{A}) \) be defined by

\[
D_{\mathcal{L}} (\varphi, \psi) = \sup \{ \| \langle \varphi, a \rangle - \langle \psi, a \rangle \| : a = [a_{ij}] \in M_{r}(B), L_{r}(a) = \|[D, a_{ij}]\| \leq 1, r \in \mathbb{N} \}
\]

for \( \varphi, \psi \in CS_{n}(\mathfrak{A}) \). Then the \( D_{\mathcal{L}} \)-topology on \( CS(\mathfrak{A}) \) agrees with the BW-topology if and only if the image of \( L_{1}^{1} \) in \( B/\langle C_{1} \rangle \) is totally bounded for \( \| \cdot \|_{1} \).

**Proof.** First we show that \( K_{n} = M_{n}(\mathfrak{C}_{1}) \) under the conditions of the theorem.

Suppose that the \( D_{\mathcal{L}} \)-topology on \( CS(\mathfrak{A}) \) agrees with the BW-topology. For each \( n \in \mathbb{N} \), \( CS_{n}(\mathfrak{A}) \) is BW-compact implies that \( D_{\mathcal{L}}(n) \) is bounded. If \( a = [a_{ij}] \in M_{n}(B) \setminus M_{n}(\mathfrak{C}_{1}) \) and \( L_{n}(a) = 0 \), then there exists \( a_{ij} \in B \setminus \langle C_{1} \rangle \) such that \( L_{1}(a_{ij}) = 0 \) since \( L \) is a matrix seminorm by Proposition 4.1. From the definition of \( L \), we see that \( L_{n}(a^{*}) = L_{n}(a) \) for each \( a \in M_{n}(B) \). There is a self-adjoint \( b \in B \setminus \langle C_{1} \rangle \) with \( L_{1}(b) = 0 \). So there is a bounded linear functional \( f \) on \( \mathfrak{A} \) such that \( \| f \| = 1 \), \( f(y) = 0 \) for each \( y \in C_{1} \), and \( f(b) \neq 0 \). Since \( b \) is self-adjoint, there is a bounded hermitian linear functional \( h \) on \( \mathfrak{A} \) such that \( h(1) = 0 \) but \( h(b) \neq 0 \). By the Hahn-Jordan decomposition theorem [12], there are positive linear functionals \( \varphi_{1} \) and \( \varphi_{2} \) on \( \mathfrak{A} \) such that \( h = \varphi_{1} - \varphi_{2} \) and \( \| h \| = \| \varphi_{1} \| + \| \varphi_{2} \| \). From the condition \( h(1) = 0 \), we get \( \varphi_{1}(1) = \varphi_{2}(1) \). Thus there are \( \psi_{1}, \psi_{2} \in CS_{1}(\mathfrak{A}) \) and \( c > 0 \) such that \( h = c(\psi_{1} - \psi_{2}) \). Now, for any \( r \in \mathbb{R}^{+} \) we have that \( L_{1}(rb) = 0 < 1 \). Therefore, \( D_{\mathcal{L}}(\psi_{1}, \psi_{2}) \geq \| \psi_{1}(rb) - \psi_{2}(rb) \| = \frac{c}{r} \| h(b) \| \) for all \( r \in \mathbb{R}^{+} \). This contradicts the boundedness of \( D_{\mathcal{L}} \). So \( K_{n} = M_{n}(\mathfrak{C}_{1}) \) by Proposition 4.1.
Assume that the image of $L^1$ in $B/(C1)$ is totally bounded for $\| \cdot \|$. If there is an $a = [a_{ij}] \in K_n$ and $a \notin M_n(C1)$, then $\| a \|_{\infty}^\sim \neq 0$. Since $ra \in K_n$ for any $r \in \mathbb{R}$, we get that $\| r \| a \|_{\infty}^\sim \to +\infty$ ($r \to \infty$). This contradicts the boundedness of the image of $L^1$ in $B/(C1)$. Thus we also have that $K_n = M_n(C1)$ by Proposition 4.1.

For $n \in \mathbb{N}$, let $\varphi = [\varphi_{ij}], \psi = [\psi_{ij}] \in CS_n(A)$ and $\varphi \neq \psi$. Then there exist $i_0, j_0 \in \{1, 2, \cdots, n\}$ such that $\varphi_{i_0,j_0} \neq \psi_{i_0,j_0}$. Since $B$ is norm dense in $A$ and $\varphi_{i_0,j_0}, \psi_{i_0,j_0} \in A^+$, there is a $b \in B$ such that $\varphi_{i_0,j_0}(b) \neq \psi_{i_0,j_0}(b)$. Therefore, $\varphi(b) \neq \psi(b)$. So $B$ separates the points of $CS(A)$.

Since a unital and completely positive linear mapping from $A$ into $M_n$ is completely positive (Proposition 3.4 in [1]), the conditions in the theorem imply that the matrix state space $CS(A)$ of $A$ is the graded set $GCS(A)$ that was defined in section 3 with $\varphi_n^*(1) = 1_n$. Now the theorem follows from Proposition 4.2.

**Remark 4.4.** Though $DL_n$ does not always define a bounded metric (see Proposition 4 in [4]), the conditions in the theorem above imply that $DL_n$ is bounded from the proof of it.

5. **Matrix metrics from matrix Lipschitz seminorms**

A complex vector space $V$ is said to be matrix ordered if $V$ is a *-vector space, each $M_n(V)$ is partially ordered and $\gamma^* M_n(V)^+ \gamma \subseteq M_n(V)^+$ when $\gamma \in M_{n,m}$. A matrix order unit space $(V, 1)$ is a matrix ordered space $V$ together with a distinguished order unit $1$ satisfying the conditions: $V^+$ is a proper cone with the order unit $1$ and each of the cones $M_n(V)^+$ is Archimedean. Each matrix order unit space $(V, 1)$ may be provided with the norm

$$\| v \| = \inf \left\{ t \in \mathbb{R} : \begin{bmatrix} t1 & v \\ v^* & t1 \end{bmatrix} \geq 0 \right\}.$$  

As in section 3 we will not assume that $V$ is complete for the norm.

A **Lipschitz seminorm** on an order unit space $S$ is a seminorm on $S$ such that its null space is the scalar multiples of the order unit $1_{M_n}$. Here we give a non-commutative version of Lipschitz seminorms.

**Definition 5.1.** Given a matrix order unit space $(V, 1)$, a matrix Lipschitz seminorm $L$ on $(V, 1)$ is a sequence of seminorms $L_n : M_n(V) \rightarrow [0, +\infty)$ such that the null space of each $L_n$ is $M_n(C1)$, $L_{m+n}(v \oplus w) = \max\{L_m(v), L_n(w)\}$, $L_n(\alpha \varphi \beta) \leq \| \varphi \| L_m(v) \| \beta \|$ and $L_m(v^*) = L_m(v)$ for any $v \in M_n(V)$, $w \in M_n(V)$, $\alpha \in M_{n,m}$ and $\beta \in M_{m,n}$.

By an operator system one means a self-adjoint subspace $\mathcal{R}$, which is not necessarily closed, in a unital $C^*$-algebra $\mathcal{A}$ such that $\mathcal{R}$ contains the identity $1 \in \mathcal{A}$. Parallel to the abstract characterization of Kadison’s function systems as order unit space [13], Choi-Effros’s representation theorem says that every matrix order unit space is completely order isomorphic to an operator system [3]. For a matrix order unit space $(V, 1)$, the matrix state space is the collection $CS(V) = (CS_n(V))$ of matrix states

$$CS_n(V) = \{ \varphi : \varphi \text{ is a unital completely positive linear mapping from } V \text{ to } M_n \},$$
Proposition 5.2. Let \((V, 1)\) be a matrix order unit space, and let \(P_n = M_n(C1)\) and \(P = (P_n)\). Then \(V\) has the matrix 2-decomposable property about \(P\).

Proof. Since every matrix order unit space is completely order isomorphic to an operator system, there is a unital \(C^\ast\)-algebra \((A, 1_A)\) such that \(V \subseteq A\) and \(1_A = 1\). Because \(C1\) is a closed subspace of \(V\), we have the complete isometry \((V/(C1))^\ast \cong (C1)^\perp\), where \((C1)^\perp = \{f \in V : f(1) = 0\}\). So \((V)^*\) is just the subspace of \(V^*\) consisting of those \(f \in V^*\) such that \(f(1) = 0\).

For \(f \in CB(V)\), there exists a completely bounded linear mapping \(g\) from \(A\) to \(M_n\), which extends \(f\), with \(\|g\|_{cb} = \|f\|_{cb}\) by Theorem 7.2 in \([14]\). From Proposition 4.2 there are \(\psi_1, \psi_2, \psi_3, \psi_4 \in CS_n(A)\) such that \(g = (\psi_1 - \psi_2) + i(\psi_3 - \psi_4)\). Let \(\phi_i = \psi_i|_V\) \((i = 1, 2, 3, 4)\), the restriction of \(\psi_i\) to \(V\). Clearly each \(\phi_i \in CS_n(V)\).

So we obtain that \(f = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4)\). Therefore, \(V\) has the matrix 2-decomposable property about \(P\). \(\square\)

Theorem 5.3. Let \((V, 1)\) be a matrix order unit space with operator space norm \(\|\cdot\| = (\|\cdot\|_n)\), and let \(\mathcal{L} = (L_n)\) be a matrix Lipschitz seminorm on \((V, 1)\). Then the \(D_\mathcal{L}\)-topology on \(CS(V)\) agrees with the BW-topology if and only if the image of \(L_1^1\) in \(V\) is totally bounded for \(\|\cdot\|_1\).

Proof. If \(f\) is a completely bounded linear mapping from \(V\) to \(M_n\) and \(f(1) = 1_n\) and \(\|f\|_{cb} = 1\), then \(f\) is completely positive by Proposition 3.4 in \([14]\). So the matrix state space \(CS(V)\) of \(V\) is the graded set \(\mathcal{G}CS(V)\) that was defined in section 3 with \(\phi_n(0) = 1_n\). Now by Proposition 5.2 Theorem 3.4 and Theorem 3.9 the \(D_\mathcal{L}\)-topology on \(CS(V)\) agrees with the BW-topology if and only if the image of \(L_1^1\) in \(V\) is totally bounded for \(\|\cdot\|_1\). \(\square\)

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References


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