ANGULAR SELF-INTERSECTIONS
FOR CLOSED GEODESICS ON SURFACES

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Abstract. In this note we consider asymptotic results for self-intersections of closed geodesics on surfaces for which the angle of the intersection occurs in a given arc. We do this by extending Bonahon’s definition of intersection forms for surfaces.

0. Introduction

In recent years, many authors have considered the self-intersection of geodesics [BS], [Bo1], [Po], [La]. In this note we consider the problem of estimating the number of self-intersections for which the angle of intersection lies in a given arc. Let \( V \) be a compact surface with negative curvature, and let \( \gamma \) denote a closed geodesic on \( V \) of length \( l(\gamma) \). It is a classical result of Huber [Hu1] and Margulis [Ma] that the number \( N(T) \) of closed geodesics of length at most \( T \) satisfies the asymptotic formula \( N(T) \sim e^{hT}/hT \), as \( T \to +\infty \), i.e., the ratio of the two sides converges to one. (Here \( h > 0 \) denotes the topological entropy of the geodesic flow over \( V \).) In the case where \( V \) has constant curvature, Huber [Hu2] obtained the stronger estimate \( N(T) = \log(e^{hT})(1 + O(e^{-\delta T})) \), for some \( \delta > 0 \), while for variable curvature this was established only in recent years [PS], [Do].

For \( 0 \leq \theta_1 < \theta_2 \leq \pi \) we let \( i_{\theta_1,\theta_2}(\gamma) \) denote the number of self-intersections of the closed geodesic \( \gamma \) such that the absolute value of the angle of intersection lies in the interval \( [\theta_1,\theta_2] \).

Theorem 1. Given \( 0 \leq \theta_1 < \theta_2 \leq \pi \), there exists \( I = I(\theta_1, \theta_2) \) and \( \delta > 0 \) such that, for any \( \epsilon > 0 \),

\[
\# \left\{ \gamma : l(\gamma) \leq T, \frac{i_{\theta_1,\theta_2}(\gamma)}{l(\gamma)^2} \in (I - \epsilon, I + \epsilon) \right\} \equiv \log \left( e^{hT} \right) \left( 1 + O(e^{-\delta T}) \right).
\]

If \( \theta_1 = 0 \) and \( \theta_2 = \pi \), then \( i_{0,\pi}(\gamma) \) is simply the total number of self-intersections of \( \gamma \). In this particular case, Anantharaman [An] has observed this result with \( I(0, \pi) = i(\tilde{m}, \tilde{m})/2 \), where \( \tilde{m} \) is the transverse measure associated to the measure of maximal entropy for the geodesic flow over \( V \) and \( i(\cdot, \cdot) \) is the intersection form introduced by Bonahon [Bo1] (see also [Ot]). Our proof follows the same basic approach as that of [La] and [An], in that we apply a standard large deviation result.
for closed orbits. The distinction is that we need to introduce two new ingredients: the angular intersection bundle and the associated angular intersection form. This allows us to restrict to counting the self-intersections of geodesics to those whose angle of intersection lies in a given interval.

We can interpret the constant $I(\theta_1, \theta_2)$ in terms of an angular intersection form, generalising $i(\cdot, \cdot)$. For a surface of constant negative curvature, it can be easily described explicitly, and we have the formula

$$I(\theta_1, \theta_2) = \frac{1}{2\pi \text{Area}(V)} \int_{\theta_1}^{\theta_2} \sin \theta d\theta.$$ 

In particular, using $\text{Area}(V) = 4\pi (g - 1)$, where $g \geq 2$ is the genus of $V$, we can verify a heuristic estimate of Sieber and Richter [SR] that

$$\sum_{l(\gamma) \leq T} i_{\theta_1, \theta_2}(\gamma) \sim \left( \int_{\theta_1}^{\theta_2} \sin \theta d\theta \right) \frac{T^2}{8\pi^2 (g - 1)} N(T).$$

Quantum Chaos provides an interesting motivation for this result associated to the study of pair correlations of energy levels [Ke]. The associated form factor, or Fourier transform, leads to correlations of closed geodesics, and (0.1) is relevant to the leading term of the off-diagonal contribution.

We can also consider the distribution of the self-intersection points of the geodesic $\gamma$ on the surface $V$. We let $S(\gamma; \theta_1, \theta_2) \subset V$ denote the set of self-intersection points of $\gamma$ such that the absolute value of the angle of intersection lies in the interval $[\theta_1, \theta_2]$. In particular, $i_{\theta_1, \theta_2}(\gamma) = \#S(\gamma; \theta_1, \theta_2)$. Given a continuous function $f : V \to \mathbb{R}$, we let $A(f, \gamma; \theta_1, \theta_2)$ denote the sum of $f$ over the points of self-intersection of $\gamma$ lying in $S(\gamma; \theta_1, \theta_2)$, i.e.,

$$A(f, \gamma; \theta_1, \theta_2) = \sum_{x \in S(\gamma; \theta_1, \theta_2)} f(x).$$

**Theorem 2.** Given $0 \leq \theta_1 < \theta_2 \leq \pi$, there exists $I_f = I(f; \theta_1, \theta_2)$ and $\delta > 0$ such that, for any $\epsilon > 0$,

$$\# \left\{ \gamma : l(\gamma) \leq T, \frac{A(f, \gamma; \theta_1, \theta_2)}{l(\gamma)^2} \in (I_f - \epsilon, I_f + \epsilon) \right\} = \text{li}(e^{hT}) \left( 1 + O(e^{-\delta T}) \right).$$

In particular, when $f = 1$, Theorem 2 reduces to Theorem 1.

Sections 1 and 2 contain preliminary material. Section 3 contains the proofs of Theorem 1 and Theorem 2.

1. **INTERSECTION BUNDLES AND FORMS**

1.1 The geodesic flow. Let $SV$ be the unit tangent bundle of $V$ and let $\pi : SV \to V$ denote the canonical projection. For $x \in V$, we write $S_x V = \pi^{-1}(x)$. Consider the geodesic flow $\phi_t : SV \to SV$ on the unit tangent bundle of $V$. This is an example of a weak-mixing hyperbolic flow [As]. There is a one-to-one correspondence between oriented closed geodesics on $V$ and periodic orbits for $\phi$. We shall write $h > 0$ for the topological entropy of $\phi$ and, given a $\phi$-invariant probability measure $\mu$, we shall write $h(\mu)$ for the measure theoretic entropy of $\phi$ with respect to $\mu$.

There is a unique invariant probability measure $m$, called the measure of maximal entropy, for which $h(m) = h$. 

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We let $\mathcal{F}$ denote the foliation of $SV$ by $\phi$-orbits. Given any $\phi$-invariant finite measure $\mu$ (not necessarily a probability measure) we can consider the associated transverse measure $\hat{\mu}$ for $\mathcal{F}$. The set of these transverse measures $\mathcal{C} = \{\hat{\mu} : \mu$ is a $\phi$-invariant measure$\}$ is called the space of currents. Each $\hat{\mu} \in \mathcal{C}$ is normalized by the requirement that (locally) $\hat{\mu} = \mu \times dt$, where $dt$ is one-dimensional Lebesgue measure along orbits in $\mathcal{F}$.

1.2 Bundles. Let $E = SV \oplus SV - \Delta$ be the Whitney sum of the bundle $SV$ with itself minus the diagonal $\Delta = \{(x, v, v) : x \in V, v \in S_x V\}$, and let $p : E \to V$ denote the canonical projection. In particular, points of the four-dimensional vector bundle $E$ (with two-dimensional fibres) consist of triples $(x, v, w)$, where $x \in V$ and $v, w \in S_x V$. Let $p_1 : E \to SV$ be defined by $p_1(x, v, w) = (x, v)$, and let $p_2 : E \to SV$ be defined by $p_2(x, v, w) = (x, w)$. Following Bonahon [Bo1], we consider the two transverse foliations (with two-dimensional leaves) of $E$ given by $\mathcal{F}_1 = p_1^{-1}(\mathcal{F}) = \mathcal{F}_2 = p_2^{-1}(\mathcal{F})$.

**Definition.** Given $0 \leq \theta_1 < \theta_2 \leq \pi$, we define the angular intersection bundle $E_{\theta_1, \theta_2}$ by $E_{\theta_1, \theta_2} = \{(x, v, w) \in E : \angle v, w \in [\theta_1, \theta_2]\}$, where $0 \leq \angle v, w \leq \pi$ denotes the angle between the two vectors. This is a closed subbundle of $E$.

It is sometimes convenient to consider an equivalent geometric definition of the bundle $E_{\theta_1, \theta_2}$ in terms of pairs of oriented geodesics on the universal cover $\tilde{V}$ of $V$. In particular, consider the space $G_{\theta_1, \theta_2}$ of pairs of oriented geodesics $(\tilde{\gamma}_1, \tilde{\gamma}_2)$ on $\tilde{V}$ which intersect, at some point $\tilde{x} \in \tilde{V}$, say, and with an angle of intersection in the interval $[\theta_1, \theta_2]$. There is a natural identification $E_{\theta_1, \theta_2}$ with $G_{\theta_1, \theta_2}/\pi_1(V)$, where the quotient is with respect to the diagonal action of the fundamental group (cf. [Bo2]).

1.3 Intersection forms. Given currents $\hat{\mu}, \hat{\mu}' \in \mathcal{C}$, we can take the lifts $\hat{\mu}_1 := p_1^{-1}\hat{\mu}$ and $\hat{\mu}_2 := p_2^{-1}\hat{\mu}'$, which are transverse measures to the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ for $E$, respectively. Bonahon defined the intersection form $i : \mathcal{C} \times \mathcal{C} \to \mathbb{R}^+$ to be the total mass of the $E$ with respect to the product measure $\hat{\mu}_1 \times \hat{\mu}_2$, i.e., $i(\hat{\mu}, \hat{\mu}') = (\hat{\mu}_1 \times \hat{\mu}_2)(E)$ [Bo1]. By analogy, we can define the following.

**Definition.** We define an angular intersection form $i_{\theta_1, \theta_2} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}^+$ to be the total mass of the $E_{\theta_1, \theta_2}$ with respect to the product measure $\hat{\mu}_1 \times \hat{\mu}_2$, i.e.,

$$
i_{\theta_1, \theta_2}(\hat{\mu}, \hat{\mu}') = (\hat{\mu}_1 \times \hat{\mu}_2)(E_{\theta_1, \theta_2}).$$

In the geometric picture, the space of currents corresponds to measures on $G_{\theta_1, \theta_2}/\pi_1(V)$. For example, given a geodesic $\gamma$ on $V$, the associated measure is simply a finite sum of Dirac measures supported on the quotient of pairs of closed geodesics on $\tilde{V}$ consisting of lifts of $\gamma$ and its $\pi_1(V)$ images (cf. [Bo2]).

More generally, in order to study the spatial distribution of the intersection points, we can introduce a weighted form. Let $f : V \to \mathbb{R}$ be a continuous function.

**Definition.** We define the weighted angular intersection form $i_{f, \theta_1, \theta_2} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}^+$ to be the integral of $f \circ p$ restricted to $E_{\theta_1, \theta_2}$ with respect to the product measure $\hat{\mu}_1 \times \hat{\mu}_2$, i.e.,

$$
i_{f, \theta_1, \theta_2}(\hat{\mu}, \hat{\mu}') = \int (f \circ p) d(\hat{\mu}_1 \times \hat{\mu}_2).$$
2. Intersection forms and closed geodesics

Given a closed geodesic $\gamma$, let $\mu_\gamma$ denote the unique invariant measure of total mass $l(\gamma)$ supported on the corresponding periodic orbit and let $\hat{\mu}_\gamma$ be the corresponding transverse measures for the orbit foliation $\mathcal{F}$, which are normalized to be a finite sum of Dirac measures on transverse sections. We shall write $\hat{\mu}_{\gamma,1} = p_1^{-1} \hat{\mu}_\gamma$ and $\hat{\mu}_{\gamma,2} = p_2^{-1} \hat{\mu}_\gamma$.

Lemma 1. Let $\gamma$ and $\gamma'$ be a pair of closed geodesics. Then:

(i) $(\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma',2})(E_{\theta_1,\theta_2})$ is equal to the number of intersections between $\gamma$ and $\gamma'$ with angle between $\theta_1$ and $\theta_2$, and

(ii) given a continuous function $f : V \to \mathbb{R}$, $\int_{E_{\theta_1,\theta_2}} f \circ p \, d(\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma',2})$ is equal to the summation of $f$ over the points of intersection of $\gamma$ and $\gamma'$ with angle between $\theta_1$ and $\theta_2$.

Proof. The proof is modelled on [Bo1] p. 111. We can choose flow boxes $B_1, B_2 \subset \Sigma$ for $\phi$ such that, provided they are sufficiently small, for $\nu_1 \in B_1$ and $\nu_2 \in B_2$ the associated geodesic arcs intersect transversally at one point at most. We denote

$$B_1 \oplus B_2 := p_1^{-1}(B_1) \cap p_2^{-1}(B_2) \subset E.$$

As in [Bo1], one can see that $(\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma',2})(B_1 \oplus B_2)$ is precisely the number of times $\gamma$ crosses $\gamma'$ with tangent vectors in $B_1$ and $B_2$, respectively. Covering $\Sigma$ by a partition consisting of such flow boxes, and summing over those $B_1 \oplus B_2$ which approximate $E_{\theta_1,\theta_2}$ gives an estimate to the number of times the geodesics intersect with angle in $[\theta_1, \theta_2]$, i.e., if $d(\cdot, \cdot)$ denotes the Hausdorff distance between closed subsets of $E$,

$$\sum_{d(E_{\theta_1,\theta_2}, B_1 \oplus B_2) < \epsilon} (\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma',2})(B_1 \oplus B_2)$$

can be made arbitrarily close to $i_{\theta_1,\theta_2}(\hat{\mu}_{\gamma}, \hat{\mu}_{\gamma'})$ by choosing $\epsilon > 0$ sufficiently small. By taking the size of the flow boxes arbitrarily small, the result follows.

For part (ii), it suffices to show that, for any small disk $D \subset V$, we have that $(\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma',2})(E_{\theta_1,\theta_2} \cap p^{-1}(D))$ is equal to the number of intersections between $\gamma$ and $\gamma'$ lying inside $D$ and with angle between $\theta_1$ and $\theta_2$. However, this follows from the preceding argument if we further restrict the summation to those $B_1 \oplus B_2$ whose projections $\pi(B_1)$ and $\pi(B_2)$ approximate $D$. $\square$

As immediate consequences of the above lemma we have that the number of self-intersections of $\gamma$ with angle lying in the interval $[\theta_1, \theta_2]$ is given by $i_{\theta_1,\theta_2}(\gamma) = (\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma',2})(E_{\theta_1,\theta_2})/2$ and that $A(f; \gamma; \theta_1, \theta_2) = \int_{E_{\theta_1,\theta_2}} f \circ p \, d(\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma',2})/2$.

The next result will show that $\tilde{\mu} \mapsto i_{\theta_1,\theta_2}(\tilde{\mu}, \tilde{\mu})$ is continuous in a neighbourhood of the transverse measure $\tilde{m}$ associated to the measure of maximal entropy $m$. Let $\tilde{m}_1 = p_1^{-1}\tilde{m}$ and $\tilde{m}_2 = p_2^{-1}\tilde{m}$.

Lemma 2. Let $\gamma_n$ be a sequence of closed geodesics and let $\mu_{\gamma_n}/l(\gamma_n)$ be the (normalised) probability measure on $\Sigma$. Assume that $\mu_{\gamma_n}/l(\gamma_n)$ converges in the weak* topology to the measure of maximal entropy $m$. Then

(i) $(\hat{\mu}_{\gamma_n,1} \times \hat{\mu}_{\gamma_n,2})(E_{\theta_1,\theta_2}) \to (\tilde{m}_1 \times \tilde{m}_2)(E_{\theta_1,\theta_2})$, as $n \to +\infty$; and

(ii) given a continuous function $f : V \to \mathbb{R}$, $\int_{E_{\theta_1,\theta_2}} (f \circ p) \, d(\hat{\mu}_{\gamma_n,1} \times \hat{\mu}_{\gamma_n,2})(E_{\theta_1,\theta_2}) \to \int_{E_{\theta_1,\theta_2}} (f \circ p) \, d(\tilde{m}_1 \times \tilde{m}_2)$, as $n \to +\infty$. 
Proof. We recall Bonahon’s proof that \((\hat{\mu}_{\gamma,n,1} \times \hat{\mu}_{\gamma,n,2})(E) \to (\hat{m}_1 \times \hat{m}_2)(E)\), as \(n \to +\infty\) [Bo1 pp. 112-114]. In his proof, he approximated \(E - U\), where \(U\) is a neighbourhood of the diagonal \(\Delta\), by unions of sets \(B_1 \oplus B_2\), where \(B_1\) and \(B_2\) are sufficiently small flow boxes. He then showed that \((\hat{\mu}_{\gamma,n,1} \times \hat{\mu}_{\gamma,n,2})(B_1 \oplus B_2) \to (\hat{m}_1 \times \hat{m}_2)(B_1 \oplus B_2)\), as \(n \to +\infty\). By similarly approximating \(E_{\theta_1,\theta_2}\) by unions \(B_1 \oplus B_2\), we see that part (1) holds.

For part (2), it suffices to show that, for any small disk \(D \subset V\), we have that \((\hat{\mu}_{\gamma,n,1} \times \hat{\mu}_{\gamma,n,2})(E_{\theta_1,\theta_2} \cap p^{-1}(D)) \to (\hat{m}_1 \times \hat{m}_2)(E_{\theta_1,\theta_2} \cap p^{-1}(D))\), as \(n \to +\infty\). In the preceding argument we may further restrict the union to those \(B_1 \oplus B_2\) whose projections \(\pi(B_1)\) and \(\pi(B_2)\) approximate \(D\). This completes the proof. \(\square\)

3. PROOF OF THE THEOREMS

Theorems 1 and 2 will follow from a large deviation result for periodic orbit measures established by Kifer [Ki]. Kifer’s result is valid for any hyperbolic flow and so, in particular, for the geodesic flow \(\phi_t : SV \to SV\).

Lemma 3 ([Ki]). Let \(U\) be an open neighbourhood of the measure of maximal entropy \(m\) in the set of \(\phi\)-invariant probability measures on \(SV\). Then

\[
\frac{1}{N(T)} \# \{ \gamma : l(\gamma) \leq T \text{ and } \mu_{\gamma}/l(\gamma) \notin U \} = O(e^{-\delta T}),
\]
as \(T \to +\infty\), where \(\delta = \inf_{\nu \in U} \{ h(h(\nu)) \} \).

In particular, if \(m \in U\), then \(\delta > 0\). In our context, Lemma 3 gives the following estimates.

Lemma 4. (1) Given \(\epsilon > 0\), there exists \(\delta > 0\) such that

\[
\frac{1}{N(T)} \# \{ \gamma : l(\gamma) \leq T \text{ and } l(\gamma)^{-2}(\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma,2})(E_{\theta_1,\theta_2}) - (\hat{m}_1 \times \hat{m}_2)(E_{\theta_1,\theta_2}) \geq \epsilon \} = O(e^{-\delta T}),
\]
as \(T \to +\infty\); and

(2) Let \(f : V \to \mathbb{R}\) be a continuous function. Given \(\epsilon > 0\), there exists \(\delta > 0\) such that

\[
\frac{1}{N(T)} \# \left\{ \gamma : l(\gamma) \leq T \text{ and } \left| \frac{1}{l(\gamma)^2} \int_{E_{\theta_1,\theta_2}} f \circ p \, d(\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma,2}) - \int_{E_{\theta_1,\theta_2}} f \circ p \, d(\hat{m}_1 \times \hat{m}_2) \right| \geq \epsilon \right\} = O(e^{-\delta T}),
\]
as \(T \to +\infty\).

Proof. To prove part (1), we cannot directly apply Lemma 3 with

\[
\mathcal{U} = \{ \nu : |(\hat{\nu}_1 \times \hat{\nu}_2)(E_{\theta_1,\theta_2}) - (\hat{m}_1 \times \hat{m}_2)(E_{\theta_1,\theta_2})| < \epsilon \},
\]
since, as the indicator function for \(E_{\theta_1,\theta_2}\) is not continuous, this set is not open. The measure \(m\) is known to be non-atomic, and so we can deduce that \(\hat{m}_1 \times \hat{m}_2\) is also non-atomic. In particular, we can choose continuous functions \(\psi_1 \leq \chi_{E_{\theta_1,\theta_2}} \leq \psi_2\)
such that
\[ \int \psi_2 d(\hat{m}_1 \times \hat{m}_2) - \tau \leq (\hat{m}_1 \times \hat{m}_2) (E_{\theta_1, \theta_2}) \leq \int \psi_1 d(\hat{m}_1 \times \hat{m}_2) + \tau, \]
for some \( 0 < \tau < \epsilon \), and we can then deduce the required bound by considering
\[ \mathcal{U}' = \left\{ \nu : \int \psi_2 d(\hat{\nu}_1 \times \hat{\nu}_2) - \int \psi_2 d(\hat{\nu}_1 \times \hat{\nu}_2) < \epsilon - \tau \right\} \]
\[ \cap \left\{ \nu : \int \psi_1 d(\hat{\nu}_1 \times \hat{\nu}_2) - \int \psi_1 d(\hat{\nu}_1 \times \hat{\nu}_2) > - \epsilon + \tau \right\} \]
and noting that
\[ \# \{ \gamma : l(\gamma) \leq T \text{ and } \mu_{\gamma}/l(\gamma) \notin \mathcal{U} \} \leq \# \{ \gamma : l(\gamma) \leq T \text{ and } \mu_{\gamma}/l(\gamma) \notin \mathcal{U}' \}. \]
Clearly, \( m \in \mathcal{U}' \), so \( \delta > 0 \), as required.

If \( f \) is non-negative, then the proof of part (2) is similar. By choosing continuous functions \( \psi_3(x, \nu) \leq (f \circ p)(x) \chi_{E_{\theta_1, \theta_2}}(\nu) \leq \psi_4(x, \nu) \) such that
\[ \int \psi_4 d(\hat{m}_1 \times \hat{m}_2) - \tau \leq \int f \circ p d(\hat{m}_1 \times \hat{m}_2) \leq \int \psi_3 d(\hat{m}_1 \times \hat{m}_2) + \tau, \]
for some \( 0 < \tau < \epsilon \), we can deduce the required bound by considering
\[ \mathcal{U}'' = \left\{ \gamma : \frac{1}{l(\gamma)} \int \psi_4 d(\hat{\nu}_{\gamma,1} \times \hat{\nu}_{\gamma,2}) - \int \psi_4 d(\hat{m}_1 \times \hat{m}_2) < \epsilon - \tau \right\} \]
\[ \cap \left\{ \gamma : \frac{1}{l(\gamma)} \int \psi_3 d(\hat{\nu}_{\gamma,1} \times \hat{\nu}_{\gamma,2}) - \int \psi_3 d(\hat{m}_1 \times \hat{m}_2) > - \epsilon + \tau \right\} \]
and noting that
\[ \# \{ \gamma : l(\gamma) \leq T \text{ and } \mu_{\gamma}/l(\gamma) \notin \mathcal{U} \} \leq \# \{ \gamma : l(\gamma) \leq T \text{ and } \mu_{\gamma}/l(\gamma) \notin \mathcal{U}'' \}. \]
Clearly, \( m \in \mathcal{U}'' \), so \( \delta > 0 \), as required. To obtain the general case, consider the positive and negative parts of \( f \) separately.

**Proof of Theorem 1.** Write \( I(\theta_1, \theta_2) = i_{\theta_1, \theta_2}(\hat{m}, \hat{m})/2 \) and recall that \( i_{\theta_1, \theta_2}(\gamma) = (\hat{\mu}_{\gamma,1} \times \hat{\mu}_{\gamma,2}) (E_{\theta_1, \theta_2})/2 \). We can apply part (1) of Lemma 4 to deduce that, except for an exceptional set with cardinality of order \( O(e^{(h-\delta)T}) \), the set of closed geodesics of length at most \( T \) satisfy \( |l(\gamma)^{-2}i_{\theta_1, \theta_2}(\gamma) - I(\theta_1, \theta_2)| < \epsilon \). Theorem 1 then follows by applying the asymptotic counting results described in the introduction.

**Proof of Theorem 2.** This is similar to the proof of Theorem 1. We can write \( I(\theta_1, \theta_2, f) = i_{\theta_1, \theta_2,f}(\hat{m}, \hat{m})/2 \) and denote \( i_{\theta_1, \theta_2,f}(\gamma) = i_{\theta_1, \theta_2}(\gamma)A(f, \gamma ; \theta_1 \theta_2) \). We can apply part (1) of Lemma 4 to deduce that, except for an exceptional set with cardinality of order \( O(e^{(h-\delta)T}) \), the set of closed geodesics of length at most \( T \) satisfies \( |l(\gamma)^{-2}i_{\theta_1, \theta_2,f}(\gamma) - I(\theta_1, \theta_2, f)| < \epsilon \). Theorem 2 then follows by applying the asymptotic counting results described in the Introduction.

An expression for \( I(\theta_1, \theta_2) = (\hat{m}_1 \times \hat{m}_2)(E_{\theta_1, \theta_2})/2 \) can be given fairly explicity. Let \( m_V \) be the projection onto the surface of \( m \). For each \( x \in V \), let \( m_x \) be the induced measure on the fibre \( S_x V \). Note that \( m_x \) cannot have atoms (for \( m_V \)-a.e. \( x \)), since otherwise invariance of the measure would imply the existence of closed orbits with non-zero \( m \) measure. However, since the measure of maximal entropy \( m \) is fully supported, such orbits would immediately contradict the ergodicity of \( m \).
Lemma 5. We can write

\[ I(\theta_1, \theta_2) = \int_V \int \left( \int_{(v_1, v_2) \in S_\theta V \times S_\theta V} \sin(\angle v_1, v_2) \, d(m_x \times m_x)(v_1, v_2) \right) \, dm_V(x), \]

where \( \angle v_1, v_2 \) is the angle between the vectors \( v_1, v_2 \in S_\theta V \) in the same fibre.

Proof. The intersection form is defined in terms of the total mass of the angular intersection bundle with respect to a product measure. One can approximate this value by approximating the bundle \( E_{\theta_1, \theta_2} \) by the unions of small sets of the form \( B_1 \oplus B_2 = p_1^{-1}(B_1) \cap p_2^{-1}(B_2) \) for pairs of flow boxes \( B_1, B_2 \) in \( SV \) whose projections onto \( V \) intersect transversely at a single point. By choosing the pairs of flow boxes \( B_1, B_2 \) sufficiently small, we can choose the angle of intersection of any geodesic arcs associated to \( B_1 \) and \( B_2 \), respectively, to be arbitrarily close to a constant \( \theta \), say. The mass with respect to the product measure of the small set \( B_1 \oplus B_2 \) is now particularly easy to estimate. It is given by the usual formula for the area of a parallelogram (height \( \times \) base \( \times \sin \theta \)). Finally, we can take the union over all such sets \( B_1 \oplus B_2 \) in the approximation to \( E_{\theta_1, \theta_2} \). The estimate then follows by approximation. \( \square \)

In the particular case of constant curvature, one easily sees that the expression in Lemma 5 reduces to \( (0, 1) \).

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