

## OPTIMAL WEYL INEQUALITY IN BANACH SPACES

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*Dedicated to Professor Albrecht Pietsch on the occasion of his 70th birthday*

ABSTRACT. A well-known multiplicative Weyl inequality states that the sequence of eigenvalues  $(\lambda_k(T))$  and the sequence of approximation numbers  $(a_k(T))$  of any compact operator  $T$  in a Banach space satisfy

$$\prod_{k=1}^n |\lambda_k(T)| \leq n^{n/2} \prod_{k=1}^n a_k(T)$$

for all  $n$ . We prove here that the constant  $n^{n/2}$  is optimal, which solves a longstanding problem.

### 1. INTRODUCTION

Estimating the eigenvalue distribution of compact operators via approximation quantities is a classic theme in approximation theory with an extensive literature. The starting point was the inequality

$$(1) \quad \prod_{k=1}^n |\lambda_k(T)| \leq \prod_{k=1}^n s_k(T)$$

proved by H. Weyl in [Wey49] for any compact operator  $T$  acting on a Hilbert space. Here  $(\lambda_k(T))$  and  $(s_k(T))$  are the sequences of eigenvalues and singular numbers of  $T$ , respectively. A great deal of effort went into proving similar inequalities for operators on Banach spaces. For a recent survey article we refer to [Koe01].

The most useful inequality for the investigation of the asymptotic behavior of the eigenvalue sequence is due to A. Pietsch [Pie80a]. It states that there exists some constant  $c > 1$  such that for any continuous linear operator  $T$  on some Banach space and all  $n = 1, 2, \dots$ ,

$$(2) \quad \prod_{k=1}^n |\lambda_k(T)| \leq c^n \prod_{k=1}^n \dot{x}_k(T).$$

Here  $(\dot{x}_k(T)) = (x_1(T), x_1(T), x_2(T), x_2(T), x_3(T), x_3(T), \dots)$  is the doubled sequence of Weyl numbers of  $T$ . For the definition of Weyl numbers see below. This inequality is sufficient to prove that the eigenvalue sequence is in  $l_p$  for some  $p > 0$

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provided that the sequence of Weyl numbers is in  $l_p$ . In particular, for any  $p > 0$  there exists a constant  $c_p$  such that

$$(3) \quad \left( \sum_{k=1}^{\infty} |\lambda_k(T)|^p \right)^{1/p} \leq c_p \left( \sum_{k=1}^{\infty} x_k(T)^p \right)^{1/p}.$$

This is also due to A. Pietsch [Pie80a].

Two obvious unaesthetic features of (2) compared with (1) are the appearance of the constant  $c > 1$  and the replacement of the sequence of singular numbers with the *doubled* sequence of Weyl numbers instead of the more natural original sequence of Weyl numbers. An example showing that  $c > 1$  is necessary in the general case was given in [Koe86, 2.a.6]. Whether the doubled Weyl number sequence can be replaced by the usual sequence was an open problem (stated e.g. in [Koe86, Koe01, Pie87]), which we are going to solve in this note.

This is not merely an aesthetic problem for two reasons. First, it was shown by H. König in [Koe86, 2.a.7] that if inequality (2) would be true with  $x_k(T)$  instead of  $\dot{x}_k(T)$ , then the constant  $c_p$  in (3) could be chosen independently of  $p > 0$ . The best known (for  $p \rightarrow 0$ ) upper bound  $c_p \leq 2e/\sqrt{p}$  is also due to H. König [Koe84] and tends to infinity as  $p$  goes to 0. Second, if one wants to estimate the first few eigenvalues of an operator, inequality (2) is practically of no help. Instead, one would need inequalities of the type

$$(4) \quad \prod_{k=1}^n |\lambda_k(T)| \leq c_n \prod_{k=1}^n x_k(T)$$

with optimal  $c_n$ . This inequality with  $c_n = n^{n/2}$  seems to be well known although it does not appear in print. We will give a proof of this inequality in due time. The main result of this note is that the constant  $c_n = n^{n/2}$  is actually optimal even if we allow the Weyl numbers to be replaced by the approximation numbers.

We now introduce the necessary notation. Following A. Pietsch [Pie74, Pie80b, Pie87], a sequence  $(s_n)_{n=1}^{\infty}$  of maps assigning to each operator  $T \in \mathcal{L}(E, F)$  between Banach spaces  $E$  and  $F$  numbers  $s_n(T)$  with the properties

- (i)  $\|T\| = s_1(T) \geq s_2(T) \geq \dots \geq 0$  for  $T \in \mathcal{L}(E, F)$ ,
- (ii)  $s_{m+n-1}(S+T) \leq s_m(S) + s_n(T)$  for  $S, T \in \mathcal{L}(E, F)$ ,
- (iii)  $s_n(BTA) \leq \|B\|s_n(T)\|A\|$  for  $A \in \mathcal{L}(E_0, E), T \in \mathcal{L}(E, F), B \in \mathcal{L}(F_0, F)$ ,
- (iv)  $s_n(T) = 0$  if  $\text{rank } T < n$ ,
- (v)  $s_n(I_n) = 1$  for the identity maps  $I_n : l_2^n \rightarrow l_2^n$  on  $l_2^n$

is called an  $s$ -number sequence. Moreover,  $s_n(T)$  is called the  $n$ -th  $s$ -number of  $T$ . Basic examples for our purposes are the approximation numbers given by

$$a_n(T) := \inf\{\|T - A\| : A \in \mathcal{L}(E, F), \text{rank } A < n\},$$

the Weyl numbers given by

$$x_n(T) := \sup\{a_n(TA) : A \in \mathcal{L}(l_2, E), \|A\| \leq 1\},$$

and the Hilbert numbers given by

$$h_n(T) := \sup\{a_n(BTA) : A \in \mathcal{L}(l_2, E), B \in \mathcal{L}(F, l_2), \|A\|, \|B\| \leq 1\}.$$

For compact operators on Hilbert spaces and any  $s$ -number sequence,  $s_n(T)$  is just the  $n$ -th singular value of  $T$ ; see [Pie87, 2.11.9]. The Hilbert numbers form the smallest  $s$ -number sequence, the approximation numbers the largest, i.e.  $h_n(t) \leq$

$s_n(T) \leq a_n(T)$  for all operators  $T$  and all  $n$ ; see [Pie87, 2.3.4 and 2.6.3]. For a compact operator acting on some Banach space,  $(\lambda_n(T))$  denotes the sequence of eigenvalues of  $T$  counted according to their multiplicity and arranged by non-increasing modulus. For more information on  $s$ -numbers and their relations to eigenvalues we refer to [Koe86, Koe01, Pie87].

Our main result is formulated in the following theorem.

**Theorem 1.1.** *Let  $(s_n)$  be an  $s$ -number sequence. Then*

$$\prod_{k=1}^n |\lambda_k(T)| \leq n^{n/2} \prod_{k=1}^n s_k(T)$$

for any compact operator  $T$  on a Banach space and all  $n$ . The constant  $n^{n/2}$  in this inequality is optimal.

By the already mentioned result of H. König from [Koe86, 2.a.7] we have

**Corollary 1.2.** *The constants  $c_p$  in (3) cannot be chosen independent of  $p > 0$  even if the Weyl numbers in the inequality are replaced with the approximation numbers.*

## 2. THE PROOFS

We start with the proof of the inequality in Theorem 1.1. Since the Hilbert numbers form the smallest  $s$ -scale, it is enough to consider the case  $s_k(T) = h_k(T)$ . If  $\lambda_n(T) = 0$ , there is nothing to prove. So we assume  $\lambda_n(T) \neq 0$ . Then by [Koe86, 1.a.4] or [Pie87, 3.2.23] we can find an  $n$ -dimensional subspace  $E_n$  of  $E$  invariant under  $T$  such that  $T_n$  has the eigenvalues  $\lambda_1(T), \dots, \lambda_n(T)$ . By a result of S. Kwapien we can find a projection  $P_n$  from  $E$  onto  $E_n$  with 2-summing norm  $\pi_2(P_n) = \sqrt{n}$ . Now the Pietsch factorization theorem [Pie66, Pie80b] implies that  $P_n$  can be factored as  $P_n = B_n A_n$  with a norm 1 operator  $B_n \in \mathcal{L}(l_2^n, E_n)$  and an operator  $A_n \in \mathcal{L}(E, l_2^n)$  with  $\|A_n\| \leq \pi_2(A_n) = \sqrt{n}$ . Let  $J_n$  be the embedding of  $E_n$  into  $E$ . Applying Weyl's inequality to the Hilbert space operator  $S_n = A_n J_n T_n B_n = A_n T J_n B_n$  together with the principle of related operators (cf. [Pie87, 3.3.4]) we obtain that

$$\begin{aligned} \prod_{k=1}^n |\lambda_k(T)| &= \prod_{k=1}^n |\lambda_k(T_n)| = \prod_{k=1}^n |\lambda_k(S_n)| = \prod_{k=1}^n h_k(S_n) \\ &\leq \prod_{k=1}^n \|A_n\| h_k(T) \|J_n\| \|B_n\| \leq n^{n/2} \prod_{k=1}^n h_k(T). \end{aligned}$$

For the proof of the optimality in Theorem 1.1, let  $c_n$  denote the smallest possible constant in the inequality

$$(5) \quad \prod_{k=1}^n |\lambda_k(T)| \leq c_n \prod_{k=1}^n a_k(T).$$

To show that  $c_n \geq n^{n/2}$ , we need an auxiliary lemma.

**Lemma 2.1.** *For any  $n = 1, 2, \dots$ ,*

$$\sup \frac{\prod_{k=1}^n \sigma_k}{\prod_{k=1}^n \sum_{l=k}^n \sigma_l} = 1$$

where the supremum is taken over all real  $\sigma_1, \dots, \sigma_n$  with  $\sigma_1 > \sigma_2 > \dots > \sigma_n > 0$ .

*Proof.* Denote the supremum by  $a_n$ . Obviously,  $a_1 = 1$ . Now assume that  $n > 1$  and  $a_{n-1} = 1$  is already proved. Observe that

$$a_n = \sup \frac{\prod_{k=1}^{n-1} \sigma_k}{\prod_{k=1}^{n-1} \sum_{l=k}^n \sigma_l}.$$

The expression in the supremum is a continuous nonincreasing function of  $\sigma_n > 0$ , which is also continuous for  $\sigma_n = 0$ . Hence the sup is attained for  $\sigma_n = 0$ , which just gives  $a_n = a_{n-1} = 1$ .  $\square$

We now construct our example. Let  $E_n = (e_{hk})$  be the  $n \times n$  Fourier matrix which has entries  $e_{hk} = n^{-1/2} \exp(2\pi i h k / n)$ . Fix numbers  $\sigma_1 > \sigma_2 > \dots > \sigma_n > 0$  and let  $D$  be the diagonal matrix with diagonal  $(\sigma_1, \dots, \sigma_n)$ . Let  $T = E_n D$  and consider  $T$  as an operator in  $l_\infty^n$ .

Since  $E_n$  is a unitary matrix, we conclude that

$$\prod_{k=1}^n |\lambda_k(T)| = |\det(T)| = |\det(D)| = \prod_{k=1}^n \sigma_k.$$

For  $k = 1, \dots, n$ , let  $A_k$  be the matrix which coincides with  $E_n D$  on the first  $k - 1$  columns and has only zero entries in the remaining columns. Obviously,  $\text{rank}(A_k) < k$ . Hence

$$a_k(T) \leq \|T - A_k\| = \sum_{l=k}^n n^{-1/2} \sigma_l.$$

Inequality (5) used for our example operator  $T$  shows that

$$\prod_{k=1}^n \sigma_k \leq c_n n^{-n/2} \prod_{k=1}^n \sum_{l=k}^n \sigma_l.$$

Since this holds for all  $\sigma_1 > \sigma_2 > \dots > \sigma_n > 0$ , a final glimpse at the above lemma proves that  $c_n \geq n^{n/2}$ .

*Remark.* In a subsequent paper of B. Carl and the author [CH04] we further investigate optimal inequalities of Weyl type. In particular, we show a scale of Weyl type inequalities useful for estimating single eigenvalues with the help of  $s$ -numbers. We also show that the Weyl numbers are, in some sense, optimal  $s$ -numbers for inequalities of the type (2) and (3).

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