HYPERELLIPTIC CURVES OVER $\mathbb{F}_2$ OF EVERY 2-RANK WITHOUT EXTRA AUTOMORPHISMS

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Abstract. We prove that for any pair of integers $0 \leq r \leq g$ such that $g \geq 3$ or $r > 0$, there exists a (hyper)elliptic curve $C$ over $\mathbb{F}_2$ of genus $g$ and 2-rank $r$ whose automorphism group consists of only identity and the (hyper)elliptic involution. As an application, we prove the existence of principally polarized abelian varieties $(A, \lambda)$ over $\mathbb{F}_2$ of dimension $g$ and 2-rank $r$ such that $\text{Aut}(A, \lambda) = \{\pm 1\}$.

1. Introduction

In this paper curves are smooth, projective, and geometrically integral algebraic varieties of dimension one defined over fields. Let $k$ be a field, and let $\overline{k}$ be its algebraic closure. If $C$ is a curve over $k$, let $\text{Aut} C$ denote the group of automorphisms of $C$ defined over $\overline{k}$. Let $J(C)$ denote the Jacobian of $C$, and let $\text{End} J(C)$ denote the endomorphism ring of $J(C)$ over $\overline{k}$. Let $\mathbb{F}_p$ be a finite field of $p$ elements for some prime $p$, and let $\overline{\mathbb{F}}_p$ be its algebraic closure.

A supersingular curve $C$ over $\mathbb{F}_p$ is a curve whose Jacobian is isogenous over $\mathbb{F}_p$ to a product of supersingular elliptic curves. Hence a supersingular curve $C$ is a cover of these supersingular elliptic curves. It has $p$-rank 0, but the converse is not true for $g \geq 3$. Supersingular curves are intimately connected to curves with large automorphism groups. For instance, in the seminal paper [1], the authors constructed supersingular curves over a finite field of characteristic 2 by taking quotients of some families of (2-rank 0) curves over $\mathbb{F}_2$ with large automorphism groups. It is well known that curves over fields of positive characteristic achieving maximal automorphism groups are all supersingular curves [13]. Is it a myth or truth that a curve over $\mathbb{F}_p$ of lower $p$-rank has larger automorphism groups in general?

In the moduli space of curves, the subset corresponding to the curves with trivial automorphism group is open (see [9, Introduction] or [2, Remark 10.6.24]). In a recent paper this fact was proved constructively [9] (see also [10], [11]). It is desirable to understand how this subset stratifies by the $p$-rank of the curves.

Question 1. Let $p$ be a prime number. Given integers $g \geq 3$ and $0 \leq r \leq g$, is there a curve $C$ over $\mathbb{F}_p$ of genus $g$ and $p$-rank $r$ such that $\text{Aut} C = \{1\}$?
There is not any constructive way to obtain curves over $\mathbb{F}_p$ of prescribed genus and $p$-rank, so we do not know the answer to this question.

On the other hand, for every prime $p$ and positive integer $g$, Poonen \[10\] has constructed (hyper)elliptic curves $C$ over $\mathbb{F}_p$ of genus $g$ with $\text{Aut}(C) = \{1, \iota\}$, where $\iota$ is the unique (hyper)elliptic involution of $C$. Automorphisms other than these two are referred to as extra automorphisms.

If $g = 1$, it is well known that for every prime $p$ a supersingular elliptic curve (i.e., with zero $p$-rank) over $\mathbb{F}_p$ has extra automorphisms, while there exist ordinary elliptic curves (i.e., with non-zero $p$-rank) over $\mathbb{F}_p$ with $\text{Aut}(C) = \{1, \iota\}$. (See [12 Chapter III].)

Every curve over $\mathbb{F}_2$ of genus 2 and 2-rank 0 can be written in the form $y^2 + y = x(x^4 + a_1 x^2 + a_0 x)$ for $a_0, a_1 \in \mathbb{F}_2$, and hence has extra automorphisms. It is easy to check this fact by hand. In fact, every curve of the form $y^2 + y = x(\sum_{i=0}^{n} a_i x^{2i})$ for some integer $n$ and $a_i \in \mathbb{F}_2$ has extra automorphisms (see [11]).

**Question 2.** Let $p$ be a prime number. Given integers $g \geq 1$ and $0 \leq r \leq g$, is there a (hyper)elliptic curve $C$ over $\mathbb{F}_p$ of genus $g$ and $p$-rank $r$ without extra automorphisms?

The present paper gives a complete answer to this question for the case $p = 2$. We hope this provides evidence for a more general theorem or conjecture in the future.

**Theorem 3.** For any integers $0 \leq r \leq g$ such that $g \geq 3$ or $r > 0$, there exists a (hyper)elliptic curve $C$ over $\mathbb{F}_2$ of genus $g$ and 2-rank $r$ such that $\text{Aut}(C) = \{1, \iota\}$, where $\iota$ is the unique (hyper)elliptic involution of $C$.

The proof of the theorem, divided in two parts, is presented in the next two sections. This theorem has the following application. For an abelian variety $A$ with polarization $\lambda$ defined over $\mathbb{F}_2$, let $\text{Aut}(A, \lambda)$ denote the group of automorphisms of $A$ over $\mathbb{F}_2$ respecting the polarization. The corollary below follows immediately from the theorem by applying Torelli’s theorem [6 Theorem 12.1]. Detailed discussion on related results can be found in the Introduction of [10].

**Corollary 4.** For any integers $0 \leq r \leq g$ such that $g \geq 3$ or $r > 0$, there exists a $g$-dimensional principally polarized abelian variety $(A, \lambda)$ over $\mathbb{F}_2$ of 2-rank $r$ such that $\text{Aut}(A, \lambda) = \{\pm 1\}$.

Finally we remark that these two questions above will be resolved if we know what algebras can become $\text{End} J(C)$ for curves $C$ over $\mathbb{F}_p$ of prescribed genus (see [8 Question (8.6)]) and $p$-rank. By [7] (see also [13]), one knows that for every $g \geq 1$ there exists a hyperelliptic curve $C$ of any genus $g \geq 1$ with $\text{End} J(C) = \mathbb{Z}$. However, this does not hold for curves over finite fields, in which case we have that $\text{End} J(C)$ strictly contains $\mathbb{Z}$.

2. **Construction for $r > 0$**

Suppose $g \geq 2$ and $r \leq g$ are two positive integers. Let $q(x)$ be a polynomial in $\mathbb{F}_2[x]$ of degree $< 2g + 1 - r$ (resp. $= 2g + 1 - r$) with $r$ (resp. $r + 1$) distinct roots, and let $f(x)$ be a polynomial in $\mathbb{F}_2[x]$ of degree $2g + 1 - r$ (resp. $\leq 2g + 1 - r$), such that $f(x)$ and $q(x)$ has no common roots. Let $C$ be the hyperelliptic curve over $\mathbb{F}_2$.
defined by the affine equation

\[ C : y^2 + y = \frac{f(x)}{q(x)} \]

Then the curve \( C \) over \( \mathbb{F}_2 \) is of genus \( g \) by the Riemann-Hurwitz formula and of 2-rank \( r \) by the Deuring-Shafarevich formula in \([2]\), which we shall explain immediately (see details in \([4]\) or \([5]\)). Let \( k \) be an algebraically closed field of characteristic \( p \). Let \( \pi : X \to Y \) be a finite Galois covering of curves over \( k \) whose Galois group \( G \) is a \( p \)-group. Let \( r_X \) and \( r_Y \) denote the \( p \)-ranks of \( X \) and \( Y \), respectively. Let \( Q_1, \cdots, Q_n \) be the set of ramification points on \( Y \) with respect to \( \pi \). For each point \( Q_i \), let \( p^{e_i} \) (here \( e_i \geq 1 \)) be its ramification index. Then

\[ r_X - 1 = \#G \cdot (r_Y - 1 + \sum_{i=1}^{n} (1 - p^{-e_i})). \]

Let \( D \) be the ramification divisor of the canonical double cover \( C \to \mathbb{P}^1 \). Write \( q := \prod_{i=1}^{r} (x - \alpha_i)^{b_i} \) (resp. \( q := \prod_{i=1}^{r+1} (x - \alpha_i)^{b_i} \)) for distinct \( \alpha_i \in \mathbb{F}_2 \) and \( b_i \in \mathbb{Z}_{>0} \).

The set \( S \) of ramification points consists of those points \( P_{\alpha_i} \) corresponding to the zeroes of \( q \) and possibly the point \( P_{\infty} \) at infinity. We have

\[
D = \left\{ \begin{array}{l}
(2g + 2 - \deg(q) - r)P_{\infty} + \sum_{i=1}^{r} (b_i + 1)P_{\alpha_i}, \\
\sum_{i=1}^{r+1} (b_i + 1)P_{\alpha_i},
\end{array} \right\}
\]

Every automorphism of \( C \) gives rise to an automorphism of \( \mathbb{P}^1 \) preserving \( D \) under the canonical double cover \( C \to \mathbb{P}^1 \). To construct curves \( C \) without extra automorphisms, it suffices to find monic polynomials \( f \) and \( q \) in \( \mathbb{F}_2[x] \) such that every automorphism of \( \mathbb{P}^1 \) preserving \( D \) is the identity map on \( \mathbb{P}^1 \).

Our construction below follows the following idea: for every pair of integers \( 0 < r \leq g \), we shall construct polynomials \( q \) such that \( q \) has \( r \) (or \( r + 1 \) resp.) distinct roots and of degree \( < 2g + 1 - r \) (or \( 2g + 1 - r \), resp.) in \( \mathbb{F}_2[x] \). We always let \( f \) be any polynomial in \( \mathbb{F}_2[x] \) of degree \( 2g + 1 - r \) (or \( \leq 2g + 1 - r \), resp.) which has no common roots with \( q \). We remark that we shall use the construction that \( q \) has \( r \) distinct roots except in Case 5 and Case 6.

In the construction below we use the notation \( f_n \) for a \( n \)-th degree irreducible polynomial in \( \mathbb{F}_2[x] \). It is a basic fact in algebra that \( f_n \) exists for every positive integer \( n \) (see \([3]\) Chapter V). For example, \( f_2 = x^2 + x + 1 \) and \( f_3 = x^3 + x^2 + 1 \) or \( x^3 + x + 1 \). For any \( f_3 \) of our choice, we denote by \( \beta_1, \beta_2, \beta_3 \) its roots in \( \mathbb{F}_2 \) in an order such that \( \beta_1^2 = \beta_2 \).

Case 1. Suppose \( r \geq 8 \):

Let \( q = f_3 f_r - 3 \) if \( 3 \nmid r \) and \( q = x f_3 f_r - 4 \) otherwise.

Let \( \sigma \) be an automorphism of \( \mathbb{P}^1 \) which acts as a 3-cycle on the three roots of \( f_3 \) in \( \mathbb{F}_2 \). Since 3 points determine an automorphism of \( \mathbb{P}^1 \), \( \sigma \) is defined over the field \( k \) generated by roots of \( f_3 \) over \( \mathbb{F}_2 \). Hence, \( \sigma(P_{\infty}) \) corresponds to a point in \( k \).

Let \( \mathbb{F} \) denote the composition of all finite extensions of \( \mathbb{F}_2 \) of degrees coprime to 3. There are exactly \( r - 2 \) distinct \( \mathbb{F} \)-rational points in the set of ramification points \( S \). Suppose \( \lambda \) is a non-trivial automorphism of \( \mathbb{P}^1 \) preserving \( D \). Then \( \lambda \) must map at least \( r - 2 \) from \( 3 \geq 3 \) of these \( \mathbb{F} \)-rational points to other \( \mathbb{F} \)-rational points of \( S \). But \( \lambda \) is determined by its values at 3 points, so \( \lambda \) must be defined over \( \mathbb{F} \). In particular, \( \lambda \) preserves the set of 3 non-\( \mathbb{F} \)-rational points of \( S \), the roots of \( f_3 \). If \( \lambda \) fixes any one of them, as they are Galois conjugates over \( \mathbb{F} \), then \( \lambda \) would
fix them all, hence \( \lambda \) would be trivial. So \( \lambda \) acts as a 3-cycle, and after replacing \( \lambda \) by \( \lambda^{-1} \) if necessary, we may assume \( \lambda = \sigma \). Since \( \lambda \) permutes the roots of \( f_3 \), it fixes its coefficients, hence \( \lambda \) fixes 0 and 1. So \( \lambda(P_\infty) \neq P_\infty \) and \( \lambda(P_\infty) \neq P_1 \). But \( D \) is preserved, so \( \lambda \) maps \( P_\infty \) to a root of \( f_{r-3} \) (or \( f_{r-4} \)), which lies in \( \mathbb{F} \) and does not lie in \( k \). This contradicts our assumption above about \( \sigma \).

Case 2. Suppose \( r = 1 \) and \( g \geq 2 \), or \( r = 2 \) and \( g \geq 4 \):

For \( r = 1 \) and \( g \geq 2 \), let \( q = x \).

Then the ramification divisor is \( D = 2gP_\infty + 2P_0 \). Since \( g \geq 2 \) every automorphism of \( \mathbb{P}^1 \) preserving \( D \) fixes \( \infty \) and 0, hence it is of the form \( x \mapsto cx \) for some non-zero \( c \in \mathbb{F}_2 \). A simple computation shows that \( c = 1 \). This resembles Case 1 in Section 2 of \([10]\).

For \( r = 2 \) and \( g \geq 4 \), let \( q = x^2(x + 1) \).

Then \( D = (2g - 3)P_\infty + 3P_0 + 2P_1 \). Every automorphism of \( \mathbb{P}^1 \) preserving \( D \) has three points \( \infty, 0 \) and 1 all fixed, hence is an identity.

Case 3. Suppose \( r = 3 \) and \( g \geq 4 \):

Let \( q = f_3 \).

Then \( D = (2g - 4)P_\infty + 2(P_{\beta_1} + P_{\beta_2} + P_{\beta_3}) \). Let \( \lambda \) be a non-trivial automorphism of \( \mathbb{P}^1 \) that preserves \( D \). By assumption \( 2g - 4 > 2 \), so \( \lambda \) fixes \( P_\infty \) and \( \lambda \) permutes the roots of \( f_3 \). Thus \( \lambda \) fixes 0 and 1. But then it fixes all three points 0, 1 and \( \infty \), therefore it must be an identity. This leads to a contradiction.

These following three cases follow the same scheme, so we shall elaborate on Case 4 and only sketch the other two cases.

Case 4. Suppose \( r = 4 \) and \( g \geq 5 \), or \( r \geq 4 \) and \( g \geq r + 3 \):

For \( r = 4 \) and \( g \geq 5 \), let \( q = x^2f_3 \).

Then the ramification divisor is \( D = (2g - 7)P_\infty + 3P_0 + 2(P_{\beta_1} + P_{\beta_2} + P_{\beta_3}) \). Let \( \lambda \) be a non-trivial automorphism of \( \mathbb{P}^1 \) which preserves \( D \). Then \( \lambda \) permutes the roots of \( f_3 \), hence it fixes 0 and 1. If \( \lambda \) fixes \( P_\infty \) and \( P_0 \), then it is an identity. If \( \lambda \) swaps \( P_\infty \) and \( P_0 \), it is of the form \( \lambda(\alpha) = c/\alpha \) for some non-zero \( c \in \mathbb{F}_2 \). It can be checked quickly that this map cannot preserve the roots of \( f_3 \).

For \( r \geq 4 \) and \( g \geq r + 3 \), let \( q = x^2(x + 1)^2f_{r-2} \).

Then \( D = (2g - 2r - 1)P_\infty + 4P_0 + 3P_1 + 2\sum(f_{r-2}P) \). Since \( 2g - 2r - 1 \geq 5 \) and \( r - 2 \geq 2 \), every automorphism of \( \mathbb{P}^1 \) preserving \( D \) has three points \( \infty, 0 \) and 1 all fixed, hence is an identity.

Case 5. Suppose \( r = 5 \) and \( g \geq 5 \):

Let \( q = f_3(x + 1)^2(x^2 + x + 1) \). Let \( \alpha_1, \alpha_2 \) be roots of \( x^2 + x + 1 \) in \( \mathbb{F}_2 \).

Then the ramification divisor is

\[
D = 2(P_{\beta_1} + P_{\beta_2} + P_{\beta_3}) + (2g - 8)P_1 + 2(P_{\alpha_1} + P_{\alpha_2}).
\]

Label the roots of \( \beta_1, \beta_2, \beta_3, 1, \alpha_1, \alpha_2 \) by 1, 2, 3, 4, 5, 6, respectively, such that the absolute Frobenius acts on \( S \) as the permutation \( \sigma = (123)(56) \). Let \( H \) be the subgroup of the automorphism of \( \mathbb{P}^1 \) preserving \( D \), which we may view as a faithful subgroup of \( S_6 \), since automorphisms are determined already by 3 values. Any automorphism of \( \mathbb{P}^1 \) which fixes \( \alpha_1 \) and \( \alpha_2 \) has to fix 1, so \( \beta_3 \) cannot be mapped to 1. Therefore, \( (12)(34) \not\in H \). The group theoretical lemma below, due to Poonen (see \([10] \) Lemma 3), indicates that \( H \) is trivial.
**Lemma 5.** Suppose $H$ is a subgroup of $S_6$ such that

1. Each non-trivial element of $H$ has at most 2 fixed points;
2. $\sigma H \sigma^{-1} \subset H$ for every $\sigma \in \text{Gal}(\mathbb{F}_2/\mathbb{F}_2)$;
3. The permutation $(12)(34)$ is not in $H$.

Then $H = \{1\}$.

**Case 6.** Suppose $r = 6$ and $g \geq 7$:

Let $q = f_3(x + 1)(x^2 + x + 1)x^{2g-11}.$

Then the ramification divisor is $D = 2(P_{\beta_1} + P_{\beta_2} + P_{\beta_3}) + 2P_1 + 2(P_{\alpha_1} + P_{\alpha_2}) + (2g - 10)P_0.$ Note that every automorphism of $\mathbb{P}^1$ preserving $D$ fixes $P_0$. Then we apply the same argument as in Case 5.

**Case 7.** Suppose $r = 7$ and $g \geq 8$:

Let $q = f_3(x + 1)(x^2 + x + 1)^2.$

Then the ramification divisor is $D = 2(P_{\beta_1} + P_{\beta_2} + P_{\beta_3}) + 2P_1 + 2(P_{\alpha_1} + P_{\alpha_2}) + 3P_0 + (2g - 13)P_{\infty}.$

Let $\lambda$ be a non-trivial automorphism of $\mathbb{P}^1$ preserving $D$. If $\lambda$ fixes $P_0$ and $P_{\infty}$, then we use the same argument as in Case 5. This is the case when $g \geq 9$. It remains to prove the case $g = 8$ and $\lambda$ swaps $P_0$ and $P_{\infty}$. Then $\lambda(\alpha) = c/\alpha$ for some non-zero $c \in \mathbb{F}_2$. If $\lambda$ fixes $P_1$, then it is defined over $\mathbb{F}_2$, hence it permutes the roots of $f_3$ and fixes $P_0$, a contradiction. If $\lambda$ swaps $P_1$ with one root of $f_3$, then it also preserves the roots of $f_3$. If $\lambda$ swaps $P_1$ with a root of $f_2$, then it permutes the roots of $f_2$. So it has to fixes $P_1$, which is absurd.

**Case 8.** Remaining cases:

For $g = r = 4, 6$, let $C : y^2 + y = x + \frac{1}{x^2 + x + 1}.$

For $g = r = 3, 5, 7$, let $C : y^2 + y = x + \frac{1}{x^2 + x + 1}.$

For $g = 2, 3$ and $r = 2$, let $C : y^2 + y = x + \frac{1}{x^2 + x + 1}$ and $C : y^2 + y = x^3 + \frac{1}{x^2 + x + 1},$ respectively.

It is an elementary computation to show that these curves have no extra automorphisms.

**3. Construction for $r = 0$**

We still assume $g \geq 2$. In this section let $C$ be a hyperelliptic curve defined by the affine equation

$$C : y^2 + y = f(x),$$

where $f(x)$ is a polynomial in $\mathbb{F}_2[x]$ of degree $2g + 1$. This is the same as letting $q = 1$ in (4). So $C$ is of genus $g$ and 2-rank 0. We remark that every curve in (4) is isomorphic to a curve with only odd-degree terms in $f(x)$ because the base field is $\mathbb{F}_2$.

Any automorphism of $C$ is of the form $x \mapsto ax + b$ and $y \mapsto cy + h(x)$ for some $a, b, c \in \mathbb{F}_2$ and some polynomial $h(x)$ in $\mathbb{F}_2[x]$ of degree $h \leq g$. Let $\mathcal{H}$ be the set of polynomials $p(x)^2 + p(x)$ for all polynomial $p(x)$ in $\mathbb{F}_2[x]$ of degree $\leq g$. It is easy to show that it is a $\mathbb{F}_2$-vector space of dimension $g + 1$. It follows that $c = a^{2g+1} = 1$ and $f(ax + b) + f(x) = h(x)^2 + h(x)$. That is,

$$a^{2g+1} = 1 \quad \text{and} \quad f(ax + b) + f(x) \in \mathcal{H}.$$
Lemma 6. Let $g = 4$ or $g \geq 7$. Let $p(x)$ be a polynomial in $\mathbb{F}_2[x]$ of degree $\leq 2g - 6$. The hyperelliptic curve $C$ defined by the affine equation

$$C : y^2 + y = f(x) := x^{2g+1} + x^{2g-1} + x^{2g-3} + p(x)$$

has Aut $C = \{1, \iota\}$ if and only if either $g \not\equiv 2 \pmod{4}$, or $g \equiv 2 \pmod{4}$ and

(i) $g - 2$ is a 2-power and $p(x+1) + p(x) \not\in \mathcal{H}$;
(ii) $g - 2$ is not a 2-power and $p(x+1) + p(x) \not\in \mathcal{H}$.

Proof. Suppose $x \mapsto ax + b$ gives rise to a non-extra automorphism $\lambda$ of $C$.

First we suppose $g \geq 7$. If $b = 0$, then 6 implies that $a = 1$ and so $\lambda$ is not extra. Otherwise, since deg($f(ax + b) + f(x)$) = $2g$, all odd-degree terms in $f(ax + b) + f(x)$ of degree $> g$ vanish. Because $2g - 5 > g$ by our assumption, the coefficients of $x^{2g-1}$, $x^{2g-3}$ and $x^{2g-5}$ are zero. That is,

$$\begin{align*}
(2g + 1) b^2 + 1 + a^2 &= 0, \\
(2g + 1) b^4 + (2g - 1) b^2 + 1 + a^4 &= 0, \\
(2g + 1) b^4 + (2g - 1) b^2 + (2g - 3) &= 0.
\end{align*}$$

Simplifying, we get respectively

$$\begin{align*}
(2g + 1) b^2 + 1 + a^2 &= 0, \\
\frac{g(g - 1)}{2} b^4 + (g - 1) b^2 + 1 + a^4 &= 0, \\
\frac{g(g - 1)(g - 2)}{2} b^4 + \frac{(g - 1)(g - 2)}{2} b^2 + g &= 0.
\end{align*}$$

Substituting (9) into (11) we get

$$\frac{g(g - 1)}{2} b^4 + (g - 1) b^2 + g^2 b^4 = 0,$$

and so

$$\frac{g(3g - 1)}{2} b^2 + (g - 1) = 0.$$

Thus $\frac{g(3g - 1)}{2} = g - 1$ and $g \equiv 1, 2 \pmod{4}$. But (11) implies $g \not\equiv 1 \pmod{4}$.

From now on we assume $g \equiv 2 \pmod{4}$. Under this condition we get $a = b = 1$ by 9 and 10. Once again, we use 5 to get

$$f(x + 1) + f(x) = (p(x + 1) + p(x)) + \gamma(x) \in \mathcal{H},$$

where $\gamma(x) = (x^4 + x^2 + 1)((x + 1)^{2g-3} + x^{2g-3})$. 

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We claim that $\gamma(x) \in \mathcal{H}$ if and only if $g - 2$ is a 2-power. Suppose $\gamma(x) \in \mathcal{H}$. We have $\deg(\gamma) = 2g$, and its odd-degree terms are
\[
\left(\frac{2g-3}{2}\right)x^{2g-1} + \left(\frac{2g-3}{2} + \frac{2g-3}{4}\right)x^{2g-3} + \left(\frac{2g-3}{2} + \frac{2g-3}{4} + \frac{2g-3}{6}\right)x^{2g-5} + \left(\frac{2g-3}{4} + \frac{2g-3}{6} + \frac{2g-3}{8}\right)x^{2g-7} + \ldots + \left(\frac{2g-3}{2m-4} + \frac{2g-3}{2m-2} + \frac{2g-3}{2m}\right)x^{2g-(2m-1)}.
\]
Setting the odd-degree terms of degree $> g$ zero, and using the identity $(\frac{2g-3}{2n}) = \binom{g-2}{n}$ over $\mathbb{F}_2$ for all $n$, we have
\[
\binom{g-2}{1} = 0,
\binom{g-2}{2} + \binom{g-2}{1} = 0,
\binom{g-2}{3} + \binom{g-2}{2} + \binom{g-2}{1} = 0,
\binom{g-2}{4} + \binom{g-2}{3} + \binom{g-2}{2} + \binom{g-2}{1} = 0,
\vdots
\binom{g-2}{m-2} + \binom{g-2}{m-1} + \binom{g-2}{m} = 0
\]
for $m < \frac{2g+1}{2}$. But we already have $\binom{g-2}{1} = \binom{g-2}{2} = \binom{g-2}{3} = 0$, so this system of equations has a solution if and only if $\binom{g-2}{m} = 0$ for all $m \leq \frac{g}{2}$. That is, $g - 2$ is a 2-power. This proved parts (i) and (ii).

When $g = 4$, we follow the same argument but only simpler. Namely, any non-trivial automorphism $\lambda$ will lead to (9) and (11) and hence $g \equiv 1, 2 \mod 4$; a contradiction.

Case 9. Suppose $r = 0$ and $g = 4$ or $g \geq 7$:

Let $f(x) = x^{2g+1} + x^{2g-1} + x^{2g-3} + p(x)$, where $p(x)$ is any polynomial in $\mathbb{F}_2[x]$ of degree $\leq 2g - 6$ such that $g \not\equiv 2 \mod 4$, or $g \equiv 2 \mod 4$ and

(i) if $g - 2$ is a 2-power, then let $p = x^n + x^{n-2} + (\text{lower-degree terms})$ where $n \equiv 3 \mod 4$; or

(ii) if $g - 2$ is not a 2-power, then let $p \in \mathcal{H}$.

We shall verify our construction above. If $g \not\equiv 2 \mod 4$ it follows from Lemma 6. Suppose $g \equiv 2 \mod 4$. It can be easily checked that part (i) implies $p(x+1) + p(x) \not\in \mathcal{H}$ so it follows from part (i) of the same lemma. In part (ii) $p \in \mathcal{H}$ implies that $p(x+1) + p(x) \in \mathcal{H}$. Since $g - 2$ is not a 2-power, $\mathcal{H} + (x^4 + x^2 + 1)((x+1)^{2g-3} + x^{2g-3})$
is a non-trivial coset of $H$, hence is disjoint from $H$. So part (ii) follows from part (ii) of the same lemma again.

**Case 10.** Suppose $r = 0$ and $g = 6$: 

Let $f = x^{2g+1} + x^{2g-3} + x^{2g-5} + p(x)$, where $p(x)$ is a polynomial in $\mathbb{F}_2[x]$ of degree $\leq 2g - 6$.

Suppose $x \mapsto ax + b$ gives rise to an automorphism $\lambda$ of $C$. For any $g \equiv 2 \mod 4$ we show that the only possible extra automorphism is the one given by $a, b$, which are both 3-rd roots of unity over $\mathbb{F}_2$. Apply (5) to coefficients of $x^{2g}, x^{2g-1}, x^{2g-3}, x^{2g-5}$, those are $a^{2g}b, 1 + a^{2g-3}(1 + b^4), 1 + a^{2g-5}$. If $b = 0$, then $a = 1$ so it is trivial. If $b \neq 0$, then $a = b + 1$ and $a^3 = 1$. If it is not trivial, then $a, b$ are 3-rd roots of unity.

When $g = 6$ we have $a^{2g-5} = a^7 = 1$ and $a^6 = 1$ so $a = 1$. This implies $b = 0$. So $\lambda$ is trivial.

**Case 11.** Suppose $r = 0$ and $g = 3$ or 5:

Let $f = x^{2g+1} + x^{2g-3} + p(x)$, where $p(x)$ is a polynomial in $\mathbb{F}_2[x]$ of degree $2g - 5$. In fact, this construction works for every odd $g \geq 3$.

Suppose $x \mapsto ax + b$ gives rise to an automorphism $\lambda$ of $C$. The coefficient of $x^{2g}$ and $x^{2g-1}$ in $f(ax + b) + f(x)$ are $a^{2g}b$ and $a^{2g-1}b$, respectively. At least one of them has to vanish by (5), so $b = 0$. This implies $a = 1$ by applying (5) again.

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**REFERENCES**


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