

SMOOTH ACTIONS OF R^n

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ABSTRACT. Given a smooth action of R^n on a k -dimensional differentiable manifold M^k , for each $0 \leq r \leq k$ we associate with “almost all” oriented orbits of dimension r an element of $H_r(M^k, R)$.

1. INTRODUCTION

In [4] the present author considered actions of R^1 on compact polyhedra. It turned out to be possible to associate with each point x in a subset of our polyhedron X that has measure one with respect to each invariant probability measure an element of $H_1(X, R)$ which we denoted by A_x . We called A_x the asymptotic cycle determined by x , and for each finite invariant measure μ on X we called $\int A_x d\mu(x)$ the μ asymptotic cycle.

In [6] the author considered smooth oriented foliations on a compact smooth manifold M^k . He defined what they meant by a transversal invariant measure and associated with any transversal invariant measure to an oriented foliation of dimension r an element of $H_r(M^k, R)$. If our foliation is one-dimensional and arises from a smooth flow without stationary points, there is an obvious 1-1 correspondence between transversal invariant measures and finite invariant measures defined on the Borel subsets of M^k . Moreover the element of $H_1(M^k, R)$ associated with a transversal invariant measure is the same as the element A_μ in $H_1(M^k, R)$ that is associated with the corresponding invariant measure μ . Thus for smooth oriented foliations one has a generalization of the notion of a μ asymptotic cycle. However there is no such generalization of the notion of the asymptotic cycle A_x associated with a point x .

For one-dimensional oriented foliations on a compact smooth manifold that are associated with a smooth nowhere vanishing vector field it is easy to give examples of transversal invariant measures, because these arise from finite measures defined by the vector field. For higher dimensional foliations it is much more difficult in general to construct such examples.

This suggests that one should consider oriented foliations on a compact smooth manifold arising from a smooth action of a connected Lie group. In [5] this was done and it was shown that in the presence of what was called a positive quantifier

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there was a canonical 1-1 correspondence between finite invariant measures defined on the Borel subsets of our manifold and transversal invariant measures for the oriented foliation whose leaves are the orbits under the action of our group.

A positive quantifier was defined to be a nowhere vanishing field of r -vectors, (r being the dimension of the leaves of our foliation), where at each point the corresponding r -vector is tangent to the orbit through the point and has the same orientation. In addition it was assumed that this field of r -vectors is invariant under the action of the group.

In case our Lie group is R_1 the positive quantifier that gives rise to the 1-1 correspondence between finite invariant measures and transversal invariant measures is just the vector field that defines the flow.

For a finite invariant measure μ the element of $H_r(M^k, R)$ can be obtained by integrating with respect to μ the interior product of closed r -forms with our positive quantifier. It was shown in [5] that if our closed r -form is bounding, this integral is zero. Thus we get a linear functional on the r -dimensional De Rham cohomology group of M^k and therefore an element of $H_r(M^k, R)$.

It would certainly be of interest if we could, for an r -dimensional oriented foliation arising from the action of a connected Lie group, associate with each point x in a subset of M^k having measure one with respect to every invariant probability measure an element A_x of $H_r(M^k, R)$. One would then hope that if A_μ is the element of $H_r(M^k, R)$ associated with a finite invariant measure μ , then $A_\mu = \int_{M^k} A_x d\mu(x)$, just as in the case when our group is R^1 .

Because there is no pointwise ergodic theorem that applies to actions of arbitrary connected Lie groups it does not seem possible to achieve this in general. However for actions of R^n there is a pointwise ergodic theorem [1] and therefore in this case we can attain the desired result. In fact we can do more.

Suppose we have a smooth action of R^n on a smooth compact manifold M^k . For any r between 0 and k let B'_r be the set of all pairs (p, o) where p is a point in M^k lying on an orbit of dimension r and o is an orientation of the tangent space to this orbit at p . We will be able to associate an element of $H_r(M^k, R)$ with "almost all" such pairs.

We begin our derivation by assuming we have a compact k -dimensional differentiable manifold M^k of class C^1 and a C^1 action of R^n on M^k . We denote by A_r the compact subspace consisting of all points in M^k that lie on orbits of dimension less than or equal to r . We recall that if B_r is the subset of A_r consisting of all points lying on orbits of dimension r , then B_r is open in the relative topology on A_r .

If G is the fiber bundle over B_r whose fiber at each p is the Grassmannian of r -planes in the tangent space to M^k at p , it is well known that the map sending $p \in B_r$ into the r -plane tangent at p to the orbit through p is a continuous map into G .

We are going to consider a space A'_r that contains one point for each point of A_r that lies on an orbit of dimension less than r and two points for each point of A_r that lies on an orbit of dimension r . The two points in A'_r that are associated with a point p in B_r correspond to the two possible orientations of the orbit through p . The set of such pairs is B'_r .

We are going to specify a topology on A'_r by associating with each x in A'_r a family $\{U_x\}$ that will be the set of all neighborhoods of x in this topology.

If $x \in A'_r$ lies on an orbit of dimension less than r , a set U will belong to $\{U_x\}$ provided there is an open set V in A_r containing x such that if $y \in V$, then $y \in U$ if y is on an orbit of dimension less than r and $(y, o) \in U$ if $y \in V$ is on an orbit of dimension R and o is any orientation of the orbit through y .

If $p \in B_r$ and o is an orientation of the orbit through p , let $x \in B'_r$ be the point (p, o) . A set U will belong to $\{U_x\}$ provided it contains a set W satisfying the following properties:

- (1) $x \in W \subset B'_r$.
- (2) There is an open set $V \subset B_r$ such that for each $q \in V$ there is exactly one orientation o_q of the orbit through q such that (q, o_q) belongs to W and W is the set of all these pairs (q, o_q) .
- (3) The function that assigns to each q in V the orientation o_q is continuous.

According to [2, p. 56] the $\{U_x\}$ will be the families of neighborhoods of a topological space. The only property of the $\{U_x\}$ listed there that needs verification is the condition that if U_1 and U_2 belong to $\{U_x\}$, there is a U in $\{U_x\}$ such that $U \subset U_1 \cap U_2$.

It is only the case in which $x \in B'_r$ that needs verification. However, if we only seek to topologize B'_r rather than all of A'_r , it is clear how to do this. When this is done one sees that a set U in A'_r belongs to $\{U_x\}$ if and only if U contains a neighborhood of x in the topology we have defined on B'_r . From this the desired verification follows.

Theorem. *The topology we have described on A'_r is compact and metrizable.*

It is obvious that A'_r is a Hausdorff space. To see that A'_r is compact we need the following.

Lemma. *Let $\{O'_\alpha\}$ be an open covering of A'_r . For any $p \in B_r$ there is an open set $V_p \subset B_r$ such that $p \in V_p$ and such that there exist two sets O'_{α_1} and O'_{α_2} in our covering whose union contains (q, o) for any $q \in V_p$ and any orientation o of the orbit through q .*

Proof. We begin by remarking that if T is the map of A'_r into itself, leaving the complement of A'_r fixed and sending any point (p, o) in B'_r into (p, \acute{o}) , where \acute{o} is the orientation of the orbit through p opposite to o , then T is a homeomorphism of A'_r onto itself of period two.

Now suppose $\{O'_\alpha\}$ is an open covering of A'_r and suppose $p \in B_r$. If o and \acute{o} are the two orientations of the orbit through p , choose O'_{α_1} and O'_{α_2} from $\{O'_\alpha\}$ such that $(p, o) \in O'_{\alpha_1}$ and $(p, \acute{o}) \in O'_{\alpha_2}$. Since $O'_{\alpha_1} \cup O'_{\alpha_2}$ is an open set containing (p, o) and (p, \acute{o}) it follows that there is an open set $O \subset O'_{\alpha_1} \cup O'_{\alpha_2}$ containing (p, o) and (p, \acute{o}) that is invariant under the involution T . From this the lemma follows.

If p belongs to the complement of B_r in A_r , choose an open set $V_p \subset A_r$ such that $p \in V_p$ and there is a set U_α in our covering containing all points of A'_r that come from points in V_p . Because A_r is compact and the collection of all the sets V_p we have chosen covers A_r there is a finite number of these sets V_p that suffices to cover A_r . It follows that it takes at most twice this many sets O'_α from our original covering to cover A'_r . Thus we have proved that A'_r is compact.

To prove that A'_r is metrizable we need only show that it has a countable basis. To start our proof, let $\{O_\alpha\}$ be a countable basis for A_r , and for each α let O'_α be the set of all points in A'_r arising from points in O_α . Then for each point p

in the complement of B'_r the collection $\{O'_\alpha\}$ contains a basis for the system of neighborhoods of p . Since B'_r is an open subset of A'_r it only remains to show that B'_r in its relative topology possesses a countable basis.

If we let E_r be the fiber space over A_r whose fiber over any point of A_r is the set of oriented r -dimensional subspaces of the tangent space at the point, then certainly E_r has a countable basis. But B'_r is a subset of E_r and its relative topology as a subspace of E_r is the same as its relative topology as a subspace of A'_r . Thus B'_r in its relative topology has a countable basis which completes our proof that A'_r is metrizable.

We now know that A'_r is a compact metrizable space. The action of R^n on A_r gives rise in an obvious way to an action of R^n on A'_r . There is a pointwise ergodic theorem for actions of R^n [1]. It follows that if f is any continuous real valued function on A'_r then

$$\lim_{a \rightarrow \infty} \frac{1}{a^n} \int_0^a \dots \int_0^a f((t_1, \dots, t_n)(x)) dt_1 \dots dt_n$$

exists for a set of points x in A'_r that has measure one with respect to any invariant probability measure on A'_r . Because A'_r is compact and metrizable the set of points x in A'_r such that this limit exists for every continuous real valued function f on A'_r still has measure one with respect to every invariant probability measure. The proof is exactly the same as that given in [3] for the case $n = 1$. In keeping with the terminology in [3] we will call these points the quasi-regular points of A'_r . We will denote by Q'_r the set of quasi-regular points in A'_r .

For any point p in B_r let K_p be the orthocomplement in R^n of the kernel of the differential of the map sending (t_1, \dots, t_n) into $(t_1, \dots, t_n)(p)$. This map makes K_p a covering space of the orbit in A_r through p . If o is an orientation of the orbit through p and $x = (p, o)$, let V_x be the r -vector field on K_p with norm one everywhere with respect to the norm induced by the usual Euclidean metric on R^n and with an orientation such that the map of K_p onto the orbit through p sends V_x onto the r -vector field on this orbit whose orientation agrees with o .

Next let w be any continuous r -form on M^k . For any point p in A_r let w_p^* be the r -form we get at the origin in R^n by pulling back w via the map sending $(t_1, \dots, t_n) \in R^n$ into $(t_1, \dots, t_n)(p)$.

For any $x = (p, o)$ in B'_r we now define $f_w(x)$ to be the interior product of w_p^* with V_x . If $x \in A'_r$ belongs to the complement of B'_r , we define $f_w(x)$ to equal zero. We are going to show that f_w is a continuous function on A'_r for any w . Consequently it will follow that for any $x \in Q'_r$ the average value of f_w on the orbit through x exists. We will also show that if $w = d\alpha$ for any $(r - 1)$ form α on M^k this average value is zero. In this way for any $x \in Q'_r$ we will get a linear functional on the De Rham cohomology group of M^k . Thus for "almost all" pairs (p, o) where p is a point in B_r and o is an orientation of the orbit through p we will get an element of $H_r(M^k, R)$. It is to be noted that if we had not introduced the space A'_r we would not be able to define f_w , so we would not get elements of $H_r(M^k, R)$.

To see that $f_w(x)$ is continuous at each point $x \in A'_r$ first observe that this is clear if $x \in B'_r$, since B'_r is open and w_p and V_x are continuous functions of p and x respectively. If x belongs to the complement of B'_r , let q be the point in A_r corresponding to x . Then for any $\epsilon > 0$ there is a neighborhood N of q such that if p belongs to this neighborhood, $\|w_p\| < \epsilon$. Since $\|V_y\| = 1$ for any $y \in B'_r$ and

$f_w(y) = 0$ if $y \notin B'_r$ it follows that if y is sufficiently close to x , then $|f_w(y)| < \epsilon$. Thus f_w is continuous everywhere in A'_r .

We now know that if x is any quasi-regular point in B'_r , then

$$\lim_{a \rightarrow \infty} \frac{1}{a^n} \int_0^a \dots \int_0^a f((t_1, \dots, t_n)(x)) dt_1 \dots dt_n$$

exists for any continuous r -form w on M^k . We still have to show that if $w = d\alpha$ where α is an $(r-1)$ form of class C^1 on M^k , this limit is zero.

For any continuous differentiable form λ on M^k and any y in A_r we will let λ_y^* denote the pullback of λ to $(0, 0, \dots, 0)$ via the map sending (t_1, \dots, t_n) into $(t_1, \dots, t_n)y$. If $w = d\alpha$, then for any $y \in A_r$, $w_y^* = (d\alpha)_y^* = d(\alpha_y^*)$.

Since A_r is compact and the map sending y into α_y^* is continuous for any $(r-1)$ form α , there exists a constant C such that $\|\alpha_y^*\| \leq C$ for all $y \in A_r$.

If we write α_y^* in the form

$$\sum_{i_1 < \dots < i_{r-1}} \alpha_{i_1 \dots i_{r-1}} dt_{i_1} \wedge \dots \wedge dt_{i_{r-1}},$$

this implies that for each $(i_1 \dots i_{r-1})$ we must have

$$|\alpha_{i_1 \dots i_{r-1}}(y)| < C.$$

Next we note that the r -vector V_x is constant on any orbit in B'_r . Thus if $w = d\alpha$, $f_w((t_1, \dots, t_n)(x))$ is the sum of a number of terms, each of which is the interior product of a differential form $d(\alpha_{i_1 \dots i_{r-1}}) \wedge dt_{i_1} \wedge \dots \wedge dt_{i_{r-1}}$ with a constant r -vector. From this it follows that when $w = d\alpha$, to see that

$$\lim_{a \rightarrow \infty} \frac{1}{a^n} \int_0^a \dots \int_0^a f((t_1, \dots, t_n)(x)) dt_1 \dots dt_n$$

equals zero, we need only show that

$$\lim_{a \rightarrow \infty} \frac{1}{a^n} \int_0^a \dots \int_0^a \frac{\partial \alpha_{i_1 \dots i_{r-1}}}{\partial t_j} dt_1 \dots dt_n = 0$$

for all $(i_1 \dots i_{r-1})$ and t_j . However this follows from the fact that $|\alpha_{i_1 \dots i_{r-1}}| \leq C$ throughout R^n .

Finally we return to the situation considered in [5] that we described in the introduction to this paper. Recall that we were given a compact differentiable manifold M^k and a smooth action of a connected Lie group on M^k such that all the orbits had the same dimension r . It was assumed that we had defined an orientation varying continuously throughout M^k of the orbits. We also assumed that we had what we called a positive quantifier on M^k . For any finite invariant measure μ on M^k we then described an element of $H_r(M^k, R)$ which we denoted by A_μ . This element of $H_r(M^k, R)$ determined a homomorphism of the r -dimensional De Rham cohomology group into R which assigned to the cohomology class determined by any closed r -form w the integral with respect to μ of the interior product of our positive quantifier with w .

If in this situation our Lie group is R^n , then $B_r = A_r$ and $B'_r = A'_r$. Moreover B'_r is the union of two disjoint open invariant sets, one corresponding to associating with each point $p \in A_r$ the orientation we are given of its orbit and the other corresponding to the opposite orientation. If we let C'_r be the set we get by using the given orientation, there is an equivariant homeomorphism between $A_r = B_r$ and C'_r . Corresponding to an invariant measure μ on A_r we therefore get an invariant

measure μ' on C'_r . We can extend μ' to an invariant measure on all of B'_r by defining the measure of the complement of C'_r in B'_r to be zero. For each $x \in C'_r$ the r -vector V_x we have defined determines an r -vector at the point in $A_r = B_r = M^k$ that corresponds to x that is tangent to the orbit through this point. This field of r -vectors is a positive quantifier for the action of R^n on $M^k = A_r = B_r$. The element A_μ of $H_r(M^k, R)$ determines an element of $A_{\mu'}$ of $H_r(C'_r, R)$. There is a smooth structure on C'_r determined by its homeomorphism with M^k . We can describe $A_{\mu'}$ as that element of $H_r(C'_r, R)$ that assigns to each r -form w on C'_r the integral of the interior product of w_x^* with V_x with respect to μ' .

For any element λ of $H_r(C'_r, R)$ and any r -form w on C'_r we will denote by $\lambda(w)$ the number that the homomorphism of the r -dimensional De Rham cohomology group into R determined by λ assigns to the equivalence class of w . Thus $A_{\mu'}(w) = \int_{C'_r} (w_x^* \lrcorner V_x) d\mu'(x)$.

We have associated with each quasi-regular point x in C'_r an element of $H_r(C'_r, R)$. We will denote this element by A_x .

We had initially indicated essentially that we would be able to prove that $A_{\mu'} = \int_{C'_r} A_x d\mu'(x)$. To see this we need only show that if w is any closed r -form on C'_r , then $A_{\mu'}(w) = [\int_{C'_r} A_x d\mu'(x)](w)$. However $(\int_{C'_r} A_x d\mu'(x))(w) = \int_{C'_r} A_x(w) d\mu'(x)$. Moreover $A_x(w)$ equals

$$\lim_{a \rightarrow \infty} \frac{1}{a^n} \int_0^a \dots \int_0^a f_w((t_1, \dots, t_n)(x)) dt_1 \dots dt_n$$

and $f_w(y) = w_y^* \lrcorner V_y$ for any y . However if one averages an integrable function over almost all orbits, the integral of this pointwise average with respect to any invariant measure equals the integral of the original function with respect to the invariant measure. Therefore

$$\int_{C'_r} A_x(w) d\mu'(x) = \int (w_x^* \lrcorner V_x) d\mu'(x) = A_\mu(w).$$

Thus we are done. \square

Of course if we had only wanted to consider the situation described in [5] we could have dispensed with the entire discussion given in this paper and disposed of the matter without any difficulties whatsoever.

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