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JOINT MEASURABILITY AND THE ONE-WAY FUBINI PROPERTY FOR A CONTINUUM OF INDEPENDENT RANDOM VARIABLES

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ABSTRACT. As is well known, a continuous parameter process with mutually independent random variables is not jointly measurable in the usual sense. This paper proposes an extension of the usual product measure-theoretic framework, using a natural "one-way Fubini" property. When the random variables are independent even in a very weak sense, this property guarantees joint measurability and defines a unique measure on a suitable minimal σ-algebra. However, a further extension to satisfy the usual (two-way) Fubini property, as in the case of Loeb product measures, may not be possible in general. Some applications are also given.

1. INTRODUCTION

In [7] (p. 102), Doob made the claim that processes with mutually independent random variables are only useful in the discrete parameter case. There are indeed essential measurability difficulties associated with a continuous parameter process with random variables that are independent even in a weak sense.

Two kinds of measurability problem usually arise. The first concerns joint measurability; namely, except in some trivial cases, such a process can never be jointly measurable with respect to the completion of the usual product σ-algebra on the joint space of parameters and sample points. This means that the conditions of independence and joint measurability in the usual sense are incompatible with each other. Thus, one cannot integrate the process or take its distribution as a function on the joint space.

The second problem concerns sample measurability; as noted in [6], with further elaborations in [10], the set of sample points whose corresponding sample functions are not Lebesgue measurable has outer measure one, so Lebesgue measure offers no basis for a meaningful concept of the mean or the distribution of a sample function. Nevertheless, it has been hypothesized in a vast literature in economics that an exact version of the law of large number holds for such a process, which means that

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the *observable* mean or distribution of a sample function is essentially independent of the particular sample realization (see [2], [9], and [17] for some of the references).

In [16]–[19], the Loeb product framework introduced in [12] (see also [14]) is used to formulate and prove various versions of the exact law of large numbers and their converses. The key point is that the Loeb product extends the usual measure-theoretic product, as noted by Anderson in [1], while retaining the common Fubini property (as first shown by Keisler in [11] — see also [13] and [14]). It also accommodates an abundance of nontrivial independent processes (Theorem 6.2 in [17]). In particular, Keisler's Fubini theorem implies that both the sample and joint measurability problems are automatically solved for independent processes that are Loeb product measurable.

The approach used in [16]–[19] works with a given "nice" extension of the usual measure-theoretic product, which happens to allow many nontrivial independent processes that are jointly measurable. An important question in the other direction is whether, given a particular nontrivial independent process, there exists a "nice" extension of the usual measure-theoretic product that makes this process jointly measurable. As noted in Remark 3.2 below, there may be no such extension satisfying the Fubini property. Nevertheless, this paper shows that the joint measurability problem for such a process can indeed be solved by considering a natural probability space we call the "one-way Fubini" extension (see Definition 2.3).

The rest of the paper is organized as follows. Section 2 presents some basic definitions. The main result (Theorem 3.1) concerning the unique extension is presented in Section 3. Section 4 includes some applications of the main result. Finally, the proof of Theorem 3.1 is given in Section 5.

2. Basic definitions

Let $(T, \mathcal{T}, \lambda)$ be a probability space which is to be used as a parameter space for a process. If one likes, T can be taken to be the unit interval $[0,1]$, but $(T, \mathcal{T}, \lambda)$ is not restricted to be the Lebesgue measure structure.

Let $(\Omega, \mathcal{A}, P)$ be a sample probability space. For example, it can be the product of a continuum of copies of some other basic probability space (the existence of such a continuum product is based on the Axiom of Choice and Kolmogorov's consistency theorem), or some extension of this product, or some other space entirely. As usual in probability theory, it is not necessary to specify precisely the sample probability space, provided some general existence issues are resolved.

Let $(T\times\Omega, \mathcal{T}\otimes\mathcal{A}, \lambda\times P)$ be the usual product probability space (see, for example, [15]). Let g be a process from $T\times\Omega$ to some Polish space X with Borel σ-algebra $\mathcal{B}$.

We make the following assumptions on g:

(1) For λ-almost all $t \in T$, g_t is a random variable defined on Ω whose distribution Pg_t^{-1} on X is denoted by μ_t.
(2) For every $B \in \mathcal{B}$, the mapping $t \mapsto \mu_t(B)$ is $\mathcal{T}$-measurable.
(3) The random variables g_t are almost surely pairwise independent in the sense that for λ-almost all $t_1 \in T$, g_{t_1} is independent of g_{t_2} for λ-almost all $t_2 \in T$.

As noted in Lemma 1 of [9], the second condition is equivalent to the measurability of the distribution mapping $t \mapsto \mu_t$ from T to $\mathcal{M}(X, \mathcal{B})$, where the space $\mathcal{M}(X, \mathcal{B})$ of distributions on $(X, \mathcal{B})$ is given the weak convergence topology and associated Borel σ-algebra. The third condition is an idealized version of weak dependence in probability theory (see the discussion in [18], p. 437). When λ is atomless, this condition is weaker than mutual independence. One may simply observe that if the random variables g_t are mutually independent, then they are pairwise independent and hence also almost surely pairwise independent. When λ has an atom A, then the third condition implies that for almost all $t \in A$, g_t is independent of itself and hence almost surely a constant. Note that g is not $\mathcal{T} \otimes \mathcal{A}$-measurable except in the trivial case when, for almost all $t \in T$, g_t is a.s. a constant (see [7], [18], and also Corollary 4.1 below).

Let $(T, \mathcal{T}, \lambda)$ and $(\Omega, \mathcal{A}, P)$ be probability spaces, and $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$ the usual product probability space. Below are the formal definitions of the various Fubini properties used in this paper.

Definition 2.1. (1) A probability space $(T \times \Omega, \mathcal{W}, Q)$ that extends $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$ is said to be a *Fubini extension* if, for any real-valued $\mathcal{W}$-integrable function f, the two functions f_t and f_ω are integrable respectively on $(\Omega, \mathcal{A}, P)$ for λ-almost all $t \in T$ and on $(T, \mathcal{T}, \lambda)$ for P-almost all $\omega \in \Omega$; moreover, $\int_\Omega f_t dP$ and $\int_T f_\omega dP$ are integrable respectively on $(T, \mathcal{T}, \lambda)$ and on $(\Omega, \mathcal{A}, P)$, with $\int_{T\times\Omega} f dQ = \int_T \left(\int_\Omega f_t dP\right) d\lambda = \int_\Omega \left(\int_T f_\omega d\lambda\right) dP$.

(2) The probability space $(T \times \Omega, \mathcal{W}, Q)$ is said to be a *generalized Fubini extension* of $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$ if there exist extensions $(T, \mathcal{T}', \lambda')$ and $(\Omega, \mathcal{A}', P')$ of $(T, \mathcal{T}, \lambda)$ and $(\Omega, \mathcal{A}, P)$ respectively such that $(T \times \Omega, \mathcal{W}, Q)$ is a Fubini extension of $(T \times \Omega, \mathcal{T}' \otimes \mathcal{A}', \lambda' \times P')$.

Given the process g, suppose that $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$ has a generalized Fubini extension $(T \times \Omega, \mathcal{W}, Q)$ such that g is $\mathcal{W}$-measurable. In this extended framework, the joint measurability problem has obviously been solved, as has the sample measurability problem because g_ω is $\mathcal{T}'$-measurable.

The second definition is needed because the existence of a generalized Fubini extension does not imply the existence of a Fubini extension that makes g measurable. The following example shows this.

Example 2.2. Take a sequence $\{\beta_n\}_{n\in\mathbb{N}}$ of iid random variables defined on a probability space $(\Lambda, \mathcal{G}, \mu)$ and having a uniform distribution on $[0,1]$. Then transfer this sequence to $\{{}^*\beta_n\}_{n\in{}^*\mathbb{N}}$ in the usual non-standard model, where ${}^*\mathbb{N}$ denotes the set of hyper-integers (see [14]).

Let m be an infinitely large hyper-integer, and T the set $\{1, 2, \ldots, m\}$. Then let $(T, \mathcal{T}, \lambda)$ be the Loeb counting probability space on T. Finally, let $(\Omega, \mathcal{A}, P)$ be the Loeb space of the transfer of $(\Lambda, \mathcal{G}, \mu)$, and define the random variable $b_t = {}^\circ({}^*\beta_t)$ for each $t \in T$.

This construction yields a pair of Loeb probability spaces $(T, \mathcal{T}, \lambda)$ and $(\Omega, \mathcal{A}, P)$ with the following properties (see also [2], p. 2200): (1) there is a jointly measurable process b on the Loeb product $(T \times \Omega, \mathcal{T} \boxtimes \mathcal{A}, \lambda \boxtimes P)$, which is the Loeb space of the internal product $(T \times {}^*\Lambda, \mathcal{T} \otimes {}^*\mathcal{G}, \lambda \times {}^*\mu)$; (2) the random variables b_t are pairwise independent; (3) for each $t \in T$, b_t has uniform distribution on $[0,1]$. By Theorem 5.2 and Proposition 7.16 in [17], the sample functions b_ω are almost surely pairwise independent and also almost surely have an identical uniform distribution on $[0,1]$. Take any one $\omega_0 \in \Omega$ such that b_{ω_0} is uniformly distributed on $[0,1]$ and independent of b_ω for P-almost all $\omega \in \Omega$. Let $\mathcal{T}_0$ be the σ-algebra generated by b_{ω_0}. Obviously, $\mathcal{T}_0 \subseteq \mathcal{T}$. Let λ_0 be the restriction of λ to $\mathcal{T}_0$. Then the process b does satisfy assumptions 1–3 restated for the product probability space $(T \times \Omega, \mathcal{T}_0 \otimes \mathcal{A}, \lambda_0 \times P)$. Furthermore, the Loeb product space $(T \times \Omega, \mathcal{T} \boxtimes \mathcal{A}, \lambda \boxtimes P)$ is a generalized Fubini extension of $(T \times \Omega, \mathcal{T}_0 \otimes \mathcal{A}, \lambda_0 \times P)$. Nevertheless, the latter space has no Fubini extension that makes b jointly measurable. Indeed, for it to have such an extension, b_ω would have to be $\mathcal{T}_0$-measurable for P-almost all $\omega \in \Omega$. Yet this is impossible given the fact that, for P-almost all $\omega \in \Omega$, the random variables b_{ω_0} and b_ω are independent and uniformly distributed on $[0, 1]$.

Even though a generalized Fubini extension is weaker, it may still not exist in general. Indeed, Remark 3.2 below shows that for *any* given atomless parameter space $(T, \mathcal{T}, \lambda)$, such an extension does not exist at all for an extended sample space based on the general product measure space, as discussed in [4] (p. 230). This failure motivates the following analogous definition.

Definition 2.3. A probability space $(T \times \Omega, \mathcal{W}, Q)$ that extends $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$ is said to be a *one-way Fubini extension* if, given any real-valued $\mathcal{W}$-integrable function f, the function f_t is integrable on $(\Omega, \mathcal{A}, P)$ for λ-almost all $t \in T$, and moreover $\int_\Omega f_t dP$ is integrable on $(T, \mathcal{T}, \lambda)$, with $\int_{T \times \Omega} f dQ = \int_T \left(\int_\Omega f_t dP\right) d\lambda$.

The purpose of this paper is to show how, for any process g satisfying assumptions (1)–(3), the joint measurability problem can be solved by using the one-way Fubini extension defined above. In particular, if $\mathcal{F}$ is the minimal extension of the usual product σ-algebra $\mathcal{T} \otimes \mathcal{A}$ such that g is $\mathcal{F}$-measurable, then there is a unique probability measure ν on $\mathcal{F}$ such that $(T \times \Omega, \mathcal{F}, \nu)$ is a one-way Fubini extension of the product space $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$. Moreover, if g is measurable in any one-way Fubini extension (or in any generalized one-way Fubini extension defined analogously to part (2) of Definition 2.1), then that extension is simply a further extension of the minimal one-way Fubini extension $(T \times \Omega, \mathcal{F}, \nu)$ of the same product space. Unlike in the usual (two-way) Fubini setting, nothing is really gained by considering a generalized extension.

Although the one-way Fubini extension makes the process g jointly measurable, it is in general false that the sample function g_ω is $\mathcal{T}$-measurable for P-almost all $\omega \in \Omega$. So the reverse order of integration in $(T \times \Omega, \mathcal{F}, \nu)$ is meaningless. Of course, the Fubini property idealizes the usual rules governing double or iterated sums in the discrete setting, so it should be imposed whenever possible. From another point of view, the one-way Fubini property ensures that the extended measure ν on $\mathcal{F}$ takes the correct values. Otherwise, as noted in Remark 4.4 below, one may obtain completely arbitrary and meaningless extensions.

We are now ready to state the main theorem.

3. The main theorem

Let g be the process defined in Section 2. Define the mapping $H : T \times \Omega \to T \times \Omega \times X$ by $H(t, \omega) := (t, \omega, g(t, \omega))$. Note that for each $t \in T$, the mapping $H_t : \Omega \to \Omega \times X$ satisfies $H_t(\omega) = (\omega, g_t(\omega))$.

Let $\mathcal{E} := \mathcal{T} \otimes \mathcal{A} \otimes \mathcal{B}$ denote the product σ-algebra on $T \times \Omega \times X$. Let $\mathcal{F} := \{H^{-1}(E) : E \in \mathcal{E}\}$. Then it is clear that $\mathcal{F}$ is a σ-algebra. Also, the first two components of $H(t, \omega)$ are given by the identity mapping $\mathrm{id}_{T \times \Omega}$ on $T \times \Omega$, while the last component is $g(t, \omega)$. Hence, $\mathcal{F}$ is the smallest σ-algebra such that $\mathrm{id}_{T \times \Omega}$ and g are both measurable. This means that $\mathcal{F}$ is the smallest extension of the product σ-algebra $\mathcal{T} \otimes \mathcal{A}$ such that g is measurable.

The following theorem shows that, under the assumptions (1)–(3) in Section 2, there is a unique probability measure ν on $\mathcal{F}$ which extends $\lambda \times P$ on $\mathcal{T} \otimes \mathcal{A}$ and has the property that integrating a ν-integrable function f on $T \times \Omega$ is equivalent to evaluating an iterated double integral in the particular order $\int_T \left(\int_\Omega f_t dP\right) d\lambda$; i.e., $(T \times \Omega, \mathcal{F}, \nu)$ is a one-way Fubini extension of $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$. Since $\mathcal{F}$ is the smallest extension of the product σ-algebra $\mathcal{T} \otimes \mathcal{A}$ such that g is measurable, the measure space $(T \times \Omega, \mathcal{F}, \nu)$ may not be complete. On the other hand, one can easily see that the completion of $(T \times \Omega, \mathcal{F}, \nu)$ is still a one-way Fubini extension of $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$.

Theorem 3.1. *Suppose assumptions* (1)–(3) *in Section* 2 *are satisfied.*

(1) *For any $E \in \mathcal{E} = \mathcal{T} \otimes \mathcal{A} \otimes \mathcal{B}$, for λ-a.e. $t \in T$ the set $H_t^{-1}(E_t)$ is $\mathcal{A}$-measurable, with $P(H_t^{-1}(E_t)) = (P \times \mu_t)(E_t)$; also, the mapping $t \mapsto (P \times \mu_t)(E_t)$ is λ-integrable.*
(2) *There is a unique probability measure ν on the measurable space $(T \times \Omega, \mathcal{F})$ such that for any $F \in \mathcal{F}$, the set F_t is $\mathcal{A}$-measurable for λ-almost all $t \in T$, and $t \mapsto P(F_t)$ is a λ-integrable function with $\nu(F) = \int_T P(F_t) d\lambda$.*
(3) *$(T \times \Omega, \mathcal{F}, \nu)$ is a one-way Fubini extension of $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$; moreover, ν is the unique extension of $\lambda \times P$ on $\mathcal{F}$ with this property.*

Remark 3.2. Let $(T, \mathcal{T}, \lambda)$ be *any* atomless probability space, and μ any probability distribution on $\mathbb{R}$. Let $(\Omega, \mathcal{A}, P)$ be the continuum product probability space $(\mathbb{R}, \mathcal{B}, \mu)^T$ (see [4], p. 230). Let $f : T \times \Omega \to \mathbb{R}$ be an iid process obtained from the coordinate functions $f_t(\omega) = \omega(t)$ on the sample space $\Omega = \mathbb{R}^T$. Based on the simple idea used in [6] (see also [10]), it is pointed out in [19] that, for any given real-valued function h on T, the collection M_h of those sample functions that differ from h at countably many points in T has P-outer measure one (see Remark 6.3 in [19]). Following a standard procedure (see, for example, [7], p. 69) one can extend the measure P to a new measure $\bar{P}$ on the σ-algebra $\bar{\mathcal{A}}$ generated by $\mathcal{A} \cup \{M_h\}$ so that $\bar{P}(M_h) = 1$. Thus, one establishes the absurd claim that almost all sample functions are essentially equal to an arbitrarily given function h.

Now assume that the common mean of the random variables f_t is m. Take h to be any function whose mean $a = \int_T h d\lambda$ is not equal to m. Since f_ω is h for $\bar{P}$-almost all $\omega \in \Omega$, the sample mean $\int_T f_\omega d\lambda = a$ for $\bar{P}$-almost all $\omega \in \Omega$, and hence $\int_\Omega \left[\int_T f(t,\omega) d\lambda\right] d\bar{P} = a$. On the other hand, $\int_T \int_\Omega \left[f(t,\omega) d\bar{P}\right] d\lambda = \int_T m d\lambda = m$. This means that the two iterated integrals are different; i.e., $\int_T \left[\int_\Omega f(t,\omega) d\bar{P}\right] d\lambda \neq \int_\Omega \left[\int_T f(t,\omega) d\lambda\right] d\bar{P}$.

This shows that the product space $(T \times \Omega, \mathcal{T} \otimes \bar{\mathcal{A}}, \lambda \times \bar{P})$ does not have even a generalized Fubini extension $(T \times \Omega, \mathcal{W}, Q)$ such that f is $\mathcal{W}$-measurable; otherwise, the two iterated integrals would be the same. However, the product space does have a one-way Fubini extension, as shown in Theorem 3.1.

4. Some applications of the main theorem

The following proposition shows that, for any $\mathcal{T} \otimes \mathcal{A}$-measurable function h, the two random variables g_t and h_t are independent for λ-almost all $t \in T$. This means that any $\mathcal{T} \otimes \mathcal{A}$-measurable function differs fundamentally from the process g.

Proposition 4.1. *Let h be a measurable function from the product space $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$ to a Polish space Y. Then, for λ-almost all $t \in T$, g_t and h_t are independent; i.e., $P(h_t^{-1}(D) \cap g_t^{-1}(B)) = P(h_t^{-1}(D))\, P(g_t^{-1}(B))$ for all Borel sets B in X and D in Y.*

Proof. Let $E := h^{-1}(D) \times B \in \mathcal{T} \otimes \mathcal{A} \otimes \mathcal{B} = \mathcal{E}$. Then $E_t = h_t^{-1}(D) \times B$ and $H_t^{-1}(E_t) = h_t^{-1}(D) \cap g_t^{-1}(B)$. So for λ-a.e. $t \in T$, part (1) of Theorem 3.1 implies that

$$\begin{aligned} P(h_t^{-1}(D) \cap g_t^{-1}(B)) &= P(H_t^{-1}(E_t)) = (P \times \mu_t)(E_t) = P(h_t^{-1}(D))\mu_t(B) \\ &= P(h_t^{-1}(D))P(g_t^{-1}(B)). \end{aligned}$$

Now we can use an argument such as that in the proof of Theorem 7.6 in [17]. There exist countable open bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ for the respective topologies of the Polish spaces X and Y such that each base is closed under finite intersections. Because $\mathcal{B}_X$ and $\mathcal{B}_Y$ are countable, the above paragraph implies that there exists a λ-null set $S_0 \subset T$ such that, for all $t \notin S_0$,

$$P(g_t^{-1}(O_X) \cap h_t^{-1}(O_Y)) = P(g_t^{-1}(O_X)) \cdot P(h_t^{-1}(O_Y))$$

holds simultaneously for all $O_X \in \mathcal{B}_X$ and all $O_Y \in \mathcal{B}_Y$. Thus, for any $t \notin S_0$, the joint distribution $P(g_t, h_t)^{-1}$ on $X \times Y$ agrees with the product $Pg_t^{-1} \times Ph_t^{-1}$ of its marginals on the π-system $\{O_X \times O_Y : O_X \in \mathcal{B}_X, O_Y \in \mathcal{B}_Y\}$ for the σ-algebra $\mathcal{B}_X \otimes \mathcal{B}_Y$ on $X \times Y$. So by a result on the unique extension of measures (see [8], p. 402), $P(g_t, h_t)^{-1} = Pg_t^{-1} \times Ph_t^{-1}$ on the whole product σ-algebra. This implies that h_t and g_t are independent for all $t \notin S_0$, which completes the proof. $\square$

The following obvious corollary was Proposition 1 in [18]. When λ is Lebesgue measure and the process g is iid, a similar result was already noted by Doob in [7] (p. 67).

Corollary 4.2. *If g is measurable on $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$, then for λ-almost all $t \in T$, the random variable g_t is essentially constant.*

Proof. Proposition 4.1 implies that for λ-a.e. $t \in T$, g_t is independent of itself and hence a constant. $\square$

The following result extends Theorem 4.2 in the Loeb product framework of [3] to the general case.

Proposition 4.3. *Let C be a subset of $T \times \Omega$ such that $0 < P(C_t) < 1$ for λ-almost all $t \in T$. Suppose the events C_t $(t \in T)$ are almost surely pairwise independent; i.e., for λ-almost all $t_1 \in T$, C_{t_1} is independent of C_{t_2} for λ-almost all $t_2 \in T$. Then C has outer measure one and inner measure zero with respect to $\lambda \times P$.*

Proof. Let g be the indicator function 1_C of C. Then g is a process satisfying the assumptions in Section 1. Also, the random variables g_t are almost surely pairwise independent. Take any $D \in \mathcal{T} \otimes \mathcal{A}$ and let h be 1_D. Proposition 4.1 implies that, for λ-almost all $t \in T$, the random variables g_t and h_t are independent. So therefore are the events C_t and D_t; i.e., $P(C_t \cap D_t) = P(C_t)P(D_t)$.

Thus, if $D \subseteq C$, then for λ-almost all $t \in T$, $P(D_t) = P(C_t)P(D_t)$. Since $P(C_t) < 1$ for λ-almost all $t \in T$, it follows that $P(D_t) = 0$ for λ-almost all $t \in T$. By the Fubini theorem, because D is an arbitrary $\mathcal{T} \otimes \mathcal{A}$-measurable subset of C, it follows that the inner measure $(\lambda \times P)_*(C) = 0$.

On the other hand, if $C \subseteq D$, then for λ-almost all $t \in T$, $P(C_t) = P(C_t)P(D_t)$. Since $P(C_t) > 0$ for λ-almost all $t \in T$, it follows that $P(D_t) = 1$ for λ-almost all $t \in T$. By the Fubini theorem, $(\lambda \times P)(D) = 1$. Because $D \supseteq C$ is arbitrary in $\mathcal{T} \otimes \mathcal{A}$, it follows that the outer measure $(\lambda \times P)^*(C) = 1$. $\square$

Remark 4.4. Suppose the set $C \subseteq T \times \Omega$ and the probability space $(\Omega, \mathcal{A}, P)$ are such that the events C_t $(t \in T)$ are almost surely pairwise independent with identical probability p, for some $0 < p < 1$. Let $g = 1_C$, and let

$$\mathcal{F} = \{(D_1 \cap C) \cup (D_2 \setminus C) : D_1, D_2 \in \mathcal{T} \otimes \mathcal{A}\}$$

be the smallest extension of the product σ-algebra such that g is measurable. If we require the one-way Fubini property on $(T \times \Omega, \mathcal{F})$, then the measure for C must be

$$\nu(C) = \int_T \left(\int_\Omega 1_C dP \right) d\lambda = \int_T P(C_t) d\lambda = p.$$

On the other hand, if we do not require the one-way Fubini property, then for an arbitrarily given number $r \in [0, 1]$, we can use the common procedure for extending measures (see, for example, [7], p. 69) to define

$$\sigma_r((D_1 \cap C) \cup (D_2 \setminus C)) = r\,(\lambda \times P)(D_1) + (1 - r)\,(\lambda \times P)(D_2)$$

for any $D_1, D_2 \in \mathcal{T} \otimes \mathcal{A}$. Because of Proposition 4.3, this must be a well-defined probability measure on $(T \times \Omega, \mathcal{F})$, with $\sigma_r(C) = r$. So the one-way Fubini property allows us to select the "correct" measure for the extension and to ignore other completely meaningless extensions such as σ_r for any $r \neq p$.

5. Proof of the main theorem

Lemma 5.1. *Suppose that the random variables f_t $(t \in T)$ are all square-integrable and are almost surely uncorrelated; i.e., suppose each $f_t \in L_2(\Omega, \mathcal{A}, P)$ and, for a.e. $t_1 \in T$, $E(f_{t_1} f_{t_2}) = Ef_{t_1} \cdot Ef_{t_2}$ for a.e. $t_2 \in T$. Then, for every $A \in \mathcal{A}$, $\int_A f_t dP = P(A)\, Ef_t$ for λ-a.e. $t \in T$.*

Proof. Let T' be the set of all $t' \in T$ such that the random variables $f_{t'}$ and f_t are uncorrelated for λ-a.e. $t \in T$. By hypothesis, $\lambda(T') = 1$.

Following a standard procedure for Hilbert spaces, let L be the smallest closed linear subspace of $L_2(\Omega, \mathcal{A}, P)$ containing both the family $\{f_t : t \in T'\}$ and the constant function $1 = 1_\Omega$. Let h be the orthogonal projection of the indicator function 1_A onto L, with $h^\perp$ as its orthogonal complement. Then $1_A = h + h^\perp$ where $E(h^\perp f_t) = \int_\Omega h^\perp f_t dP = 0$ for all $t \in T'$, and also $Eh^\perp = \int_\Omega h^\perp dP = 0$. It follows that $E(1_A\, f_t) = E(h\, f_t)$ for all $t \in T'$, and also $E1_A = P(A) = Eh$.

Next, because $h \in L$, there exists a sequence of functions

$$h_n = r_n + \sum_{k=1}^{i_n} \alpha_n^k\, f_{t_n^k}\ \ (n = 1, 2, \ldots)$$

with $t_n^k \in T'$, as well as r_n and α_n^k $(k = 1, \ldots, i_n)$ all real, such that $h_n \to h$ in $L_2(\Omega, \mathcal{A}, P)$.

Let $T_n^k := \{t \in T : f_t \text{ and } f_{t_n^k} \text{ are uncorrelated}\}$. By hypothesis, $\lambda(T_n^k) = 1$ because each $t_n^k \in T'$. Define $T^* := T' \cap \left(\bigcap_{n=1}^\infty \bigcap_{k=1}^{i_n} T_n^k \right)$. Then $\lambda(T^*) = 1$, because $\lambda(T') = 1$. Also, for any $t \in T^*$, because f_t and each $f_{t_n^k}$ are uncorrelated, one has

$$E(h_n\, f_t) = r_n\, Ef_t + \sum_{k=1}^{i_n} \alpha_n^k E(f_{t_n^k}\, f_t) = r_n\, Ef_t + \sum_{k=1}^{i_n} \alpha_n^k (Ef_{t_n^k})\,(Ef_t) = Eh_n \cdot Ef_t.$$

So

$$\begin{aligned}\int_A f_t dP &= E(1_A f_t) = E(h f_t) = \lim_{n\to\infty} E(h_n f_t) \\ &= E f_t \lim_{n\to\infty} E h_n = E f_t \cdot E h = P(A)\, E f_t\end{aligned}$$

for all $t \in T^*$, where $\lambda(T^*) = 1$. □

Proposition 5.2. *For every $E \in \mathcal{E} = \mathcal{T} \otimes \mathcal{A} \otimes \mathcal{B}$, the following properties hold:*

(i) *the mapping $t \mapsto (P \times \mu_t)(E_t)$ is $\mathcal{T}$-measurable;*

(ii) *for λ-a.e. $t \in T$, the set $H_t^{-1}(E_t)$ is $\mathcal{A}$-measurable, and $P(H_t^{-1}(E_t)) = (P \times \mu_t)(E_t)$.*

Proof. Let $\mathcal{D}$ be the collection of sets $E \in \mathcal{E}$ satisfying properties (i) and (ii).

First, we show that each measurable triple product set $E = S \times A \times B \in \mathcal{E}$ satisfies (i)–(ii), implying that $E \in \mathcal{D}$. Indeed:

(i) If $t \notin S$, then $E_t = \emptyset$ and $(P \times \mu_t)(E_t) = 0$. On the other hand, if $t \in S$, then $E_t = A \times B$ and $(P \times \mu_t)(E_t) = P(A)\mu_t(B)$ for all $t \in T$. Hence, $(P \times \mu_t)(E_t) = 1_S(t)P(A)\mu_t(B)$ for all $t \in T$. Because $S \in \mathcal{T}$ and $t \mapsto \mu_t(B)$ is $\mathcal{T}$-measurable, so is $t \mapsto (P \times \mu_t)(E_t)$.

(ii) If $t \notin S$, then $E_t = \emptyset$, and $P(H_t^{-1}(E_t)) = 0 = (P \times \mu_t)(E_t)$. On the other hand, if $t \in S$, then $E_t = A \times B$, so $H_t^{-1}(E_t) = A \cap g_t^{-1}(B) \in \mathcal{A}$. In this case, applying Lemma 1 to the square-integrable and almost surely uncorrelated random variables $1_{g_t^{-1}(B)}$ $(t \in T)$ implies that, for λ-a.e. $t \in S$, one has

$$\begin{aligned}P(H_t^{-1}(E_t)) &= P(A \cap g_t^{-1}(B)) = \int_A 1_{g_t^{-1}(B)} dP = P(A) \int_\Omega 1_{g_t^{-1}(B)} dP \\ &= P(A)\, \mu_t(B) = (P \times \mu_t)(A \times B) = (P \times \mu_t)(E_t).\end{aligned}$$

It remains to verify that the family $\mathcal{D}$ is a Dynkin (or λ-) class in the sense that:

(a) $T \times \Omega \times X \in \mathcal{D}$;

(b) if $E, E' \in \mathcal{D}$ with $E \supset E'$, then $E \setminus E' \in \mathcal{D}$;

(c) if E^n is an increasing sequence of sets in $\mathcal{D}$, then $\bigcup_{n=1}^\infty E^n \in \mathcal{D}$.

Then we can apply Dynkin's π–λ theorem to establish that $\mathcal{D} = \mathcal{E} = \mathcal{T} \otimes \mathcal{A} \otimes \mathcal{B}$, because the set of products of measurable sets is a π-system, i.e., closed under finite intersections (see [5], p. 44, and [8], p. 404). In fact:

(a) $T \times \Omega \times X \in \mathcal{D}$ as a triple product of measurable sets.

(b) If E, E' satisfy properties (i) and (ii) with $E \supset E'$, then $(E \setminus E')_t = E_t \setminus E'_t$ and so:

(i) the mapping

$$t \mapsto (P \times \mu_t)(E \setminus E')_t = (P \times \mu_t)(E_t) - (P \times \mu_t)(E'_t)$$

is $\mathcal{T}$-measurable.

(ii) for λ-a.e. $t \in T$, the set $H^{-1}((E \setminus E')_t) = H_t^{-1}(E_t) \setminus H_t^{-1}(E'_t)$ is $\mathcal{A}$-measurable, with

$$\begin{aligned}P(H^{-1}((E \setminus E')_t)) &= P(H_t^{-1}(E_t)) - P(H_t^{-1}(E'_t)) \\ &= (P \times \mu_t)(E_t) - (P \times \mu_t)(E'_t) = (P \times \mu_t)((E \setminus E')_t).\end{aligned}$$

Hence, $E \setminus E' \in \mathcal{D}$.

(c) If E^n is an increasing sequence in $\mathcal{D}$, then:

(i) the mapping

$$t \mapsto (P \times \mu_t)(\bigcup_{n=1}^{\infty} E_t^n) = \lim_{n\to\infty} (P \times \mu_t)(E_t^n)$$

is $\mathcal{T}$-measurable;

(ii) for λ-a.e. $t \in T$, the set $H_t^{-1}(\bigcup_{n=1}^{\infty} E_t^n) = \bigcup_{n=1}^{\infty} H_t^{-1}(E_t^n)$ is $\mathcal{A}$-measurable, and

$$\begin{aligned} P(H_t^{-1}(\bigcup_{n=1}^{\infty}(E_t^n)) &= \lim_{n\to\infty} P(H_t^{-1}(E_t^n)) = \lim_{n\to\infty} (P \times \mu_t)(E_t^n) \\ &= (P \times \mu_t)(\bigcup_{n=1}^{\infty} E_t^n). \end{aligned}$$

Hence, $\bigcup_{n=1}^{\infty} E^n \in \mathcal{D}$. □

Proof of Theorem 3.1. Part (1) was proved as part of Proposition 5.2.

To prove part (2), note that given any $F \in \mathcal{F}$, there exists at least one $E \in \mathcal{E}$ such that $F = H^{-1}(E)$. Then $F_t = H_t^{-1}(E_t) \in \mathcal{A}$ for λ-a.e. $t \in T$, by Proposition 5.2. The same result implies that $P(F_t) = P(H_t^{-1}(E_t)) = (P \times \mu_t)(E_t)$, and that this is a $\mathcal{T}$-measurable function of t. So we can define a unique set function ν on the measurable space $(T \times \Omega, \mathcal{F})$ by $\nu(F) := \int_T P(F_t) d\lambda$. Note that $\nu(T \times \Omega) = 1$ and, whenever F^n $(n = 1, 2 \ldots)$ is a disjoint countable collection of sets in $\mathcal{F}$, then

$$\begin{aligned} \nu(\bigcup_{n=1}^{\infty} F^n) &= \int_T P(\bigcup_{n=1}^{\infty} F_t^n) d\lambda = \int_T \sum_{n=1}^{\infty} P(F_t^n) d\lambda = \sum_{n=1}^{\infty} \int_T P(F_t^n) d\lambda \\ &= \sum_{n=1}^{\infty} \nu(F^n). \end{aligned}$$

So ν is a uniquely defined probability measure.

To prove part (3), note first that whenever $F \in \mathcal{T} \otimes \mathcal{A}$, then $\nu(F) = \int_T P(F_t) d\lambda = (\lambda \times P)(F)$. So $(T \times \Omega, \mathcal{F}, \nu)$ is an extension of $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$. The remainder of the proof is virtually identical to that of the usual Fubini Theorem. For the sake of completeness, we include a proof adapted from [15] (p. 308). Let $V \subseteq L_1(T \times \Omega, \mathcal{F}, \nu)$ denote the set of all ν-integrable functions f that satisfy the one-way Fubini property $\int_{T\times\Omega} f d\nu = \int_T \left(\int_\Omega f_t dP\right) d\lambda$. Then V includes every measurable indicator function 1_F $(F \in \mathcal{F})$ because

$$\nu(F) = \int_{T\times\Omega} 1_F d\nu = \int_T P(F_t) d\lambda = \int_T \left[\int_\Omega (1_F)_t dP\right] d\lambda.$$

These equations confirm that the one-way Fubini property determines ν uniquely on $(T \times \Omega, \mathcal{F})$.

Next, V is obviously closed under linear combinations; i.e., V is a linear subspace. In particular, V includes all measurable simple functions, and all differences between members of V. Also, any ν-integrable function is the difference between two non-negative ν integrable functions, and any non-negative ν-integrable function

f is the limit of an increasing sequence f^n of non-negative simple functions. So it remains only to show that V contains the integrable limit of an increasing sequence f^n of non-negative functions in V.

Indeed, suppose $f \in L_1(T \times \Omega, \mathcal{F}, \nu)$ and let f^n $(n = 1, 2, \ldots)$ be any sequence of non-negative functions in V satisfying $f^n \uparrow f$ as $n \to \infty$. Then the monotone convergence theorem implies that $\lim_{n\to\infty} \int_{T\times\Omega} f^n d\nu = \int_{T\times\Omega} f d\nu$. Since each f^n is in V, and so satisfies the one-way Fubini property, we know that f^n_t is in $L_1(\Omega, \mathcal{A}, P)$ for λ-a.e. $t \in T$. It is obvious that for λ-a.e. $t \in T$, $f^n_t \uparrow f_t$, and hence f_t is $\mathcal{A}$-measurable with $\int_\Omega f_t dP = \lim_{n\to\infty} \int_\Omega f^n_t dP$. In fact, one must have $\int_\Omega f^n_t dP \uparrow \int_\Omega f_t dP$. Hence, the monotone convergence theorem and the one-way Fubini property for f^n imply that

$$\int_T \left(\int_\Omega f_t dP \right) d\lambda = \lim_{n\to\infty} \int_T \left(\int_\Omega f^n_t dP \right) d\lambda = \lim_{n\to\infty} \int_{T\times\Omega} f^n d\nu = \int_{T\times\Omega} f d\nu.$$

So f also satisfies the one-way Fubini property.

This shows that $V = L_1(T \times \Omega, \mathcal{F}, \nu)$. $\square$

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