

ANGULAR SELF-INTERSECTIONS FOR CLOSED GEODESICS ON SURFACES

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ABSTRACT. In this note we consider asymptotic results for self-intersections of closed geodesics on surfaces for which the angle of the intersection occurs in a given arc. We do this by extending Bonahon's definition of intersection forms for surfaces.

0. INTRODUCTION

In recent years, many authors have considered the self-intersection of geodesics [BS], [Bo1], [Po], [La]. In this note we consider the problem of estimating the number of self-intersections for which the angle of intersection lies in a given arc. Let V be a compact surface with negative curvature, and let γ denote a closed geodesic on V of length $l(\gamma)$. It is a classical result of Huber [Hu1] and Margulis [Ma] that the number $N(T)$ of closed geodesics of length at most T satisfies the asymptotic formula $N(T) \sim e^{hT}/hT$, as $T \rightarrow +\infty$, i.e., the ratio of the two sides converges to one. (Here $h > 0$ denotes the topological entropy of the geodesic flow over V .) In the case where V has constant curvature, Huber [Hu2] obtained the stronger estimate $N(T) = \text{li}(e^{hT})(1 + O(e^{-\delta T}))$, for some $\delta > 0$, while for variable curvature this was established only in recent years [PS], [Do].

For $0 \leq \theta_1 < \theta_2 \leq \pi$ we let $i_{\theta_1, \theta_2}(\gamma)$ denote the number of self-intersections of the closed geodesic γ such that the absolute value of the angle of intersection lies in the interval $[\theta_1, \theta_2]$.

Theorem 1. *Given $0 \leq \theta_1 < \theta_2 \leq \pi$, there exists $I = I(\theta_1, \theta_2)$ and $\delta > 0$ such that, for any $\epsilon > 0$,*

$$\# \left\{ \gamma : l(\gamma) \leq T, \frac{i_{\theta_1, \theta_2}(\gamma)}{l(\gamma)^2} \in (I - \epsilon, I + \epsilon) \right\} = \text{li}(e^{hT}) (1 + O(e^{-\delta T})).$$

If $\theta_1 = 0$ and $\theta_2 = \pi$, then $i_{0, \pi}(\gamma)$ is simply the total number of self-intersections of γ . In this particular case, Anantharaman [An] has observed this result with $I(0, \pi) = i(\tilde{m}, \tilde{m})/2$, where \tilde{m} is the transverse measure associated to the measure of maximal entropy for the geodesic flow over V and $i(\cdot, \cdot)$ is the intersection form introduced by Bonahon [Bo1] (see also [Ot]). Our proof follows the same basic approach as that of [La] and [An], in that we apply a standard large deviation result

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for closed orbits. The distinction is that we need to introduce two new ingredients: the angular intersection bundle and the associated angular intersection form. This allows us to restrict to counting the self-intersections of geodesics to those whose angle of intersection lies in a given interval.

We can interpret the constant $I(\theta_1, \theta_2)$ in terms of an *angular intersection form*, generalising $i(\cdot, \cdot)$. For a surface of constant negative curvature, it can be easily described explicitly, and we have the formula

$$I(\theta_1, \theta_2) = \frac{1}{2\pi \text{Area}(V)} \int_{\theta_1}^{\theta_2} \sin \theta d\theta.$$

In particular, using $\text{Area}(V) = 4\pi(g-1)$, where $g \geq 2$ is the genus of V , we can verify a heuristic estimate of Sieber and Richter [SR] that

$$(0.1) \quad \sum_{l(\gamma) \leq T} i_{\theta_1, \theta_2}(\gamma) \sim \left(\int_{\theta_1}^{\theta_2} \sin \theta d\theta \right) \frac{T^2}{8\pi^2(g-1)} N(T).$$

Quantum Chaos provides an interesting motivation for this result associated to the study of pair correlations of energy levels [Ke]. The associated form factor, or Fourier transform, leads to correlations of closed geodesics, and (0.1) is relevant to the leading term of the off-diagonal contribution.

We can also consider the distribution of the self-intersection points of the geodesic γ on the surface V . We let $\mathcal{S}(\gamma; \theta_1, \theta_2) \subset V$ denote the set of self-intersection points of γ such that the absolute value of the angle of intersection lies in the interval $[\theta_1, \theta_2]$. In particular, $i_{\theta_1, \theta_2}(\gamma) = \#\mathcal{S}(\gamma; \theta_1, \theta_2)$. Given a continuous function $f : V \rightarrow \mathbb{R}$, we let $A(f, \gamma; \theta_1, \theta_2)$ denote the sum of f over the points of self-intersection of γ lying in $\mathcal{S}(\gamma; \theta_1, \theta_2)$, i.e.,

$$A(f, \gamma; \theta_1, \theta_2) = \sum_{x \in \mathcal{S}(\gamma; \theta_1, \theta_2)} f(x).$$

Theorem 2. *Given $0 \leq \theta_1 < \theta_2 \leq \pi$, there exists $I_f = I(f; \theta_1, \theta_2)$ and $\delta > 0$ such that, for any $\epsilon > 0$,*

$$\#\left\{ \gamma : l(\gamma) \leq T, \frac{A(f, \gamma; \theta_1, \theta_2)}{l(\gamma)^2} \in (I_f - \epsilon, I_f + \epsilon) \right\} = \text{li}(e^{hT}) (1 + O(e^{-\delta T})).$$

In particular, when $f = 1$, Theorem 2 reduces to Theorem 1.

Sections 1 and 2 contain preliminary material. Section 3 contains the proofs of Theorem 1 and Theorem 2.

1. INTERSECTION BUNDLES AND FORMS

1.1 The geodesic flow. Let SV be the unit tangent bundle of V and let $\pi : SV \rightarrow V$ denote the canonical projection. For $x \in V$, we write $S_x V = \pi^{-1}(x)$. Consider the geodesic flow $\phi_t : SV \rightarrow SV$ on the unit tangent bundle of V . This is an example of a weak-mixing hyperbolic flow [As]. There is a one-to-one correspondence between oriented closed geodesics on V and periodic orbits for ϕ . We shall write $h > 0$ for the topological entropy of ϕ and, given a ϕ -invariant *probability* measure μ , we shall write $h(\mu)$ for the measure theoretic entropy of ϕ with respect to μ . There is a unique invariant probability measure m , called the measure of maximal entropy, for which $h(m) = h$.

We let \mathcal{F} denote the foliation of SV by ϕ -orbits. Given any ϕ -invariant finite measure μ (not necessarily a probability measure) we can consider the associated transverse measure $\tilde{\mu}$ for \mathcal{F} . The set of these transverse measures $\mathcal{C} = \{\tilde{\mu} : \mu \text{ is a } \phi\text{-invariant measure}\}$ is called the *space of currents*. Each $\tilde{\mu} \in \mathcal{C}$ is normalized by the requirement that (locally) $\mu = \tilde{\mu} \times dt$, where dt is one-dimensional Lebesgue measure along orbits in \mathcal{F} .

1.2 Bundles. Let $E = SV \oplus SV - \Delta$ be the Whitney sum of the bundle SV with itself minus the diagonal $\Delta = \{(x, v, v) : x \in V, v \in S_x V\}$, and let $p : E \rightarrow V$ denote the canonical projection. In particular, points of the four-dimensional vector bundle E (with two-dimensional fibres) consist of triples (x, v, w) , where $x \in V$ and $v, w \in S_x V$. Let $p_1 : E \rightarrow SV$ be defined by $p_1(x, v, w) = (x, v)$, and let $p_2 : E \rightarrow SV$ be defined by $p_2(x, v, w) = (x, w)$. Following Bonahon [Bo1], we consider the two transverse foliations (with two-dimensional leaves) of E given by $\mathcal{F}_1 = p_1^{-1}(\mathcal{F}) = \mathcal{F}_2 = p_2^{-1}(\mathcal{F})$.

Definition. Given $0 \leq \theta_1 < \theta_2 \leq \pi$, we define the *angular intersection bundle* E_{θ_1, θ_2} by $E_{\theta_1, \theta_2} = \{(x, v, w) \in E : \angle v, w \in [\theta_1, \theta_2]\}$, where $0 \leq \angle v, w \leq \pi$ denotes the angle between the two vectors. This is a closed subbundle of E .

It is sometimes convenient to consider an equivalent geometric definition of the bundle E_{θ_1, θ_2} in terms of pairs of geodesics on the universal cover \tilde{V} of V . In particular, consider the space $\mathcal{G}_{\theta_1, \theta_2}$ of pairs of oriented geodesics $(\tilde{\gamma}_1, \tilde{\gamma}_2)$ on \tilde{V} which intersect, at some point $\tilde{x} \in \tilde{V}$, say, and with an angle of intersection in the interval $[\theta_1, \theta_2]$. There is a natural identification E_{θ_1, θ_2} with $\mathcal{G}_{\theta_1, \theta_2} / \pi_1(V)$, where the quotient is with respect to the diagonal action of the fundamental group (cf. [Bo2]).

1.3 Intersection forms. Given currents $\tilde{\mu}, \tilde{\mu}' \in \mathcal{C}$, we can take the lifts $\hat{\mu}_1 := p_1^{-1}\tilde{\mu}$ and $\hat{\mu}'_2 := p_2^{-1}\tilde{\mu}'$, which are transverse measures to the foliations \mathcal{F}_1 and \mathcal{F}_2 for E , respectively. Bonahon defined the *intersection form* $i : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}^+$ to be the total mass of the E with respect to the product measure $\hat{\mu}_1 \times \hat{\mu}'_2$, i.e., $i(\tilde{\mu}, \tilde{\mu}') = (\hat{\mu}_1 \times \hat{\mu}'_2)(E)$ [Bo1]. By analogy, we can define the following.

Definition. We define an *angular intersection form* $i_{\theta_1, \theta_2} : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}^+$ to be the total mass of the E_{θ_1, θ_2} with respect to the product measure $\hat{\mu}_1 \times \hat{\mu}'_2$, i.e., $i_{\theta_1, \theta_2}(\tilde{\mu}, \tilde{\mu}') = (\hat{\mu}_1 \times \hat{\mu}'_2)(E_{\theta_1, \theta_2})$.

In the geometric picture, the space of currents corresponds to measures on $\mathcal{G}_{\theta_1, \theta_2} / \pi_1(V)$. For example, given a geodesic γ on V , the associated measure is simply a finite sum of Dirac measures supported on the quotient of pairs of closed geodesics on \tilde{V} consisting of lifts of γ and its $\pi_1(V)$ images (cf. [Bo2]).

More generally, in order to study the spatial distribution of the intersection points, we can introduce a weighted form. Let $f : V \rightarrow \mathbb{R}$ be a continuous function.

Definition. We define the *weighted angular intersection form* $i_{f; \theta_1, \theta_2} : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}^+$ to be the integral of $f \circ p$ restricted to E_{θ_1, θ_2} with respect to the product measure $\hat{\mu}_1 \times \hat{\mu}'_2$, i.e., $i_{f; \theta_1, \theta_2}(\tilde{\mu}, \tilde{\mu}') = \int f \circ p d(\hat{\mu}_1 \times \hat{\mu}'_2)$.

2. INTERSECTION FORMS AND CLOSED GEODESICS

Given a closed geodesic γ , let μ_γ denote the unique invariant measure of total mass $l(\gamma)$ supported on the corresponding periodic orbit and let $\tilde{\mu}_\gamma$ be the corresponding transverse measures for the orbit foliation \mathcal{F} , which are normalized to be a finite sum of Dirac measures on transverse sections. We shall write $\hat{\mu}_{\gamma,1} = p_1^{-1}\tilde{\mu}_\gamma$ and $\hat{\mu}_{\gamma,2} = p_2^{-1}\tilde{\mu}_\gamma$.

Lemma 1. *Let γ and γ' be a pair of closed geodesics. Then:*

- (i) $(\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma',2})(E_{\theta_1,\theta_2})$ is equal to the number of intersections between γ and γ' with angle between θ_1 and θ_2 , and
- (ii) given a continuous function $f : V \rightarrow \mathbb{R}$, $\int_{E_{\theta_1,\theta_2}} f \circ p \, d(\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma',2})$ is equal to the summation of f over the points of intersection of γ and γ' with angle between θ_1 and θ_2 .

Proof. The proof is modelled on [Bo1, p. 111]. We can choose flow boxes $B_1, B_2 \subset SV$ for ϕ such that, provided they are sufficiently small, for $v_1 \in B_1$ and $v_2 \in B_2$ the associated geodesic arcs intersect transversally at one point at most. We denote

$$B_1 \oplus B_2 := p_1^{-1}(B_1) \cap p_2^{-1}(B_2) \subset E.$$

As in [Bo1], one can see that $(\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma',2})(B_1 \oplus B_2)$ is precisely the number of times γ crosses γ' with tangent vectors in B_1 and B_2 , respectively. Covering SV by a partition consisting of such flow boxes, and summing over those $B_1 \oplus B_2$ which approximate E_{θ_1,θ_2} gives an estimate to the number of times the geodesics intersect with angle in $[\theta_1, \theta_2]$, i.e., if $d(\cdot, \cdot)$ denotes the Hausdorff distance between closed subsets of E ,

$$\sum_{d(E_{\theta_1,\theta_2}, B_1 \oplus B_2) < \epsilon} (\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma',2})(B_1 \oplus B_2)$$

can be made arbitrarily close to $i_{\theta_1,\theta_1}(\tilde{\mu}_\gamma, \tilde{\mu}_{\gamma'})$ by choosing $\epsilon > 0$ sufficiently small. By taking the size of the flow boxes arbitrarily small, the result follows.

For part (ii), it suffices to show that, for any small disk $D \subset V$, we have that $(\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma',2})(E_{\theta_1,\theta_2} \cap p^{-1}(D))$ is equal to the number of intersections between γ and γ' lying inside D and with angle between θ_1 and θ_2 . However, this follows from the preceding argument if we further restrict the summation to those $B_1 \oplus B_2$ whose projections $\pi(B_1)$ and $\pi(B_2)$ approximate D . □

As immediate consequences of the above lemma we have that the number of self-intersections of γ with angle lying in the interval $[\theta_1, \theta_2]$ is given by $i_{\theta_1,\theta_2}(\gamma) = (\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma,2})(E_{\theta_1,\theta_2})/2$ and that $A(f, \gamma; \theta_1, \theta_2) = \int_{E_{\theta_1,\theta_2}} f \circ p \, d(\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma,2})/2$.

The next result will show that $\tilde{\mu} \mapsto i_{\theta_1,\theta_2}(\tilde{\mu}, \tilde{\mu})$ is continuous in a neighbourhood of the transverse measure \tilde{m} associated to the measure of maximal entropy m . Let $\hat{m}_1 = p_1^{-1}\tilde{m}$ and $\hat{m}_2 = p_2^{-1}\tilde{m}$.

Lemma 2. *Let γ_n be a sequence of closed geodesics and let $\mu_{\gamma_n}/l(\gamma_n)$ be the (normalised) probability measure on SV . Assume that $\mu_{\gamma_n}/l(\gamma_n)$ converges in the weak* topology to the measure of maximal entropy m . Then*

- (1) $(\hat{\mu}_{\gamma_n,1} \times \hat{\mu}_{\gamma_n,2})(E_{\theta_1,\theta_2}) \rightarrow (\hat{m}_1 \times \hat{m}_2)(E_{\theta_1,\theta_2})$, as $n \rightarrow +\infty$; and
- (2) given a continuous function $f : V \rightarrow \mathbb{R}$, $\int_{E_{\theta_1,\theta_2}} (f \circ p) \, d(\hat{\mu}_{\gamma_n,1} \times \hat{\mu}_{\gamma_n,2})(E_{\theta_1,\theta_2}) \rightarrow \int_{E_{\theta_1,\theta_2}} (f \circ p) \, d(\hat{m}_1 \times \hat{m}_2)$, as $n \rightarrow +\infty$.

Proof. We recall Bonahon’s proof that $(\widehat{\mu}_{\gamma_n,1} \times \widehat{\mu}_{\gamma_n,2})(E) \rightarrow (\widehat{m}_1 \times \widehat{m}_2)(E)$, as $n \rightarrow +\infty$ [Bo1, pp. 112-114]. In his proof, he approximated $E - U$, where U is a neighbourhood of the diagonal Δ , by unions of sets $B_1 \oplus B_2$, where B_1 and B_2 are sufficiently small flow boxes. He then showed that $(\widehat{\mu}_{\gamma_n,1} \times \widehat{\mu}_{\gamma_n,2})(B_1 \oplus B_2) \rightarrow (\widehat{m}_1 \times \widehat{m}_2)(B_1 \oplus B_2)$, as $n \rightarrow +\infty$. By similarly approximating E_{θ_1, θ_2} by unions $B_1 \oplus B_2$, we see that part (1) holds.

For part (2), it suffices to show that, for any small disk $D \subset V$, we have that $(\widehat{\mu}_{\gamma_n,1} \times \widehat{\mu}_{\gamma_n,2})(E_{\theta_1, \theta_2} \cap p^{-1}(D)) \rightarrow (\widehat{m}_1 \times \widehat{m}_2)(E_{\theta_1, \theta_2} \cap p^{-1}(D))$, as $n \rightarrow +\infty$. In the preceding argument we may further restrict the union to those $B_1 \oplus B_2$ whose projections $\pi(B_1)$ and $\pi(B_2)$ approximate D . This completes the proof. \square

3. PROOF OF THE THEOREMS

Theorems 1 and 2 will follow from a large deviation result for periodic orbit measures established by Kifer [Ki]. Kifer’s result is valid for any hyperbolic flow and so, in particular, for the geodesic flow $\phi_t : SV \rightarrow SV$.

Lemma 3 ([Ki]). *Let \mathcal{U} be an open neighbourhood of the measure of maximal entropy m in the set of ϕ -invariant probability measures on SV . Then*

$$\frac{1}{N(T)} \#\{\gamma : l(\gamma) \leq T \text{ and } \mu_\gamma/l(\gamma) \notin \mathcal{U}\} = O(e^{-\delta T}),$$

as $T \rightarrow +\infty$, where $\delta = \inf_{\nu \in \mathcal{U}^c} \{h - h(\nu)\}$.

In particular, if $m \in \mathcal{U}$, then $\delta > 0$. In our context, Lemma 3 gives the following estimates.

Lemma 4. (1) *Given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\begin{aligned} \frac{1}{N(T)} \#\{\gamma : l(\gamma) \leq T \text{ and } |l(\gamma)^{-2}(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma,2})(E_{\theta_1, \theta_2}) \\ - (\widehat{m}_1 \times \widehat{m}_2)(E_{\theta_1, \theta_2})| \geq \epsilon\} = O(e^{-\delta T}), \end{aligned}$$

as $T \rightarrow +\infty$; and

(2) *Let $f : V \rightarrow \mathbb{R}$ be a continuous function. Given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\begin{aligned} \frac{1}{N(T)} \#\left\{ \gamma : l(\gamma) \leq T \text{ and } \left| \frac{1}{l(\gamma)^2} \int_{E_{\theta_1, \theta_2}} f \circ p \, d(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma,2}) \right. \right. \\ \left. \left. - \int_{E_{\theta_1, \theta_2}} f \circ p \, d(\widehat{m}_1 \times \widehat{m}_2) \right| \geq \epsilon \right\} = O(e^{-\delta T}), \end{aligned}$$

as $T \rightarrow +\infty$.

Proof. To prove part (1), we cannot directly apply Lemma 3 with

$$\mathcal{U} = \{\nu : |(\widehat{\nu}_1 \times \widehat{\nu}_2)(E_{\theta_1, \theta_2}) - (\widehat{m}_1 \times \widehat{m}_2)(E_{\theta_1, \theta_2})| < \epsilon\},$$

since, as the indicator function for E_{θ_1, θ_2} is not continuous, this set is not open. The measure m is known to be non-atomic, and so we can deduce that $\widehat{m}_1 \times \widehat{m}_2$ is also non-atomic. In particular, we can choose continuous functions $\psi_1 \leq \chi_{E_{\theta_1, \theta_2}} \leq \psi_2$

such that

$$\int \psi_2 d(\widehat{m}_1 \times \widehat{m}_2) - \tau \leq (\widehat{m}_1 \times \widehat{m}_2)(E_{\theta_1, \theta_2}) \leq \int \psi_1 d(\widehat{m}_1 \times \widehat{m}_2) + \tau,$$

for some $0 < \tau < \epsilon$, and we can then deduce the required bound by considering

$$\begin{aligned} \mathcal{U}' = & \left\{ \nu : \int \psi_2 d(\widehat{\nu}_1 \times \widehat{\nu}_2) - \int \psi_2 d(\widehat{\nu}_1 \times \widehat{\nu}_2) < \epsilon - \tau \right\} \\ & \cap \left\{ \nu : \int \psi_1 d(\widehat{\nu}_1 \times \widehat{\nu}_2) - \int \psi_1 d(\widehat{\nu}_1 \times \widehat{\nu}_2) > -\epsilon + \tau \right\} \end{aligned}$$

and noting that

$$\#\{\gamma : l(\gamma) \leq T \text{ and } \mu_\gamma/l(\gamma) \notin \mathcal{U}\} \leq \#\{\gamma : l(\gamma) \leq T \text{ and } \mu_\gamma/l(\gamma) \notin \mathcal{U}'\}.$$

Clearly, $m \in \mathcal{U}'$, so $\delta > 0$, as required.

If f is non-negative, then the proof of part (2) is similar. By choosing continuous functions $\psi_3(x, v) \leq (f \circ p)(x)\chi_{E_{\theta_1, \theta_2}}(v) \leq \psi_4(x, v)$ such that

$$\int \psi_4 d(\widehat{m}_1 \times \widehat{m}_2) - \tau \leq \int_{E_{\theta_1, \theta_2}} f \circ p d(\widehat{m}_1 \times \widehat{m}_2) \leq \int \psi_3 d(\widehat{m}_1 \times \widehat{m}_2) + \tau,$$

for some $0 < \tau < \epsilon$, we can deduce the required bound by considering

$$\begin{aligned} \mathcal{U}'' = & \left\{ \gamma : \frac{1}{l(\gamma)^2} \int \psi_4 d(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma,2}) - \int \psi_4 d(\widehat{m}_1 \times \widehat{m}_2) < \epsilon - \tau \right\} \\ & \cap \left\{ \gamma : \frac{1}{l(\gamma)^2} \int \psi_3 d(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma,2}) - \int \psi_3 d(\widehat{m}_1 \times \widehat{m}_2) > -\epsilon + \tau \right\} \end{aligned}$$

and noting that

$$\#\{\gamma : l(\gamma) \leq T \text{ and } \mu_\gamma/l(\gamma) \notin \mathcal{U}\} \leq \#\{\gamma : l(\gamma) \leq T \text{ and } \mu_\gamma/l(\gamma) \notin \mathcal{U}''\}.$$

Clearly, $m \in \mathcal{U}''$, so $\delta > 0$, as required. To obtain the general case, consider the positive and negative parts of f separately. \square

Proof of Theorem 1. Write $I(\theta_1, \theta_2) = i_{\theta_1, \theta_2}(\tilde{m}, \tilde{m})/2$ and recall that $i_{\theta_1, \theta_2}(\gamma) = (\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma,2})(E_{\theta_1, \theta_2})/2$. We can apply part (1) of Lemma 4 to deduce that, except for an exceptional set with cardinality of order $O(e^{(h-\delta)T})$, the set of closed geodesics of length at most T satisfy $|l(\gamma)^{-2}i_{\theta_1, \theta_2}(\gamma) - I(\theta_1, \theta_2)| < \epsilon$. Theorem 1 then follows by applying the asymptotic counting results described in the introduction. \square

Proof of Theorem 2. This is similar to the proof of Theorem 1. We can write $I(\theta_1, \theta_2, f) = i_{\theta_1, \theta_2, f}(\tilde{m}, \tilde{m})/2$ and denote $i_{\theta_1, \theta_2, f}(\gamma) = i_{\theta_1, \theta_2}(\gamma)A(f, \gamma; \theta_1, \theta_2)$. We can apply part (1) of Lemma 4 to deduce that, except for an exceptional set with cardinality of order $O(e^{(h-\delta)T})$, the set of closed geodesics of length at most T satisfies $|l(\gamma)^{-2}i_{\theta_1, \theta_2, f}(\gamma) - I(\theta_1, \theta_2, f)| < \epsilon$. Theorem 2 then follows by applying the asymptotic counting results described in the Introduction. \square

An expression for $I(\theta_1, \theta_2) = (\widehat{m}_1 \times \widehat{m}_2)(E_{\theta_1, \theta_2})/2$ can be given fairly explicitly. Let m_V be the projection onto the surface of m . For each $x \in V$, let m_x be the induced measure on the fibre $S_x V$. Note that m_x cannot have atoms (for m_V -a.e. x), since otherwise invariance of the measure would imply the existence of closed orbits with non-zero m measure. However, since the measure of maximal entropy m is fully supported, such orbits would immediately contradict the ergodicity of m .

Lemma 5. *We can write*

$$I(\theta_1, \theta_2) = \int_V \int \left(\int_{\substack{\{(v_1, v_2) \in S_x V \times S_x V \\ \angle v_1, v_2 \in [\theta_1, \theta_2]\}}} \sin(\angle v_1, v_2) d(m_x \times m_x)(v_1, v_2) \right) dm_V(x),$$

where $\angle v_1, v_2$ is the angle between the vectors $v_1, v_2 \in S_x V$ in the same fibre.

Proof. The intersection form is defined in terms of the total mass of the angular intersection bundle with respect to a product measure. One can approximate this value by approximating the bundle E_{θ_1, θ_2} by the unions of small sets of the form $B_1 \oplus B_2 = p_1^{-1}(B_1) \cap p_2^{-1}(B_2)$ for pairs of flow boxes B_1, B_2 in SV whose projections onto V intersect transversely at a single point. By choosing the pairs of flow boxes B_1, B_2 sufficiently small, we can choose the angle of intersection of any geodesic arcs associated to B_1 and B_2 , respectively, to be arbitrarily close to a constant θ , say. The mass with respect to the product measure of the small set $B_1 \oplus B_2$ is now particularly easy to estimate. It is given by the usual formula for the area of a parallelogram (height \times base $\times \sin \theta$). Finally, we can take the union over all such sets $B_1 \oplus B_2$ in the approximation to E_{θ_1, θ_2} . The estimate then follows by approximation. \square

In the particular case of constant curvature, one easily sees that the expression in Lemma 5 reduces to (0.1).

REFERENCES

- [An] N. Anantharaman, *Distribution of closed geodesics on a surface, under homological constraints*, preprint, 1999.
- [As] D. Anosov, *Geodesic flows of closed Riemann manifolds with negative curvature*, Proceedings of the Steklov Institute of Mathematics, Vol. 90, Amer. Math. Soc., Providence, RI, 1969. MR0242194 (39:3527)
- [BS] J. Birman and C. Series, *Geodesics with bounded intersection number on surfaces are sparsely distributed*, Topology **24** (1985), 217-225. MR0793185 (87f:57012)
- [Bo1] F. Bonahon, *Bouts des variétés hyperboliques de dimension 3*, Annals of Math. **124** (1986), 71-158. MR0847953 (88c:57013)
- [Bo2] F. Bonahon, *The geometry of Teichmüller spaces via geodesic currents*, Invent. Math. **92** (1988), 139-162. MR0931208 (90a:32025)
- [Do] D. Dolgopyat, *On statistical properties of geodesic flows on negatively curved surfaces*, Ph.D. thesis, Princeton, 1997.
- [Hu1] H. Huber, *Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen*, Math. Ann. **138** (1959), 1-26. MR0109212 (22:99)
- [Hu2] H. Huber, *Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen, II*, Math. Ann. **142** (1961), 385-398. MR0126549 (23:A3845)
- [Ke] J. Keating, *Periodic orbits, spectral statistics, and the Riemann zeros*, Supersymmetry and trace formulae (I. Lerner, J. Keating and D. Khmelnitskii, eds.), Kluwer, New York, 1999, pp. 1-15.
- [Ki] Y. Kifer, *Large deviations, averaging and periodic orbits of dynamical systems*, Comm. Math. Phys. **162** (1994), 33-46. MR1272765 (95b:58091)
- [La] S. Lalley, *Self-intersections of closed geodesics on a negatively curved surface: statistical regularities*, Convergence in ergodic theory and probability (Columbus, OH, 1993), Ohio State Univ. Math. Res. Inst. Publ., 5, de Gruyter, Berlin, 1996, pp. 263-272. MR1412610 (97h:58103)
- [Ma] G. Margulis, *Certain applications of ergodic theory to the study of manifolds of negative curvature*, Functional Anal. Appl. **3** (1969), 89-90. MR0257933 (41:2582)
- [Ot] J.-P. Otal, *Le théorème d'hyperbolisation pour les variétés fibreés de dimension 3*, Astérisque **235** (1996), 1-159. MR1402300 (97e:57013)

- [Po] M. Pollicott, *Asymptotic distribution of closed geodesics*, Israel J. Math. **52** (1985), 209-224. MR0815810 (87g:58105)
- [PS] M. Pollicott and R. Sharp, *Exponential error terms for growth functions on negatively curved surfaces*, Amer. J. Math. **120** (1998), 1019-1042. MR1646052 (99h:58148)
- [SR] M. Sieber and K. Richter, *Correlations between periodic orbits and their rôle in spectral statistics*, Physica Scripta **T90** (2001), 128-133.

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