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MULTIPLE POSITIVE SOLUTIONS OF SINGULAR PROBLEMS BY VARIATIONAL METHODS

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Abstract. The purpose of this paper is to use an appropriate variational framework to obtain positive solutions of some singular boundary value problems.

1. Introduction

Consider the boundary value problem

$$\begin{cases} -y'' = f(t, y), \quad 0 < t < 1, \\ y(0) = y(1) = 0 \end{cases} \tag{1.1}$$

where we only assume that $f \in C((0,1) \times (0,\infty), [0,\infty))$ satisfies

$$2\varepsilon \le f(t,y) \le C y^{-\gamma}, \quad (t,y) \in (0,1) \times (0,\varepsilon), \tag{1.2}$$

for some $\varepsilon, C > 0$ and $\gamma \in (0,1)$, so that it may be singular at $y = 0$ (here of course C could depend on ε). A typical example is

$$f(t,y) = y^{-\gamma} + g(t,y) \tag{1.3}$$

with $g \in C((0,1) \times [0,\infty), [0,\infty))$.

Define $f_\varepsilon \in C((0,1) \times \mathbb{R}, [0,\infty))$ by

$$f_\varepsilon(t,y) = f(t, (y - \varphi_\varepsilon(t))^+ + \varphi_\varepsilon(t)) \tag{1.4}$$

where $\varphi_\varepsilon(t) = \varepsilon t(1-t)$ is the solution of

$$\begin{cases} -y'' = 2\varepsilon, \quad 0 < t < 1, \\ y(0) = y(1) = 0 \end{cases} \tag{1.5}$$

and $y^\pm = \max\{\pm y, 0\}$, and consider

$$\begin{cases} -y'' = f_\varepsilon(t, y), \quad 0 < t < 1, \\ y(0) = y(1) = 0. \end{cases} \tag{1.6}$$

By (1.2),

$$2\varepsilon \le f_\varepsilon(t,y) \le C \varphi_\varepsilon(t)^{-\gamma}, \quad (t,y) \in (0,1) \times (-\infty,\varepsilon). \tag{1.7}$$

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We observe that if y is a solution of (1.6), then $y \geq \varphi_\varepsilon$ and hence also a solution of (1.1). To see this suppose

$$y(t) < \varphi_\varepsilon(t) \quad \text{for some } t. \tag{1.8}$$

By Lemma 2.8.1 of Agarwal and O'Regan [1],

$$y(t) \geq t(1-t)|y|_0, \quad t \in [0,1], \tag{1.9}$$

where $|y|_0 = \max_{t\in[0,1]} |y(t)|$, so (1.8) implies $|y|_0 < \varepsilon$. But then $-y'' \geq 2\varepsilon = -\varphi_\varepsilon''$ by (1.7), so $y \geq \varphi_\varepsilon$, contradicting (1.8). Conversely, every solution of (1.1) is a solution of (1.6).

Since $\varphi_\varepsilon^{-\gamma} \in L^1(0,1)$, we see from (1.7) that solutions of (1.6) are the critical points of the C^1 functional

$$\Phi(y) = \int_0^1 \left(\frac{1}{2}\,|y'(t)|^2 - F_\varepsilon(t,y(t))\right) dt, \quad y \in H = H_0^1(0,1), \tag{1.10}$$

where $F_\varepsilon(t,y) = \displaystyle\int_\varepsilon^y f_\varepsilon(t,x)\,dx$ and $H_0^1(0,1)$ is the usual Sobolev space, normed by

$$\|y\| = \left(\int_0^1 |y'(t)|^2\,dt\right)^{1/2}. \tag{1.11}$$

The purpose of this paper is to use this variational framework to obtain positive solutions of (1.1).

We will show that, under additional assumptions on the behavior of f at infinity, Φ satisfies the compactness condition of Cerami [2]:

(C): every sequence $\{y_m\} \subset H$ such that

$$\Phi(y_m) \text{ is bounded}, \quad (1+\|y_m\|)\|\Phi'(y_m)\| \to 0 \tag{1.12}$$

has a convergent subsequence.

This condition is weaker than the usual Palais-Smale condition, but can be used in place of it when constructing deformations of sublevel sets via negative pseudo-gradient flows, and therefore also in minimax theorems such as the mountain pass lemma.

By a standard argument it suffices to show that $\{y_m\}$ is bounded when verifying (C). Moreover,

$$\begin{aligned}\|y_m^-\|^2 &= -\left(\int_0^1 f_\varepsilon(t,y_m(t))y_m^-(t)\,dt + \left\langle \Phi'(y_m), y_m^-\right\rangle\right)\\ &\leq |y_m^-|_0 \int_{y_m<0} f_\varepsilon(t,y_m(t))\,dt + \|\Phi'(y_m)\|\,\|y_m^-\|\end{aligned} \tag{1.13}$$

and hence

$$\|y_m^-\| \leq C \int_0^1 \varphi_\varepsilon(t)^{-\gamma}\,dt + o(1) \tag{1.14}$$

since $|\cdot|_0 \leq \|\cdot\|$ and by (1.7), so we will only need to check that $\{y_m^+\}$ is bounded.

We refer the reader to Agarwal and O'Regan [1] for a broad introduction to singular problems and to Rabinowitz [3] for variational methods.

2. An existence principle

Assume

(f): there is a constant $M > 0$, independent of λ, such that $\|y\| \neq M$ for every solution $y > 0$ to

$$\begin{cases} -y'' = \lambda f(t, y), & 0 < t < 1, \\ y(0) = y(1) = 0 \end{cases} \tag{2.1}$$

for each $\lambda \in (0, 1]$.

Note that (f) holds if there exists an a priori bound of the norm of the solutions of the problem.

Proposition 2.1. *If* (1.2) *and* (f) *hold, then* (1.1) *has a positive solution.*

Proof. We will show that Φ assumes its infimum on

$$B = \Big\{ y \in H : \|y\| \leq M \Big\} \tag{2.2}$$

at some point $y_0 \in \overset{\circ}{B}$, which is then a local minimizer, if ε is chosen small enough.

Clearly, $\inf \Phi(B) > -\infty$. Let $\{y_m\}$ be a minimizing sequence. Passing to a subsequence we may assume that y_m converges to some $y_0 \in B$ weakly in H, strongly in $L^2(0, 1)$, and a.e. in $(0, 1)$. Then

$$\begin{aligned} \Phi(y_0) &= \frac{1}{2} \|y_0\|^2 - \int_0^1 F_\varepsilon(t, y_0(t))\, dt \\ &\leq \liminf \frac{1}{2} \|y_m\|^2 - \lim \int_0^1 F_\varepsilon(t, y_m(t))\, dt \\ &= \lim \Phi(y_m) = \inf_{y \in B} \Phi(y), \end{aligned} \tag{2.3}$$

so $\Phi(y_0) = \inf \Phi(B)$.

Suppose that $y_0 \in \partial B$. Then it is also a minimizer of $\Phi|_{\partial B}$, so the gradient of Φ at y_0 points in the direction of the inward normal to ∂B, i.e.,

$$\Phi'(y_0) = -\nu y_0 \tag{2.4}$$

or

$$-y_0'' = \frac{1}{1+\nu} f_\varepsilon(t, y_0), \tag{2.5}$$

for some $\nu \geq 0$. If $y_0 \geq \varphi_\varepsilon$, (2.5) reduces to (2.1) with $\lambda = \dfrac{1}{1+\nu} \in (0, 1]$, so, as in the introduction, it follows that $y_0 < \varepsilon$. But then multiplying (2.5) by y_0 and integrating by parts gives

$$M^2 = \frac{1}{1+\nu} \int_0^1 y_0(t) f_\varepsilon(t, y_0(t))\, dt \leq C\varepsilon \int_0^1 \varphi_\varepsilon(t)^{-\gamma}\, dt = C\varepsilon^{1-\gamma} \tag{2.6}$$

by (1.7), where C is a generic positive constant, which is impossible if ε is sufficiently small. □

For example, consider

$$\begin{cases} -y'' = \mu f(t, y), & 0 < t < 1, \\ y(0) = y(1) = 0 \end{cases} \tag{2.7}$$

where $\mu > 0$ is a parameter and $f \in C((0,1) \times (0,\infty), [0,\infty))$ satisfies (1.2). If $y > 0$ is a solution of

$$\begin{cases} -y'' = \lambda\mu f(t,y), & 0 < t < 1, \\ y(0) = y(1) = 0 \end{cases} \tag{2.8}$$

with $\|y\| = M$,

$$M^2 = \lambda\mu \int_0^1 y(t) f(t,y(t))\, dt \le \mu \sup_{(t,y)\in(0,1)\times(0,M]} y f(t,y), \tag{2.9}$$

so (2.7) has a positive solution for

$$\mu < \sup_{M>0} \frac{M^2}{\displaystyle\sup_{(t,y)\in(0,1)\times(0,M]} y f(t,y)}. \tag{2.10}$$

Similarly,

$$\begin{cases} -y'' = y^{-\gamma} + \mu g(t,y), & 0 < t < 1, \\ y(0) = y(1) = 0 \end{cases} \tag{2.11}$$

where $\gamma \in (0,1)$, $\mu > 0$ is a parameter, and $g \in C((0,1) \times [0,\infty), [0,\infty))$ has a positive solution for

$$\mu < \sup_{M>0} \frac{M^{1-\gamma}\left(M^{1+\gamma} - 1\right)}{\displaystyle\sup_{(t,y)\in(0,1)\times(0,M]} y g(t,y)}. \tag{2.12}$$

3. ASYMPTOTICALLY LINEAR CASE

Assume

$$f(t,y) \le Cy, \quad (t,y) \in (0,1) \times [\varepsilon,\infty), \tag{3.1}$$

for some $C > 0$. We say that (1.1) is resonant if

$$\frac{f(t,y)}{y} \to \lambda_1 \quad \text{as } y \to \infty \tag{3.2}$$

where $\lambda_1 = \pi^2$ is the first eigenvalue of

$$\begin{cases} -y'' = \lambda y, & 0 < t < 1, \\ y(0) = y(1) = 0. \end{cases} \tag{3.3}$$

Denote by

$$H(t,y) = F_\varepsilon(t,y) - \frac{1}{2} y f_\varepsilon(t,y) \tag{3.4}$$

the nonquadratic part of F_ε.

Theorem 3.1. *If* (1.2) *and* (3.1) *hold, then* (1.1) *has a positive solution in the following cases:*

(i) *Nonresonance below* λ_1*:*

$$f(t,y) \le ay + C, \quad (t,y) \in (0,1) \times [\varepsilon,\infty), \tag{3.5}$$

for some $a < \lambda_1$ *and* $C > 0$*,*

(ii) *Resonance:* (3.2) *holds,*

$$H(t,y) \le C, \quad (t,y) \in (0,1) \times [\varepsilon,\infty), \tag{3.6}$$

for some $C > 0$, *and*

$$H(t,y) \to -\infty \quad \textit{as } y \to \infty. \tag{3.7}$$

Proof. (i) By (1.7) and (3.5),

$$F_\varepsilon(t,y) \le \begin{cases} 0, & y < \varepsilon, \\ \dfrac{a}{2}\, y^2 + Cy, & y \ge \varepsilon, \end{cases} \tag{3.8}$$

and, since $a < \lambda_1$, it follows from Wirtinger's inequality that Φ is bounded from below and coercive, and hence satisfies (C) and admits a global minimizer.

(ii) For $y \ge \varepsilon$,

$$\frac{\partial}{\partial y}\left(\frac{F_\varepsilon(t,y)}{y^2}\right) = -\frac{2H(t,y)}{y^3}, \qquad \frac{F_\varepsilon(t,y)}{y^2} \to \frac{\lambda_1}{2} \quad \text{as } y \to \infty \tag{3.9}$$

by (3.2), so

$$F_\varepsilon(t,y) = \left(\frac{\lambda_1}{2} + 2\int_y^\infty \frac{H(t,x)}{x^3}\, dx\right) y^2 \le \frac{\lambda_1}{2}\, y^2 + C \tag{3.10}$$

by (3.6), and hence Wirtinger's inequality implies Φ is bounded from below.

To verify (C), let $\{y_m\}$ satisfy (1.12) and suppose $\rho_m := \|y_m\| \to \infty$. Since $\{y_m^-\}$ is bounded, for a subsequence, $\widetilde{y}_m := \dfrac{y_m}{\rho_m}$ converges to some $\widetilde{y} \ge 0$ weakly in H, strongly in $L^2(0,1)$, and a.e. in $(0,1)$. Then

$$\int_0^1 \widetilde{y}_m'(t)\big(\widetilde{y}_m'(t) - \widetilde{y}'(t)\big)\, dt = \int_0^1 g_m(t)\, dt + \frac{\langle \Phi'(y_m), \widetilde{y}_m - \widetilde{y}\rangle}{\rho_m} \tag{3.11}$$

where $g_m(t) = \dfrac{f(t, y_m(t))}{\rho_m}\big(\widetilde{y}_m(t) - \widetilde{y}(t)\big)$. By (1.7) and (3.1),

$$|g_m(t)| \le C\left(\varphi_\varepsilon(t)^{-\gamma} + 1\right)\left|\widetilde{y}_m(t) - \widetilde{y}(t)\right| \tag{3.12}$$

and hence $g_m \to 0$ a.e. and $|g_m| \le C\big(\varphi_\varepsilon^{-\gamma} + 1\big) \in L^1(0,1)$, so passing to the limit in (3.11) gives $\|\widetilde{y}\| = 1$; in particular, $\widetilde{y} \ne 0$.

By (1.7) and (3.6),

$$H(t,y) \le \begin{cases} C\varphi_\varepsilon(t)^{-\gamma}|y|, & y < 0, \\ 0, & 0 \le y < \varepsilon, \\ C, & y \ge \varepsilon, \end{cases} \tag{3.13}$$

and $|y_m^-|_0$ is bounded, so

$$\int_{\widetilde{y} > 0} H(t, y_m(t))\, dt \to -\infty \tag{3.14}$$

by (3.7) and $\int_{\widetilde{y}=0} H(t, y_m(t))\, dt$ is bounded from above. Hence

$$\begin{aligned} \frac{1}{2} \langle \Phi'(y_m), y_m \rangle - \Phi(y_m) &= \int_0^1 H(t, y_m(t))\, dt \\ &= \int_{\widetilde{y}>0} H(t, y_m(t))\, dt + \int_{\widetilde{y}=0} H(t, y_m(t))\, dt \to -\infty, \end{aligned} \tag{3.15}$$

contrary to assumption. □

Theorem 3.2. *If* (1.2), (3.1), *and* (f) *hold, then* (1.1) *has two positive solutions in the following cases:*

(i) *Resonance:* (3.2) *holds,*

$$H(t, y) \geq -C, \quad (t, y) \in (0, 1) \times [\varepsilon, \infty), \tag{3.16}$$

for some $C > 0$, *and*

$$H(t, y) \to +\infty \quad \text{as } y \to \infty, \tag{3.17}$$

(ii) *Nonresonance above* λ_1*:*

$$f(t, y) \geq by - C, \quad (t, y) \in (0, 1) \times [\varepsilon, \infty), \tag{3.18}$$

for some $b > \lambda_1$ *and* $C > 0$.

Proof. By (see the proof of) Proposition 2.1, Φ has a local minimizer $y_0 \in \overset{\circ}{B}$ and $\inf \Phi(\partial B) \geq \Phi(y_0)$. We will show that $\Phi(R\varphi_1) \leq \inf \Phi(\partial B)$ if $R > M$ is sufficiently large, where $\varphi_1 > 0$ is the normalized eigenfunction associated with λ_1, and we will verify (C). Then the mountain pass lemma will give a second critical point at the level

$$c := \inf_{\gamma \in \Gamma} \max_{y \in \gamma([0,1])} \Phi(y) \tag{3.19}$$

where

$$\Gamma = \Big\{ \gamma \in C([0, 1], H) : \gamma(0) = y_0,\ \gamma(1) = R\varphi_1 \Big\} \tag{3.20}$$

is the class of paths joining y_0 and $R\varphi_1$.

(i) By (3.10), (3.16), and (3.17),

$$\begin{aligned} \frac{\lambda_1}{2} y^2 - F_\varepsilon(t, y) = -2y^2 \int_y^\infty \frac{H(t, x)}{x^3}\, dx \leq C, \quad y \geq \varepsilon \\ \text{and } \to -\infty \quad \text{as } y \to \infty, \end{aligned} \tag{3.21}$$

and by (1.7),

$$\frac{\lambda_1}{2} y^2 - F_\varepsilon(t, y) \leq C \left(\varphi_\varepsilon(t)^{-\gamma} + 1 \right), \quad 0 \leq y < \varepsilon, \tag{3.22}$$

so

$$\Phi(R\varphi_1) = \int_0^1 \left(\frac{\lambda_1}{2} R^2 \varphi_1(t)^2 - F_\varepsilon(t, R\varphi_1(t)) \right) dt \to -\infty \quad \text{as } R \to \infty. \tag{3.23}$$

The verification of (C) is similar to that in the proof of Theorem 3.1.

(ii) By (1.7) and (3.18),

$$F_\varepsilon(t,y) \geq \begin{cases} -C\varphi_\varepsilon(t)^{-\gamma}, & 0 \leq y < \varepsilon, \\ \dfrac{b}{2}\,y^2 - Cy, & y \geq \varepsilon, \end{cases} \tag{3.24}$$

and, since $b > \lambda_1$, it follows that $\Phi(R\varphi_1) \to -\infty$ as $R \to \infty$.

If (1.12) holds and $\rho_m := \|y_m\| \to \infty$, passing to a subsequence $\widetilde{y}_m := \dfrac{y_m}{\rho_m}$ converges to a nontrivial $\widetilde{y} \geq 0$ weakly in H, strongly in $L^2(0,1)$, and a.e. in $(0,1)$ as in the proof of Theorem 3.1. Then

$$\begin{aligned} (b-\lambda_1)\int_0^1 \widetilde{y}_m(t)\varphi_1(t)\,dt &= \frac{1}{\rho_m}\int_0^1 \big(by_m(t)\varphi_1(t) - y_m'(t)\varphi_1'(t)\big)\,dt \\ &= \frac{1}{\rho_m}\left(\int_0^1 \big(by_m(t) - f_\varepsilon(t,y_m(t))\big)\varphi_1(t)\,dt - \langle \Phi'(y_m),\varphi_1\rangle\right) \\ &\leq \frac{1}{\rho_m}\left(\int_{y_m\geq\varepsilon} C\varphi_1(t)\,dt + \int_{y_m<\varepsilon} by_m(t)\varphi_1(t)\,dt + \|\Phi'(y_m)\|\right) \end{aligned} \tag{3.25}$$

by (3.18), and $|y_m^-|_0$ is bounded, so

$$(b-\lambda_1)\int_0^1 \widetilde{y}(t)\varphi_1(t)\,dt \leq 0, \tag{3.26}$$

which is impossible. □

4. Superlinear case

Assume

$$0 < \theta F_\varepsilon(t,y) \leq yf(t,y), \quad (t,y) \in (0,1)\times[y_0,\infty), \tag{4.1}$$

for some $\theta > 2$ and $y_0 > \varepsilon$.

Theorem 4.1. *If* (1.2), (4.1), *and* (f) *hold, then* (1.1) *has two positive solutions.*

Proof. As in the proof of Theorem 3.2 it suffices to show that $\Phi(R\varphi_1) \to -\infty$ as $R \to \infty$ and to verify (C). The former follows since

$$F_\varepsilon(t,y) \geq \begin{cases} -C\varphi_\varepsilon(t)^{-\gamma}, & 0 \leq y < \varepsilon, \\ F_\varepsilon(t,y_0)\left(\dfrac{y}{y_0}\right)^\theta, & y \geq y_0, \end{cases} \tag{4.2}$$

by (1.7) and (4.1). As for the latter,

$$\begin{aligned} \left(\frac{\theta}{2} - 1\right)\|y_m\|^2 &= \int_0^1 \big(\theta F_\varepsilon(t,y_m(t)) - y_m(t)f_\varepsilon(t,y_m(t))\big)\,dt \\ &\quad + \theta\Phi(y_m) - \langle \Phi'(y_m), y_m\rangle \\ &\leq C\left(\int_{y_m<\varepsilon} \varphi_\varepsilon(t)^{-\gamma}|y_m(t)|\,dt + 1\right), \end{aligned} \tag{4.3}$$

and the last integral is bounded since $|y_m^-|_0$ is bounded. □

For example, Theorem 4.1 guarantees that

$$\begin{cases} -y'' = \mu(y^{-\gamma} + y^{\beta}), \quad 0 < t < 1, \\ y(0) = y(1) = 0 \end{cases} \tag{4.4}$$

where $\gamma \in (0,1)$ and $\beta > 1$, has two positive solutions for

$$0 < \mu < \frac{\Big((\gamma+1)^{\gamma+1}(\beta-1)^{\beta-1}\Big)^{1/(\gamma+\beta)}}{\gamma+\beta} \tag{4.5}$$

(see the example after Proposition 2.1 and (2.10)). The problem

$$\begin{cases} -y'' = y^{-\gamma} + \mu e^{y}, \quad 0 < t < 1, \\ y(0) = y(1) = 0 \end{cases} \tag{4.6}$$

where $\gamma \in (0,1)$, has two positive solutions for

$$0 < \mu < \sup_{M>0} \frac{M^{1+\gamma} - 1}{M^{\gamma} e^{M}} \tag{4.7}$$

(see (2.12)).

References

[1] Ravi P. Agarwal and Donal O'Regan. *Singular differential and integral equations with applications.* Kluwer Academic Publishers, Dordrecht, 2003. MR2011127 (2004h:34002)

[2] Giovanna Cerami. An existence criterion for the critical points on unbounded manifolds. *Istit. Lombardo Accad. Sci. Lett. Rend. A*, 112(2):332–336 (1979), 1978. MR0581298 (81k:58021)

[3] Paul H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, volume 65 of *CBMS Regional Conference Series in Mathematics.* Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986. MR0845785 (87j:58024)

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