THE GELFAND-KIRILLOV DIMENSION OF QUADRATIC ALGEBRAS SATISFYING THE CYCLIC CONDITION

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Abstract. We consider algebras over a field $K$ presented by generators $x_1, \ldots, x_n$ and subject to $(\binom{n}{2})$ square-free relations of the form $x_i x_j = x_k x_l$ with every monomial $x_i x_j, i \neq j$, appearing in one of the relations. It is shown that for $n \geq 1$ the Gelfand-Kirillov dimension of such an algebra is at least two if the algebra satisfies the so-called cyclic condition. It is known that this dimension is an integer not exceeding $n$. For $n \geq 4$, we construct a family of examples of Gelfand-Kirillov dimension two. We prove that an algebra with the cyclic condition with generators $x_1, \ldots, x_n$ has Gelfand-Kirillov dimension $n$ if and only if it is of $I$-type, and this occurs if and only if the multiplicative submonoid generated by $x_1, \ldots, x_n$ is cancellative.

1. Introduction

In [4] Gateva-Ivanova and Van den Bergh studied the structure of monoids of left $I$-type and their algebras. These monoids originate from the work of Tate and Van den Bergh on homological properties of Sklyanin algebras [8]. It was shown in [4] that a monoid of left $I$-type has a presentation with generators $x_1, \ldots, x_n$ and $(\binom{n}{2})$ relations of the form $x_i x_j = x_k x_l$ such that every monomial $x_i x_j$ with $1 \leq i, j \leq n$ appears at most once in one of the relations. Moreover, such monoids yield set-theoretical solutions of the quantum Yang-Baxter equation, and the corresponding monoid algebras share many properties with commutative polynomial algebras. In particular, they are noetherian domains of finite global dimension, satisfy a polynomial identity, are Koszul, Auslander-Gorenstein, Cohen-Macaulay and have Gelfand-Kirillov dimension $n$. In [6] the monoids of left $I$-type are characterized as natural submonoids of semidirect products of the free abelian monoid of rank $n$ and the symmetric group of degree $n$. As a consequence, it is proved that a monoid is of left $I$-type if and only if it is of right $I$-type, [6, Corollary 2.3].

A monoid $S$ is said to be of skew type if it has a presentation with $n \geq 2$ generators $x_1, \ldots, x_n$ and $(\binom{n}{2})$ square-free relations of the form $x_i x_j = x_k x_l$ with every monomial $x_i x_j, i \neq j$, appearing in one of the relations (see [3], where a systematic study of these monoids and their algebras was initiated). Recall that
a monoid $S = (x_1, \ldots, x_n)$ of skew type is said to be right (respectively left) non-degenerate if for every $1 \leq i, k \leq n$ there exist $1 \leq j, l \leq n$ so that $x_i x_j = x_k x_l$ (or $x_i x_k = x_j x_l$, respectively). Furthermore $S$ is said to satisfy the cyclic condition if for every relation $x_i x_j = x_k x_l$ one also has a relation $x_i x_k = x_j x_l$ for some $r$ (see [3 Lemma 2.1]). The latter is a powerful combinatorial condition that has already proved crucial in the study of monoids of $I$-type, their algebras and corresponding torsion-free groups. [3] The cyclic condition is symmetric. [3 Proposition 2.1]. Hence it is easy to see that it implies left and right non-degeneracy. It was shown in [3] that for monoids $S$ satisfying the cyclic condition we have $1 \leq \text{GK}(K[S]) \leq n$, where $\text{GK}(K[S])$ denotes the Gelfand-Kirillov dimension of the monoid algebra $K[S]$. Furthermore there exist non-degenerate monoids $S$ of skew type on $4^n$ generators (for any positive $n$) so that $\text{GK}(K[S]) = 1$. [3].

In this paper we prove that $\text{GK}(K[S]) \geq 2$ for any monoid $S$ of skew type that satisfies the cyclic condition. For any $n \geq 4$ we construct examples of such monoids on $n$ generators with $\text{GK}(K[S]) = 2$. Furthermore we show that $\text{GK}(K[S]) = n$ if and only if $S$ is of $I$-type, and this occurs if and only if $S$ is cancellative.

2. The Gelfand-Kirillov Dimension

Let $S = (x_1, x_2, \ldots, x_n)$ be a monoid of skew type that satisfies the cyclic condition.

Let $F = \langle y_1, y_2, \ldots, y_n \rangle$ be the free monoid of rank $n$ and let $\pi : F \to S$ be the natural epimorphism, that is $\pi(y_i) = x_i$ for $i = 1, \ldots, n$. Let $x \in S$. We say that a word $w \in F$ represents $x$ if $\pi(w) = x$.

It is known that two words $w, w' \in F$ represent the same element $x \in S$ if and only if there exists a finite sequence of words

$$w = w_0, w_1, w_2, \ldots, w_m = w'$$

such that $w_i$ is obtained from $w_{i-1}$ ($i = 1, \ldots, m$) by substituting a subword $y_j y_k$ by $y_l y_q$, where $x_j x_k = x_l x_q$ is a defining relation of $S$. In this case, we say that $w_i$ is obtained from $w_{i-1}$ by an $S$-relation.

**Lemma 2.1.** If $x_i x_j = x_k x_l$ is a defining relation of $S$, then there exist positive integers $r, s$ such that $r + s \leq n$ and the submonoid $(x_1^r, x_1^s)$ is free abelian of rank 2.

**Proof.** By [3] Proposition 2.1, since $S$ satisfies the cyclic condition, there exist positive integers $r$, $s$ and $s + r$ different integers

$$i_1, i_2, \ldots, i_s, j_1, j_2, \ldots, j_r \in \{1, 2, \ldots, n\}$$

such that

$$x_{i_1} x_{j_1} = x_{j_2} x_{i_2}, \quad x_{i_2} x_{j_1} = x_{j_3} x_{i_3}, \quad \ldots, \quad x_{i_k} x_{j_1} = x_{j_k} x_{i_1},
\quad x_{i_1} x_{j_2} = x_{j_3} x_{i_2}, \quad x_{i_2} x_{j_2} = x_{j_3} x_{i_3}, \quad \ldots, \quad x_{i_k} x_{j_2} = x_{j_k} x_{i_1},
\quad \vdots \quad \vdots \quad \vdots
\quad x_{i_1} x_{j_r} = x_{j_1} x_{i_2}, \quad x_{i_2} x_{j_r} = x_{j_1} x_{i_3}, \quad \ldots, \quad x_{i_k} x_{j_r} = x_{j_k} x_{i_1}.$$  

From the relations in the first column we have

$$x_{i_1}^r x_{j_1} = x_{j_1} x_{i_2}.$$  

Similarly, from the other columns, we obtain

$$x_{i_2}^r x_{j_1} = x_{j_1} x_{i_3}, \quad \ldots, \quad x_{i_k}^r x_{j_1} = x_{j_k} x_{i_1}.$$
Hence $x_i^r x_j^s = x_j^s x_i^r$, and thus the submonoid $\langle x_i^r, x_j^s \rangle$ is abelian. Note that the only words that represent $x_i^m$ and $x_j^n$ are $y_i^m$ and $y_j^n$ respectively. Let $p, q$ be positive integers and $x = x_i^p x_j^q$. We claim that any word $w \in F$ that represents $x$ is of the form
\begin{equation}
 w = y_i^{n_1} y_j^{n_2} y_i^{n_3} y_j^{n_4} \cdots y_{i_{g-1}}^{n_g-1} y_{j_{g-1}}^{n_g-1},
\end{equation}
where $g$ is an integer greater than 1; $n_1, n_g$ are non-negative integers; $l_1 = l_g = 1$, and $n_2, n_3, \ldots, n_{g-1}, m_1, m_2, \ldots, m_{g-1}$ are positive integers such that

(i) $n_1 + k_1 \equiv k_{g-1} - n_g \equiv 1 \pmod{r}$ and $l_t + 1 \equiv m_t \pmod{s}$ for all $1 \leq t \leq g - 1$;
(ii) if $g > 2$, then $k_u - k_{u+1} \equiv n_{u+1} \pmod{r}$ for all $1 \leq u \leq g - 2$;
(iii) $n_1 + n_2 + \cdots + n_g = rp$ and $m_1 + m_2 + \cdots + m_{g-1} = sq$.

Note that the word $y_i^{p} y_j^{q}$ represents $x$ and satisfies conditions (i), (ii) and (iii). Therefore, in order to prove the claim, it is sufficient to see that any word $w$ of the form (1) that satisfies conditions (i), (ii) and (iii), all the words obtained from $w$ by an $S$-relation also satisfy conditions (i), (ii) and (iii). Suppose that $g = 2$. In this case
\begin{equation}
 w = y_i^{n_1} y_j^{n_2},
\end{equation}
with $n_1 = sq > 0$, $n_1 + n_2 = rp$ and $n_1 + k_1 \equiv k_1 - n_2 \equiv 1 \pmod{r}$. If $n_1 > 0$, we can obtain by an $S$-relation (the relation $x_i x_{j_{k_1}} = x_{j_{k_1+1}} x_i$), the word
\begin{equation}
 w' = y_i^{n_1-1} y_j^{n_2} y_i^{n_1-1} y_j^{n_2} y_i^{n_1-1} y_j^{n_2} y_i^{n_1-1} y_j^{n_2},
\end{equation}
where $k_1 + 1$ is taken modulo $r$ in the set $\{1, \ldots, r\}$, and it is easy to see that $w'$ satisfies conditions (i), (ii) and (iii). If $n_2 > 0$, we can obtain by an $S$-relation the word
\begin{equation}
 w' = y_i^{n_1} y_j^{n_2} y_i^{n_1} y_j^{n_2} y_i^{n_1} y_j^{n_2} y_i^{n_1} y_j^{n_2},
\end{equation}
where $k_1 - 1$ is taken modulo $r$ in the set $\{1, \ldots, r\}$, and it is easy to see that $w'$ satisfies the conditions (i), (ii) and (iii). Similarly, it is straightforward to prove that, if $g > 2$, all the words $w'$ obtained from $w$ by an $S$-relation satisfy conditions (i), (ii) and (iii). Now condition (iii) implies that the submonoid $\langle x_i^r, x_j^s \rangle$ is free abelian of rank 2.

As a direct consequence of Lemma 2.1, we get the following result.

**Corollary 2.2.** Let $S = \langle x_1, x_2, \ldots, x_n \rangle$ be a monoid of skew type that satisfies the cyclic condition. Let $m = (n - 1)!$. Then the submonoid $A = \langle x_1^m, \ldots, x_n^m \rangle$ is commutative.

**Theorem 2.3.** Let $S = \langle x_1, x_2, \ldots, x_n \rangle$ (with $n > 1$) be a monoid of skew type that satisfies the cyclic condition. Let $A = \langle x_1^m, \ldots, x_n^m \rangle$, where $m = (n - 1)!$. If $K$ is a field, then the Gelfand-Kirillov dimension of the monoid algebra $K[S]$ is an integer such that $2 \leq \text{GK}(K[S]) = \text{GK}(K[A]) \leq n$. Moreover, $\text{GK}(K[S])$ is equal to the maximal rank of a free abelian submonoid of the form $\langle x_i^m, \ldots, x_n^m \rangle \subseteq S$.

**Proof.** By 3 Theorem 4.5 and the comment after 3 Proposition 2.4, $K[S]$ is a finite left and right module over the commutative subring $K[A]$, where $A = \langle x_1^p, \ldots, x_n^p \rangle$ for some $p \geq 1$. The proof actually shows that we may take $p = (n-1)!$. Hence $\text{GK}(K[S]) = \text{GK}(K[A]) \leq n$ and it is an integer. By Lemma 2.1, we have that $2 \leq \text{GK}(K[S])$. Let $P$ be a prime ideal of $K[A]$. Then the image $A_P$ of
A in $K[A]/P$ is a 0-cancellative monoid. Let $C = \langle z_i^m, \ldots, z_i^n \rangle \subseteq A_P$ be a free abelian submonoid of maximal rank that is generated by certain images $z_i^m$ of the elements $x_i^m$. Then $B = \langle x_i^m, \ldots, x_i^n \rangle \subseteq A$ is free abelian of rank $r$. It is easy to see that the group $G_P$ of quotients of the cancellative semigroup of nonzero elements of $A_P$ is of rank $r$, whence $\text{GK}(K[A]/P) \leq \text{GK}(K[A_P]) \leq \text{GK}(K[G_P]) = r \leq k$. Therefore $\text{GK}(A) \leq k$, since the Gelfand-Kirillov and the classical Krull dimensions coincide on finitely generated commutative algebras, [7] Theorem 4.5. The result follows.

\section{Examples of dimension two}

For $n \geq 4$, let $T^{(n)}$ be the monoid of skew type generated by $x_1, \ldots, x_n$ with defining relations

\begin{align*}
x_1 x_2 &= x_3 x_1, \quad \ldots, \quad x_1 x_{n-2} = x_{n-1} x_1, \quad x_1 x_{n-1} = x_2 x_1, \\
x_n x_1 &= x_{n-1} x_n, \quad x_n x_{n-1} = x_1 x_n, \\
x_i x_{i+1} &= x_{i+2} x_i, \quad \ldots, \quad x_{i+1} x_{n-1} = x_n x_i, \quad x_i x_n = x_{i+1} x_i,
\end{align*}

for all $2 \leq i \leq n-2$. Note that $T^{(n)}$ satisfies the cyclic condition.

\textbf{Lemma 3.1.} Let $\rho$ be the least cancellative congruence on $T^{(n)}$. If $n > 4$, then $x_2 x_1 x_2 = x_n x_1 x_2$ and $x_1 \rho x_2 \rho \ldots \rho x_n$.

\textbf{Proof.} By using the defining relations, we have

\begin{align*}
x_2 x_1 x_2 &= x_1 x_{n-1} x_2 = x_1 x_2 x_{n-2} = x_3 x_1 x_{n-2} = x_3 x_{n-1} x_1 = x_n x_3 x_1 = x_n x_1 x_2.
\end{align*}

Since $x_2 x_1 x_2 = x_n x_1 x_2$, it follows that $x_2 \rho x_n$. Now the relations

\begin{align*}
x_2 x_3 &= x_4 x_2, \quad \ldots, \quad x_2 x_{n-1} = x_n x_2
\end{align*}

imply that $x_2 \rho x_3 \rho \ldots \rho x_n$. Since $x_n x_1 = x_{n-1} x_n$, we also get

\begin{align*}
x_1 \rho x_2 \rho \ldots \rho x_n.
\end{align*}

\hfill $\square$

Let $T'_n$ be the subset of $T^{(n)}$ of all elements right divisible by all generators of $T^{(n)}$. Since $T^{(n)}$ is left non-degenerate, $T'_n$ is an ideal of $T^{(n)}$; see [3].

\textbf{Lemma 3.2.} Consider $z = x_2 x_1 x_2 \in T^{(n)}$. Then $z \in T'_n$.

\textbf{Proof.} For $n = 4$ we have

\begin{align*}
z &= x_2 x_1 x_2 = x_2 x_3 x_1 = x_4 x_2 x_1 = x_4 x_1 x_3 = x_3 x_4 x_3 = x_3 x_1 x_4 \in T'_n.
\end{align*}

Suppose that $n > 4$. By Lemma 3.1, $z = x_n x_1 x_2$ and thus

\begin{align*}
z &= x_n x_1 x_2 = x_{n-1} x_n x_2 = x_{n-1} x_2 x_{n-1} = x_2 x_n - 2 x_{n-1} \\
&= x_2 x_n x_{n-2} = x_3 x_2 x_{n-2} = x_3 x_{n-1} x_2 = x_n x_3 x_2 = x_n x_2 x_n \\
&= x_2 x_{n-1} x_n = x_2 x_n x_1 = x_3 x_2 x_1 = x_3 x_1 x_{n-1} = x_1 x_2 x_{n-1} \\
&= x_1 x_n x_2 = x_n x_{n-1} x_2 = x_n x_2 x_{n-2} = x_2 x_{n-1} x_{n-2}.
\end{align*}

We claim that $z = x_2 x_{i+1} x_i$ for all $n - 2 \geq i \geq 3$. We prove this by induction. If $n = 5$ the claim is proved. Suppose that $n > 5$ and that we know $z = x_2 x_{i+1} x_i$ for
some $4 \leq i \leq n - 2$. Then
\[
z = x_2x_{i+1}x_i = x_2x_ix_n = x_{i+1}x_2x_n = x_{i+1}x_3x_2
\]
\[
= x_3x_ix_2 = x_3x_2x_{i-1} = x_2x_nx_{i-1} = x_2x_{i-1}x_{n-1}
\]
\[
= x_ix_2x_{n-1} = x_ix_nx_2 = x_{i+1}x_2x_{i-1} = x_2x_ix_{i-1},
\]
which proves the inductive claim. It follows that $z \in T(n)x_i$, for all $3 \leq i \leq n - 2$.

Since $z = x_2x_1x_2 = x_2x_3x_1 = x_{n-1}x_2x_{n-1} = x_nx_2x_n$, we have that $z \in T'_n$. □

Let $m = (n-1)!$. By Corollary 2.2, the submonoid $A = \langle x_1^m, \ldots, x_n^m \rangle$ of $T(n)$ is commutative.

**Lemma 3.3.** If $n > 4$, then $x_k^m x_j^m x_i^m \in T'_n$ for all $1 \leq i < j < k \leq n$.

**Proof.** Note that from the relations
\[
x_1x_2 = x_3x_1, \quad \ldots, \quad x_1x_{n-2} = x_{n-1}x_1, \quad x_1x_{n-1} = x_2x_1,
\]
it follows that
\[
(2) \quad x^j_1x_i = x_{j+i}x_1^j, \quad x_1^{n-2}x_i = x_ix_1^{n-2},
\]
for all $2 \leq i \leq n - 1$ and $1 \leq j < n - i$. From the relations
\[
(3) \quad x_nx_1 = x_{n-1}x_n, \quad x_nx_{n-1} = x_1x_n,
\]
it follows that
\[
(4) \quad x_n^2x_1 = x_1x_n^2 \quad \text{and} \quad x_n^2x_{n-1} = x_{n-1}x_n^2.
\]
For each $2 \leq i \leq n - 2$, the relations
\[
x_ix_{i+1} = x_{i+2}x_i, \quad \ldots, \quad x_ix_{n-1} = x_{n}x_i, \quad x_ix_n = x_{i+1}x_i
\]
imply that
\[
(5) \quad x_i^{n-i}x_j = x_jx_i^{n-i},
\]
for all $2 \leq i < j \leq n$.

**Case 1.** $1 < i < j < k \leq n$. In this case it is easy to see that
\[
(6) \quad x_k x_j^{k-j-1} = x_j^{k-j-1}x_{j+1}
\]
and
\[
(7) \quad x_j x_i^{j-i+1} = x_i^{j-i+1}x_{n-1}.
\]
Then we have
\[
x_k^{2m} x_j^{2m} x_i^{2m} = x_{k-j+1}^{2m} x_j^{2m} x_i^{2m} \quad \text{(by (1))}
\]
\[
x_j^{k-j-1} x_j^{2m} x_i^{2m} x_j^{m-k+j+1} x_i^{2m} \quad \text{(by (2))}
\]
\[
x_j^{k-j} x_j^{m-2m} x_i^{2m} x_j^{m-k+j+1} x_i^{2m} \quad \text{(by (3))}
\]
\[
x_i^{k-j} x_j^{m} x_j^{m-k-j+1} x_j^{2m} \quad \text{(by (4))}
\]
\[
x_i^{k-j} x_j^{m} x_j^{m-k-j+1} x_j^{2m} \quad \text{(by (5))}
\]
\[
x_i^{k-j} x_j^{m} x_j^{m-k-j+1} x_j^{2m} \quad \text{(by (6))}
\]

By Lemma 3.2 we know that \( z \in T'_n \). Since \( T'_n \) is an ideal of \( T^{(n)} \), it follows that \( x_k^{2m} x_j^{2m} x_i^{2m} \in T'_n \).

**Case 2.** \( 1 < j < k \leq n \) and \( j < n - 1 \). Then we have
\[
x_k^{2m} x_j^{2m} x_i^{2m} = x_j^{k-j} x_j^{2m} x_j^{m-k+j+1} x_j^{2m} \quad \text{(by (4))}
\]
\[
x_j^{k-j+1} x_j^{m} x_j^{m-k-j+1} x_j^{2m} \quad \text{(by (2))}
\]
\[
x_j^{k-j} x_j^{m} x_j^{m-k-j+1} x_j^{2m} \quad \text{(by (4))}
\]
\[
x_j^{k-j} x_j^{m} x_j^{m-k-j+1} x_j^{2m} \quad \text{(by (5))}
\]
\[
x_j^{k-j} x_j^{m} x_j^{m-k-j+1} x_j^{2m} \quad \text{(by (6))}
\]

By Lemma 3.2 \( x_k^{2m} x_j^{2m} x_i^{2m} \in T'_n \) in this case.

**Case 3.** \( i = 1, j = n - 1 \) and \( k = n \). Then we have
\[
x_n^{2m} x_{n-1}^{2m} = x_n^{2m} x_{n-1}^{2m} x_n^{2m-1} \quad \text{(by (2))}
\]
\[
x_n^{2m} x_{n-1}^{x_n^{2m-1}} x_n^{2m-1} \quad \text{(by (2))}
\]
\[
x_n^{2m-1} x_n^{2m-3} x_n^{2m-1} x_n^{2m-1} \quad \text{(by (3))}
\]
\[
x_n^{2m-2} x_n^{2m-4} x_n^{2m-1} x_n^{2m-1} \quad \text{(by (3))}
\]
\[
x_n^{2m-1} x_n^{x_n^{2m-1}} x_n^{2m-1} x_n^{2m-1} \quad \text{(by (3))}
\]

Again, by Lemma 3.2 \( x_k^{2m} x_j^{2m} x_i^{2m} \in T'_n \) in this case.

Therefore \( x_k^{2m} x_j^{2m} x_i^{2m} \in T'_n \) for all \( 1 \leq i < j < k \leq n \).

**Theorem 3.4.** Let \( K \) be a field. Then \( \text{GK}(K[T^{(n)}]) = 2 \) for all \( n \geq 4 \).
Proof. For \( n = 4 \) the result follows from \([5]\) Proposition 2.1, because \( T^{(4)} \) coincides with the monoid \( C(1) \) of \([5]\). Suppose that \( n > 4 \). As above, let \( m = (n - 1)! \). From \([3]\) Proposition 6.3 we know that \( (T_n)^* I(\rho) = 0 \) for some \( q \), where \( I(\rho) \) is the ideal of \( K[T^{(n)}] \) determined by the least cancellative congruence \( \rho \) on \( T^{(n)} \). In particular, by Lemma \([6, Example 3.6]\), \( x_k^m x_j^m = I(\rho) \) for all \( k,j \). Therefore, from Lemma 3.3, it follows that

\[ x_k^{2mq} x_j^{2mq} x_i^{2mq} (x_k^m - x_j^m) = 0, \]

for all \( 1 \leq i < j < k \leq n \), which implies that \( x_k^m, x_j^m, x_i^m \) do not generate a free abelian semigroup. Therefore Theorem \([2,3]\) implies that \( \text{GK}(K[T^{(n)}]) = 2 \). \( \square \)

**Corollary 3.5.** For any integers \( n \geq 4 \) and \( 2 \leq j \leq n \), there exists a monoid \( M = \langle x_1, x_2, \ldots, x_n \rangle \) of skew type, satisfying the cyclic condition and such that \( \text{GK}(K[M]) = j \) for any field \( K \).

Proof. If \( j = n \), then the free abelian monoid of rank \( n \), \( M = \text{FaM}_n \), satisfies the conditions.

Suppose that \( j = n - 1 \). By \([5]\), there exists a monoid \( A \) of skew type with 4 generators that satisfies the cyclic condition such that, for any field \( K \), \( \text{GK}(K[A]) = 3 \). Let \( M = A \times \text{FaM}_{n-4} \). Then it is easy to see that \( M \) is a monoid of skew type with \( n \) generators that satisfies the cyclic condition. Since \( K[M] \) is the polynomial algebra over \( K[A] \) with \( n - 4 \) indeterminates, by \([7\) Example 3.6], \( \text{GK}(K[M]) = 3 + (n - 4) = n - 1 \).

Suppose that \( j \leq n - 2 \). Let \( M = T^{(n-j+2)} \times \text{FaM}_{j-2} \). It is easy to see that \( M \) is a monoid of skew type with \( n \) generators that satisfies the cyclic condition. Since \( K[M] \) is the polynomial algebra over \( K[T^{(n-j+2)}] \) with \( j - 2 \) indeterminates, by \([7\) Example 3.6], \( \text{GK}(K[M]) = \text{GK}(K[T^{(n-j+2)}]) + (j - 2) \). By Theorem 3.4 \( \text{GK}(K[M]) = j \). \( \square \)

4. I-TYPE MONOIDS

Let \( \text{FaM}_n \) be the multiplicative free abelian monoid of rank \( n \) with basis \( u_1, \ldots, u_n \). Recall that a monoid \( S \) generated by \( x_1, \ldots, x_n \) is said to be of left \( I \)-type if there exists a bijection (called a left \( I \)-structure)

\[ v: \text{FaM}_n \rightarrow S \]

such that

\[ v(1) = 1 \quad \text{and} \quad \{v(u_1 a), \ldots, v(u_n a)\} = \{x_1 v(a), \ldots, x_n v(a)\} \]

for all \( a \in \text{FaM}_n \). As mentioned in the introduction, it is proved in \([6]\) that a monoid \( S \) is of left \( I \)-type if and only if it is of right \( I \)-type. So we call a monoid of left or right \( I \)-type simply a monoid of \( I \)-type.

Let \( S = \langle x_1, x_2, \ldots, x_n \rangle \) be a monoid of skew type. Let \( X = \{x_1, x_2, \ldots, x_n\} \).

As in \([6]\), we define the associated bijective map \( r: X \times X \rightarrow X \times X \) by

\[ r(x_i, x_j) = (x_k, x_l) \]

if \( x_i x_j = x_k x_l \) is a defining relation of \( S \), and \( r(x_i, x_i) = (x_i, x_i) \). For each \( x \in X \), we also denote by \( f_x: X \rightarrow X \) and \( g_x: X \rightarrow X \) the mappings defined by \( f_x(x_i) = p_1(r(x, x_i)) \) and \( g_x(x_i) = p_2(r(x, x_i)) \), where \( p_1 \) and \( p_2 \) denote the projections onto the first and second component respectively. So \( r(x_i, x_j) = (f_x(x_j), g_x(x_j)) \). Suppose that \( S \) is right non-degenerate. So \( f_x \) is bijective for all \( x \in X \). We denote by \( \sigma_i \in \text{Sym}_n \) the permutation defined by \( f_{x_i}(x_j) = x_{\sigma_i(j)} \).
The next result is a partial generalization of Proposition 2.2(c) of [2].

**Theorem 4.1.** Let $S = \langle x_1, x_2, \ldots, x_n \rangle$ be a right non-degenerate monoid of skew type. Then the following conditions are equivalent.

(i) $S$ is of $I$-type.

(ii) $\sigma_i \circ \sigma_i^{-1}(j) = \sigma_j \circ \sigma_j^{-1}(i)$ for all $i, j$.

(iii) For every defining relation $x_i x_j = x_j x_i$ of $S$ we have $\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$.

**Proof.** We denote by $(\sigma_i \circ \sigma_i^{-1}(j))(k)$ for all $i, j, k$.

First, recall from [6, Corollary 3.1] that $S$ is of $I$-type if and only if $r$ yields a solution of the quantum Yang-Baxter equation, that is $r_1 \circ r_2 \circ r_1 = r_2 \circ r_1 \circ r_2$. Therefore, if $S$ is of $I$-type, then by (8) and (9), we have

$$\sigma_i \circ \sigma_i^{-1}(j)(k) = \sigma_i \circ \sigma_j(k),$$

for all $i, j, k$. Thus

$$\sigma_i \circ \sigma_i^{-1}(j) = \sigma_i \circ \sigma_j,$$

for all $i, j$. By putting $j' = \sigma_i(j)$, we can write (10) as

$$\sigma_i \circ \sigma_i^{-1}(j') = \sigma_i \circ \sigma_i^{-1}(j'),$$

for all $i, j'$. Therefore (ii) is a consequence of (i).

Suppose that

$$\sigma_i \circ \sigma_i^{-1}(j) = \sigma_j \circ \sigma_i^{-1}(i),$$

for all $i, j$. We will prove that $r$ yields a solution of the quantum Yang-Baxter equation and thus $S$ is of $I$-type. By (8) and (9), it is sufficient to prove the following equalities:

(a) $\sigma_i \circ \sigma_i^{-1}(j)(\sigma_i^{-1}(j)(k)) = \sigma_i \circ \sigma_i^{-1}(j)(k)$;

(b) $\sigma_i \circ \sigma_i^{-1}(j)(\sigma_i^{-1}(j)(k)) = \sigma_j \circ \sigma_i^{-1}(i)(\sigma_i^{-1}(j)(k))$;

(c) $\sigma_i \circ \sigma_i^{-1}(j)(\sigma_i^{-1}(i)) = \sigma_i \circ \sigma_j \circ \sigma_i^{-1}(i)$.

The equality (a) follows from

$$\sigma_i \circ \sigma_i^{-1}(i) = \sigma_i \circ \sigma_i^{-1}(i'),$$

with $j' = \sigma_i(j)$. By (a), the equality (b) is equivalent to

$$\sigma_i \circ \sigma_i^{-1}(j)(\sigma_j(k)) = \sigma_i \circ \sigma_i^{-1}(j)(\sigma_j(k)).$$

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and the latter follows from our assumption
\[ \sigma_l^{-1} \circ \sigma_i = \sigma_{\gamma^{-1}(i)} \circ \sigma_{\gamma^{-1}(l)}, \]
with \( l = \sigma_i(\sigma_j(k)) \). By (a), the equality (c) is equivalent to
\[ \sigma_{\gamma^{-1}(i)}(\sigma_{\gamma^{-1}(j)}(i)) = \sigma_{\gamma^{-1}(j)}(\sigma_{\gamma^{-1}(i)}(\sigma_{\gamma^{-1}(j)}(j)) \circ \sigma_{\gamma^{-1}(j)}(i)), \]
In view of (11), this equality is equivalent to
\[ \sigma_{\gamma^{-1}(i)}(\sigma_{\gamma^{-1}(j)}(i)) = \sigma_{\gamma^{-1}(j)}(\sigma_{\gamma^{-1}(i)}(\sigma_{\gamma^{-1}(j)}(j))) \circ \sigma_{\gamma^{-1}(j)}(i)), \]
and the latter follows from our assumption
\[ \sigma_{\gamma^{-1}(i)} \circ \sigma_{\gamma^{-1}(j)} = \sigma_{\gamma^{-1}(j)} \circ \sigma_{\gamma^{-1}(i)}, \]
with \( l = \sigma_i(\sigma_j(k)) \) and \( j' = \sigma_i(j) \). Hence \( r \) yields a solution of the quantum Yang-Baxter equation and (ii) implies (i).

Finally, notice that \( x_i x_p = x_j x_q \) if and only if \( \sigma_i(p) = j \) and \( \sigma_j(q) = i \). The latter is equivalent to \( \sigma_i^{-1}(j) = p \) and \( \sigma_j^{-1}(i) = q \). Hence, saying that \( \sigma_i \sigma_p = \sigma_j \sigma_q \) whenever \( x_i x_p = x_j x_q \) is equivalent to saying that \( \sigma_i \sigma_i^{-1}(j) = \sigma_j \sigma_j^{-1}(i) \). So conditions (ii) and (iii) are equivalent. This completes the proof. \( \square \)

5. The dimension \( n \) case

Let \( S = \langle x_1, x_2, \ldots, x_n \rangle \) be a monoid of skew type that satisfies the cyclic condition. In this section we study the second extreme case, namely the case where \( \text{GK}(K[S]) = n \) for any field \( K \). As in Section 4 we define \( \sigma_i \in \text{Sym}_n \) by
\[ \sigma_i(j) = \begin{cases} i & \text{if } j = i, \\ k & \text{if } x_i x_j = x_k x_l \end{cases} \text{ is a defining relation of } S. \]
Let \( m = (n - 1)! \). Since \( S \) satisfies the cyclic condition, for all \( i, j \) we have that
\[ i, j \]

\[ x_i x_j^m = x_i x_j^{-1}, \]

**Theorem 5.1.** Let \( S = \langle x_1, x_2, \ldots, x_n \rangle \) be a monoid of skew type that satisfies the cyclic condition. Let \( K \) be a field. Then the following conditions are equivalent:

(i) \( \text{GK}(K[S]) = n \).
(ii) \( S \) is of I-type.
(iii) \( S \) is cancellative.

**Proof.** (i) \( \Rightarrow \) (ii). Suppose that \( \text{GK}(K[S]) = n \). Let \( m = (n - 1)! \). We know that \( A = \langle x_1^m, \ldots, x_n^m \rangle \) is abelian. Moreover \( \text{GK}(K[A]) = \text{GK}(K[S]) = n \) by Theorem 2.3. This implies that \( A \) is a free abelian monoid of rank \( n \). Indeed, otherwise the natural map \( K[y_1, \ldots, y_n] \rightarrow K[A] \) has a nontrivial kernel, whence the classical Krull dimension of \( K[A] \) is smaller than \( n \), while it is equal to the Gelfand-Kirillov dimension; see [7, Theorem 4.5].

Suppose that \( x_i x_j = x_k x_l \) is a defining relation of \( S \). Then for all \( t \in \{1, \ldots, n\} \) we have, by (12),
\[ x_i x_j x_t^m = x_i x_j x_t^{m(t)} x_j = x_i x_j x_t^{m(t)} x_j. \]
Also we have

\[ x_k x_l x^m = x_k x^m_{\sigma_1(t)} x_l = x^m_{\sigma_k(\sigma_1(t))} x_k x_l. \]

Since

\[ x_i x_j x^m_{i-1} x^m_{j-1} = x_i x_j x^m_{i-1} = x^m_{i} x_j x^m_{i-1} = x^m_{i} x^m_{j}, \]

multiplying the two previous equalities by \( x^m_{j-1} x^m_{i-1} \) on the right, we get

\[ x^m_{i} x^m_{j} = x^m_{i} x^m_{j} \]

Since \( A \) is free abelian, this implies that

\[ \sigma_i(\sigma_j(t)) = \sigma_k(\sigma_l(t)). \]

By Theorem 4.1, \( S \) is of I-type.

(ii) \( \Rightarrow \) (i). The definition of a monoid of I-type implies that the growth function of \( S \) is the same as that of a free abelian monoid of rank \( n \). Hence \( \text{GK}(K[S]) = n \).

(ii) \( \Rightarrow \) (iii). This follows from [4, Corollary 1.5].

(iii) \( \Rightarrow \) (ii). Suppose that \( x_i x_j = x_k x_l \) is a defining relation of \( S \). Then for all \( t \in \{1, \ldots, n\} \), as in the proof of the implication (i) \( \Rightarrow \) (ii) we get

\[ x^m_{i} x^m_{j} = x^m_{i} x^m_{j} x^m_{k} x^m_{l}. \]

Since \( S \) is cancellative, this implies that

\[ x^m_{i} x^m_{j} = x^m_{i} x^m_{j}. \]

By the form of the defining relations of \( S \), it is then clear that

\[ \sigma_i(\sigma_j(t)) = \sigma_k(\sigma_l(t)). \]

By Theorem 4.1, \( S \) is of I-type.

Corollary 5.2. Let \( S \) be a monoid of skew type. Then \( S \) is of I-type if and only if \( S \) is cancellative and satisfies the cyclic condition.

Proof. Suppose that \( S = \langle x_1, \ldots, x_n \rangle \) is of I-type. By [4, Theorem 1.3], the associated map \( r: X^2 \to X^2 \), where \( X = \{x_1, \ldots, x_n\} \), yields a solution of the quantum Yang-Baxter equation. By [6, Corollary 3.1], \( S \) satisfies the cyclic condition. Now the result follows from Theorem 5.1. \( \square \)

References


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