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FUNCTIONAL CALCULUS AND $*$-REGULARITY OF A CLASS OF BANACH ALGEBRAS

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Abstract. Suppose that (A, G, α) is a C^*-dynamical system such that G is of polynomial growth. If A is finite dimensional, we show that any element in $K(G; A)$ has slow growth and that $L^1(G, A)$ is $*$-regular. Furthermore, if G is discrete and π is a "nice representation" of A, we define a new Banach $*$-algebra $l^1_\pi(G, A)$ which coincides with $l^1(G; A)$ when A is finite dimensional. We also show that any element in $K(G; A)$ has slow growth and $l^1_\pi(G, A)$ is $*$-regular.

1. Introduction and preliminaries

For a Banach $*$-algebra B, we denote by $B^\sim$ the unitalization of B together with the $*$-algebra norm defined by $\|b + \lambda 1\| = \|b\| + |\lambda|$. We also denote by $C^*(B)$ the enveloping C^*-algebra of B and $\Phi : B \to C^*(B)$ the canonical embedding (not necessarily injective).

In the following, we assume that $C^*(B) \neq (0)$. Moreover, throughout this paper, all $*$-representations of Banach $*$-algebras are assumed to be non-degenerate and all ideals are closed. For any $*$-representation π of B, there is a unique $*$-representation π_* of $C^*(B)$ such that $\pi = \pi_* \circ \Phi$. Let $\mathrm{Prim}\ C^*(B)$ be the space of primitive ideals of $C^*(B)$ and let $\mathrm{Prim}_* B$ be the space of kernels of topological irreducible $*$-representations of B, both equipped with the Jacobson topology. Moreover, Φ induces a continuous surjection $\Psi : \mathrm{Prim}\ C^*(B) \to \mathrm{Prim}_* B$ (see e.g. [7, Corollary 10.5.7]).

Definition 1.1 ([7, 10.5.8])**.** A Banach $*$-algebra B is said to be *$*$-regular* if the canonical map $\Psi : \mathrm{Prim}\ C^*(B) \to \mathrm{Prim}_* B$ is a homeomorphism.

Remark 1.2. B is $*$-regular if and only if $B/*\text{-rad}(B)$ is $*$-regular (where $*\text{-rad}(B)$ is the $*$-radical of B).

We now recall the following result from [2]. As we are in a more general setting, we repeat their argument here for clarity.

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Proposition 1.3 ([2, Satz 1])**.** *A Banach $*$-algebra B is $*$-regular if and only if for any $*$-representations π and ρ of B, the inclusion $\ker\pi \subseteq \ker\rho$ will imply that $\|\rho(b)\| \leq \|\pi(b)\|$ for all $b \in B$.*

Proof. Suppose that B is $*$-regular. Let

$$E := \{P \in \mathrm{Prim}\ C^*(B) : \ker\pi_* \subseteq P\}.$$

As $\Phi(\ker\pi) = \ker\pi_* \cap \Phi(B)$, we see that $\bigcap_{P\in E}\Phi(\Psi(P)) = \bigcap_{P\in E} P \cap \Phi(B) = \Phi(\ker\pi)$. Let $I \in \mathrm{Prim}\ C^*(B)$ such that $\ker\rho_* \subseteq I$. Then

$$\bigcap_{P\in E}\Phi(\Psi(P)) \ \subseteq\ \Phi(\ker\rho)\ \subseteq\ I\cap\Phi(B)\ =\ \Phi(\Psi(I)),$$

which implies that $\bigcap_{Q\in\Psi(E)} Q \subseteq \Psi(I)$ (as $\ker\Phi \subseteq J$ for any $J \in \mathrm{Prim}_*\, B$). Since Ψ is a homeomorphism and E is closed, $\Psi(I) \in \Psi(E)$ and $I \in E$. This shows that $\ker\pi_* \subseteq \ker\rho_*$ and so $\|\rho_*(x)\| \leq \|\pi_*(x)\|$ for any $x \in C^*(B)$. Conversely, the hypothesis clearly implies that Ψ is injective (as $\ker\pi = \ker\rho$ will then imply $\|\rho_*(x)\| = \|\pi_*(x)\|$ for any $x \in C^*(B)$). Let $E \subseteq \mathrm{Prim}\ C^*(B)$ be a closed subset. For any $\ker\tau \in \mathrm{hull}(\ker\Psi(E))$, we have

$$\ker\tau\ \supseteq \bigcap_{\ker\sigma_*\in E}\ker\sigma\ =\ \ker(\bigoplus_{\ker\sigma_*\in E}\sigma).$$

Therefore, by the hypothesis, $\|\tau_*(x)\| \leq \sup\{\|\sigma_*(x)\| : \ker\sigma_* \in E\}$ $(x \in C^*(B))$. Hence $\ker\tau_* \in E$ (as E is closed) and $\ker\tau \in \Psi(E)$. □

Corollary 1.4. *Let B be a Banach $*$-algebra.*

(a) *Suppose that there is a dense subset B_0 of B_{sa} such that for any $b \in B_0$ and any smooth function $\varphi : \mathbb{R}_+ \to \mathbb{C}$ with compact support and $\varphi(0) = 0$, there exists $c \in B$ with $\mu(c) = \varphi(\mu(b^2))$ for any $*$-representation μ of B. Then B is $*$-regular.*

(b) *Suppose that $\{B_i\}_{i\in I}$ is a directed family of Banach $*$-subalgebras of B (i.e. $B_i \subseteq B_j$ if $i \leq j$) such that $\bigcup_{i\in I} B_i$ is dense in B. If all B_i are $*$-regular, then so is B.*

Proof. (a) Let (π, H_π) and (ρ, H_ρ) be two $*$-representations of B such that $\ker\pi \subseteq \ker\rho$. Assume that there exists $a \in B$ such that $\|\pi(a)\| < \|\rho(a)\|$. By the density of B_0 and the C^*-identity, there exists $b \in B_0$ such that $\|\pi(b^2)\| < \|\rho(b^2)\|$. Consider a smooth function $\varphi : \mathbb{R}_+ \to \mathbb{C}$ with compact support such that

$$\varphi(\mathbb{R}_+)\ \subseteq\ [0,1], \qquad \varphi(\,[0,\|\pi(b^2)\|]\,)\ =\ \{0\} \qquad \text{and} \qquad \varphi(\|\rho(b^2)\|)\ =\ 1.$$

Let $c \in B$ be the element given by the hypothesis. Then $\pi(c) = \varphi(\pi(b^2)) \in \mathcal{L}(H_\pi)_+$. From the equalities:

$$\sigma(\pi(c))\ =\ \varphi(\sigma(\pi(b^2)))\ =\ \{0\},$$

we see that $c \in \ker\pi$, where $\sigma(x)$ denotes the spectrum of x. However, $\sigma(\rho(c)) = \varphi(\sigma(\rho(b^2))) \neq \{0\}$, which gives the contradiction that $c \notin \ker\rho$.

(b) Suppose that π and ρ are two $*$-representations of B such that $\ker\pi \subseteq \ker\rho$. It is clear that for any $i \in I$, one has

$$\ker\ (\pi\,|_{B_i}) \subseteq \ker\ (\rho\,|_{B_i})$$

(where $\pi\,|\,_{B_i}$ is the non-degenerated part of the restriction of π on B_i and so is $\rho\,|\,_{B_i}$). Therefore, by Proposition 1.3, $\|\rho(b)\| \leq \|\pi(b)\|$ $(b \in B_i)$. Now, the result follows from the density of $\bigcup_{i\in I} B_i$ as well as Proposition 1.3. □

Remark 1.5. (a) Corollary 1.4(a) is the argument in [2, Satz 2]. Note that we do not assume that the spectrum of b^2 is in $\mathbb{R}_+$.

(b) Recall that $b \in B$ is said to have *slow growth* if there exists $k \in \mathbb{N}$ such that $\|e^{itb}\| = O(|t|^k)$ for $t \in \mathbb{R}$ (see [1]).

It was proved in [2] that $L^1(G)$ is a $*$-regular Banach $*$-algebra if G is a polynomial growth group. The main step in their proof is the observation that if G is of polynomial growth, then by the argument of a main result in [3], any self-adjoint element in $K(G)$ (the space of continuous functions with compact supports) has slow growth. It is natural to ask if a similar thing holds for a C^*-dynamical system (A, G, α). In this paper, we will consider two particular cases: the case when A is finite dimensional and the case when G is discrete.

Notation 1.6. Throughout this paper, (A, G, α) is a C^*-dynamical system. For any $f, g \in K(G; A)$ (continuous maps from G to A with compact supports) and $s \in G$, we define

$$(f \star g)(s) := \int_G \alpha_t(f(st))g(t^{-1})\, dt \quad \text{and} \quad f^{\#}(s) := \Delta(s^{-1})\alpha_{s^{-1}}(f(s^{-1})^*)$$

(see [5, 2.3]).

If Θ is a map from $K(G; A)$ to itself defined by $\Theta(f)(t) = \alpha_t^{-1}(f(t))$, then we have $\Theta(f*g) = \Theta(f)\star\Theta(g)$ and $\Theta(f^*) = \Theta(f)^{\#}$ (where $f*g(s) := \int_G f(r)\alpha_r(g(r^{-1}s))\, dr$ and $f^*(t) := \Delta(t)^{-1}\alpha_t(f(t^{-1})^*)$. Moreover, any covariant representation (π, u) of (A, G, α), i.e.

$$\pi(\alpha_r(a)) \;=\; u_r\pi(a)u_{r^{-1}},$$

induces a $*$-representation $u \star \pi$ of $(K(G, A), \star, ^{\#})$ which is defined by

$$u \star \pi(f)\xi \;=\; \int_G u_t\pi(f(t))\xi\, dt.$$

2. The finite dimensional C^*-algebra case

Throughout this section, we assume that in the C^*-dynamical system (A, G, α), A is a finite dimensional C^*-algebra with the C^*-norm $\|\cdot\|_A$. Suppose that $A = \bigoplus_{k=1}^n M_{m_k}$. Then $\mathrm{Tr} := \frac{1}{n}\sum_{k=1}^n \mathrm{Tr}_{m_k}$ is a normalised trace on A (where Tr_{m_k} is the normalised trace on the component M_{m_k}). Since any automorphism of A is the composition of an inner automorphism with a swapping of components of A that have the same dimensions (i.e. m_k), Tr is α-invariant (i.e. $\mathrm{Tr}(\alpha_s(a)) = \mathrm{Tr}(a)$ for any $a \in A$ and $s \in G$).

Remark 2.1. (a) Suppose that (H_T, ρ) is the GNS representation corresponding to Tr. Let $\mathcal{H}$ be the Hilbert space $L^2(G) \otimes H_T$. If we regard $A \subseteq H_T$ and $K(G; A) \subseteq L^2(G) \otimes H_T$, then for any $f, g \in K(G; A)$, we have

$$(f, g)_{\mathcal{H}} \;=\; \int_G \mathrm{Tr}(f(t)^*g(t))\, dt$$

(note that we use the convention that the inner product is anti-linear in the first variable).

(b) Let f be a measurable map from G to A (i.e., there exists a sequence of measurable simple maps that converges to f almost everywhere). As usual, we define, for $1 \leq p < \infty$,

$$\|f\|_p := \left(\int_G \|f(t)\|_A^p \ dt \right)^{1/p}$$

and

$$\|f\|_\infty := \inf \left\{ \sup_{t\in\Delta} \|f(t)\|_A : \Delta \subseteq G; \mu_G(G \setminus \Delta) = 0 \right\}$$

where μ_G is the Haar measure on G. Let $L^p(G;A) = \{f : G \to A \mid f$ is measurable and $\|f\|_p < \infty\}$ (strictly speaking, we identify two such maps if they coincide almost everywhere). It is well known that $(L^1(G;A), \star, ^\#)$ is a Banach $*$-algebra under this norm. Moreover, since A is finite dimensional, $\|\cdot\|_2$ is equivalent to the norm $\|\cdot\|_{\mathcal{H}}$ on $K(G;A)$ and so, $L^2(G;A) \cong \mathcal{H}$ (as Banach spaces).

(c) Consider the Hilbert A-module $L^2(G) \otimes A$ with the A-inner product

$$\langle \phi \otimes a, \psi \otimes b \rangle_A := \left(\int_G \phi(t)^* \psi(t) \ dt \right) a^* b$$

($\phi, \psi \in L^2(G)$; $a, b \in A$). Then the canonical $*$-homomorphism $\mu : L^1(G;A) \to \mathcal{L}_A(L^2(G) \otimes A)$ induces an injective $*$-representation $T : L^1(G,A) \to \mathcal{L}(\mathcal{H})$ (note that $\mathcal{H} = (L^2(G) \otimes A) \otimes_\rho H_T$).

Lemma 2.2. (a) *If $f \in K(G;A)$, then $\sum_{k=1}^\infty \frac{(if)^k}{k!}$ converges to $u(f)$ in $L^1(G;A) \cap C_0(G;A)$ (and so we can regard $u(f)$ as an element in $L^2(G;A) \cong \mathcal{H}$).*

(b) *Suppose that G is unimodular. If $g, h \in L^1(G;A) \cap L^\infty(G;A)$ such that $T(g)^*T(g) \leq T(h)^*T(h)$, then $\|g\|_{\mathcal{H}} \leq \|h\|_{\mathcal{H}}$.*

Proof. (a) For any $k, l \in K(G;A)$, we have $\|k \star l\|_\infty \leq \|k\|_1 \|l\|_\infty$. Therefore,

$$\sum_{n=1}^\infty \frac{\|(if)^n\|_\infty}{n!} \ \leq \ \sum_{n=0}^\infty \frac{\|f\|_1^n}{(n+1)!} \|f\|_\infty \ \leq \ e^{\|f\|_1} \|f\|_\infty \ < \ \infty$$

and $u(f) \in C_0(G;A)$.

(b) By the assumption, for all $\xi \in K(G;A) \subseteq \mathcal{H}$,

$$\|g \star \xi\|_{\mathcal{H}}^2 \ = \ (T(g)^*T(g)\xi, \xi)_{\mathcal{H}} \ \leq \ (T(h)^*T(h)\xi, \xi)_{\mathcal{H}} \ = \ \|h \star \xi\|_{\mathcal{H}}^2. \tag{2.1}$$

If $(f_j)_{j\in I}$ is a net in $L^1(G;A) \cap L^\infty(G;A)$ such that $\|f_j\|_1 \to 0$ and there exists $\kappa \in \mathbb{R}_+$ with $\|f_j\|_\infty < \kappa$ (for all $j \in I$), then

$$\|f_j\|_{\mathcal{H}}^2 \ = \ \int_G \mathrm{Tr}[f_j(t)^* f_j(t)] \ dt \ \leq \ \kappa \ \|\mathrm{Tr}\| \int_G \|f_j(t)\|_A \ dt. \tag{2.2}$$

Now suppose that $(\xi_i) \subseteq K(G;A)$ is a contractive approximate identity for $L^1(G;A)$. Notice that as G is unimodular, $\|k \star l\|_\infty \leq \|k\|_\infty \|l\|_1$ for any $k, l \in K(G;A)$. Thus $\|g \star \xi_i - g\|_\infty \leq 2\|g\|_\infty$ and $\|g \star \xi_i - g\|_1 \to 0$. Therefore, inequality (2.2) implies that $\|g \star \xi_i - g\|_{\mathcal{H}} \to 0$ and the same is true for h. Now, the required inequality follows from (2.1). □

Proposition 2.3. *Suppose that G is of polynomial growth and $f \in K(G;A)$ with $f^\# = f$. Then f has slow growth.*

Proof. By Lemma 2.2, [3, Lemme 4] and Remark 2.1 (b), we see that $\|u(f)\|_2 \leq C_0 \|f\|_2$ for some constant $C_0 > 0$. Now, the same argument as that in [3, Lemme 6] will imply the result. □

Using the above proposition and the argument in [3, Lemme 7], the "smooth functional calculus" can be defined for any $f \in K(G;A)_{sa}$ in such a way that the hypothesis of Corollary 1.4(a) holds. Since $K(G;A)_{sa}$ is dense in $L^1(G;A)_{sa}$, we see that $L^1(G;A)$ is $*$-regular. This gives the following generalisations of [2, Satz 2]. Note that part (c) is also a partial generalisation of [6, Remark 1] (i.e. $L^1(G;A)$ is $*$-regular if G is abelian).

Theorem 2.4. *Suppose that G is a polynomial growth group, A is a finite dimensional C^*-algebra and α is an action of G on A. Let $f \in K(G;A)$ with $f = f^\#$ and let φ be a smooth and integrable complex function on $\mathbb{R}$.*

(a) *$\varphi\{f\} := \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(r)e^{irf}\, dr$ exists in the unitalisation, $L^1(G;A)^\sim$, of $L^1(G;A)$ (where $\hat{\varphi}$ is the Fourier transform of φ) and $\varphi\{f\} \in L^1(G;A)$ if $\varphi(0) = 0$.*

(b) *For any covariant representation (ν, v) of (A, G, α), we have $v \star \nu(\varphi\{f\}) = \varphi(v \star \nu(f))$.*

(c) *$L^1(G;A)$ is $*$-regular.*

3. THE DISCRETE GROUP CASE

In this section, we will consider the case when G is a discrete group (but A is a general C^*-algebra). The absence of a bounded trace that gives an equivalent norm on A makes the situation much more complicated.

Let us start with the easy case when G is a "locally finite group". We recall the well-known fact that if G is finite, then $K(G;A) = l^1(G;A) = A \times_\alpha G$ (note that if $\{f_n\}$ is a sequence in $K(G;A)$ converging to an element in $A \times_\alpha G$, then $\{f_n(t)\}$ is Cauchy for any $t \in G$ and so $\{f_n\}$ converges in $K(G;A)$).

Proposition 3.1. *Let G be the inductive limit of a system of finite groups $\{G_i\}_{i\in I}$ and let α be an action of G on a C^*-algebra A. Then for any $f \in K(G;A)$ with $f = f^\#$ and any smooth and integrable complex function φ on $\mathbb{R}$, $\varphi\{f\} := \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(r)e^{irf}\, dr$ exists in the $K(G;A)^\sim$ and $l^1(G;A)$ is $*$-regular.*

Proof. Note that $K(G_i;A) = A \times_{\alpha_i} G_i$ is $*$-regular (α_i being the restriction of α on G_i). Moreover, it is easy to see that $\bigcup_{i\in I} l^1(G_i)$ is dense in $l^1(G)$. Since $l^1(G;A) = l^1(G) \otimes^\pi A$ (where $\otimes^\pi$ is the projective tensor product), it is easy to see that $\bigcup_{i\in I} l^1(G_i;A)$ is dense in $l^1(G;A)$. Now this result follows from Corollary 1.4(b). The existence of the functional calculus follows from the fact that for any $f \in K(G;A)$ there exists $i \in I$ such that $f \in K(G_i;A) = A \times_{\alpha_i} G_i$. □

Next, we consider the case when G is discrete and has polynomial growth. As in the previous section, we want to use a similar argument to that of [3]. In order to do this, we need to replace $\{\|\cdot\|_p\}_{p\in[1,\infty]}$ by another series of norms $\{n_{\pi,p}\}_{p\in[1,\infty]}$ such that $n_{\pi,2}$ is the one given by the Hilbert C^*-module $l^2(G) \otimes A$.

Remark 3.2. Suppose that π is a representation of A on H. For any $f : G \to A$ and any $1 \le p < \infty$, we define

$$n_{\pi,p}(f) = \sup\left\{\left(\sum_{t\in G} \|\pi(f(t))\xi\|^p\right)^{1/p} : \xi \in H \text{ and } \|\xi\| \le 1\right\}.$$

(a) It is clear that $n_{\pi,p}$ is a semi-norm on $K(G;A)$ and if π is faithful, then $n_{\pi,p}$ is a norm on $K(G;A)$. Moreover, $n_{\pi,p}(f) \le \|f\|_p$.

(b) Suppose that $f \in K(G;A)$. Then

$$\sum_{t\in G} \|\pi(f(t))\xi\|^2 = \langle \pi(\sum_{t\in G} f(t)^* f(t))\xi, \xi\rangle.$$

Hence if π is faithful, then $n^2_{\pi,2}(f) = \|\sum_{t\in G} f(t)^* f(t)\|$ and $n_{\pi,2}$ is the norm on $K(G;A)$ induced from the Hilbert A-module $l^2(G)\otimes A$.

(c) Suppose that $\xi \in H$ and $f, g \in K(G;A)$. Then

$$\begin{aligned} \sum_{s\in G} \|\pi(f\star g(s))\xi\| &\leq \sum_{v\in G}\sum_{u\in G} \|\pi(\alpha_{v^{-1}}(f(u))g(v))\xi\| \\ &\leq \sum_{v\in G}\sum_{u\in G} \|f(u)\|\|\pi(g(v))\xi\| \leq \|f\|_1\ n_{\pi,1}(g)\ \|\xi\| \end{aligned} \tag{3.1}$$

and so, $n_{\pi,1}(f\star g) \leq \|f\|_1\ n_{\pi,1}(g)$.

(d) In general, it may not be true that $n_{\pi,p}(f) = n_{\pi,p}(f^{\#})$.

Proposition 3.3. *Suppose that A is finite dimensional and π is any faithful representation of A. Then $n_{\pi,1} \sim \|\cdot\|_1$.*

Proof. Let $A = \bigoplus_{k=1}^N M_{m_k}(\mathbb{C})$. For any $a = ((a^{(1)}_{ij}), ..., (a^{(N)}_{ij})) \in A$, we define

$$\|a\|_s = \sum_{k=1}^{N} \sum_{i,j=1}^{m_k} |a^{(k)}_{ij}|.$$

Since $\|\cdot\|_s$ is equivalent to the C^*-norm $\|\cdot\|_A$, on A, there exists a $\kappa > 0$ such that $\|a\|_A \leq \kappa\|a\|_s$ $(a\in A)$. Therefore, if $f(t) = ((f(t)^{(1)}_{ij}), ..., (f(t)^{(N)}_{ij})) \in A$, then

$$\|f\|_1 \ \leq\ \kappa \sum_{t\in G}\sum_{k=1}^{N}\sum_{i,j=1}^{m_k} |f(t)^{(k)}_{ij}|.$$

Let $\pi^{(k)}$ be the representation defined by $\pi^{(k)}(a^{(1)}, ..., a^{(N)}) = \pi(0, ..., 0, a^{(k)}, 0, ..., 0)$. There exists $\xi_i \in H$ such that $\|\xi_i\| \leq 1$ and $|f(t)^{(k)}_{ij}| \leq |\langle \pi^{(k)}(f(t))\xi_j, \xi_i\rangle|$ $(t\in G)$. Thus, for fixed i and j,

$$\sum_{t\in G} |f(t)^{(k)}_{ij}| \ \leq\ \sum_{t\in G} \|\pi^{(k)}(f(t))\xi_j\| \ \leq\ n_{\pi,1}(f).$$

Consequently, $n_{\pi,1}(f) \leq \|f\|_1 \leq \kappa \left(\sum_{k=1}^N m_k^2\right) n_{\pi,1}(f)$. □

Example 3.4. Suppose that $G = \mathbb{Z}$ and $A = \mathcal{K}(l^2(\mathbb{Z}))$. Let $\{e_k\}_{k\in\mathbb{Z}}$ be the canonical basis for $l^2(\mathbb{Z})$ and let $p^{(k)} \in A$ be defined by $p^{(k)}(e_l) = \delta_{k,l}\ e_k$. Fix $m \in \mathbb{N}$. Define $f_m \in K(G;A)$ by

$$f_m(k) = \begin{cases} p^{(k)} & \text{if } |k| \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $\|f_m\|_1 = 2m+1$. However, if π is the canonical representation of A on $l^2(\mathbb{Z})$, then

$$\begin{aligned} n_{\pi,1}(f_m) &= \sup\left\{ \sum_{k=-m}^{m} \|\pi(f_m(k))\xi\| : \xi \in l^2(\mathbb{Z}); \|\xi\| \leq 1 \right\} \\ &= \sup\left\{ \sum_{k=-m}^{m} |\xi_k| : (\xi_l) \in l^2(\mathbb{Z}); \sum_{k\in\mathbb{Z}} |\xi_l|^2 \leq 1 \right\} \;=\; \sqrt{2m+1}. \end{aligned}$$

Therefore, $n_{\pi,1}$ is not equivalent to $\|\cdot\|_1$.

Proposition 3.5. *Suppose that (π, u, H) is a covariant representation of (A, G, α). Then $\|f\|_{\pi,1} = \max\{n_{\pi,1}(f),\ n_{\pi,1}(f^{\#})\}$ is an involutive algebra semi-norm on $(K(G,A), \star, ^{\#})$. If $l^1_\pi(G;A)$ is the completion of the quotient of $K(G;A)$ under $\|\cdot\|_{\pi,1}$, then there exists a contractive Banach $*$-algebra homomorphism ϵ_π from $l^1(G;A)$ to $l^1_\pi(G;A)$. Moreover, there exists a contractive representation $\mu_{\pi,u}$ of $l^1_\pi(G;A)$ on H such that $\mu_{\pi,u} \circ \epsilon_\pi = u \star \pi$.*

Proof. For any $g \in K(G;A)$ and $\xi \in H$, we have

$$\begin{aligned} \sum_{s\in G} \|\pi(f \star g(s))\xi\| &\leq \sum_{r\in G}\sum_{t\in G} \|\pi(\alpha_{r^{-1}}(f(t)))\pi(g(r))\xi\| \\ &= \sum_{r\in G}\sum_{t\in G} \|\pi(f(t))u_r\pi(g(r))\xi\| \\ &\leq \sum_{r\in G} n_{\pi,1}(f)\|u_r\pi(g(r))\xi\| \;=\; n_{\pi,1}(f)\; n_{\pi,1}(g)\; \|\xi\|. \end{aligned}$$

Thus, $n_{\pi,1}$ is an algebra semi-norm on $(K(G;A), \star)$ and $\|\cdot\|_{\pi,1}$ is an involutive algebra semi-norm on $(K(G;A), \star, ^{\#})$. The second statement of the proposition follows from Remark 3.2(a). Finally, as $\|u \star \pi(f)\| \leq n_{\pi,1}(f)$, the third statement is easy to obtain. □

Remark 3.6. (a) We can regard $l^1_\pi(G;A)$ as the completion of the quotient of $l^1(G;A)$ with respect to $\|\cdot\|_{\pi,1}$. In this case, the completion of the quotient of $l^1(G;A)^{\tilde{}}$ with respect to $\|\cdot\|_{\pi,1}$ coincides with $l^1_\pi(G;A)^{\tilde{}}$.

(b) Suppose that π is faithful. Then any $x \in l^1_\pi(G;A)$ defines a map $f : G \to A$ such that

$$\sup\left\{ \sum_{t\in G} \|\pi(f(t))\xi\| : \xi \in H \text{ with } \|\xi\| \leq 1 \right\} < \infty$$

(because any sequence in $K(G;A)$ converging to x will converge pointwisely and the pointwise limits of any two such sequences are the same). Thus, $\epsilon_\pi : l^1(G;A) \to l^1_\pi(G;A)$ (Proposition 3.5) is injective. Moreover, ϵ_π extends to an injection from $l^1(G;A)^{\tilde{}}$ to $l^1_\pi(G;A)^{\tilde{}}$.

(c) Let $G = \mathbb{Z}$ and $A = \mathcal{K}(l^2)$, and let α be the trivial action. If π is the canonical representation of A and u is the trivial representation of G respectively on l^2, then (π, u, l^2) is a covariant representation. Suppose that ϵ_π is surjective. Then the Open Mapping theorem will imply that $\|\cdot\|_{\pi,1}$ is equivalent to $\|\cdot\|_1$, which contradicts the conclusion of Example 3.4 (note that $\|\cdot\|_{\pi,1} = n_{\pi,1}$ in this case). Therefore, ϵ_π is in general not surjective.

Lemma 3.7. *Suppose that G is a polynomial growth discrete group and (π, H) is a representation of A. If $f \in K(G;A)$ and $f = f^{\#}$, then $\epsilon_\pi(f)$ has slow growth.*

Proof. Let $u(f)$ be as in Lemma 2.2(a) and let $\mu : l^1(G;A) \to A \times_\alpha G$ be the canonical $*$-homomorphism. Since $\mu(u(f)) = u(\mu(f))$, we have

$$\mu(u(f)^{\#} \star u(f)) \leq \mu(f \star f) \tag{3.2}$$

(by an analogue of [3, Lemme 4] for C^*-algebras). If E is the canonical conditional expectation from $A \times_\alpha G$ to A (see e.g. [4]), then $E(\mu(g)) = g(e)$ for any $g \in l^1(G;A)$. Therefore, using (3.2),

$$\begin{aligned}\sum_{t\in G} u(f)(t)^* u(f)(t) &= (u(f)^{\#} \star u(f))(e) \\ &= E(\mu(u(f)^{\#} \star u(f))) \leq E(\mu(f \star f)) = \sum_{t\in G} f(t)^* f(t).\end{aligned}$$

Consequently, for any $\xi \in H$ with $\|\xi\| \leq 1$, we have

$$\begin{aligned}\sum_{t\in G} \|\pi(u(f)(t))\xi\|^2 &= \left\langle \pi\left(\sum_{t\in G} u(f)(t)^* u(f)(t)\right)\xi, \xi\right\rangle \leq \left\|\sum_{t\in G} u(f)(t)^* u(f)(t)\right\| \\ &\leq \left\|\sum_{t\in G} f(t)^* f(t)\right\| \leq n_{\pi,2}(f)^2\end{aligned}$$

(see Remark 3.2(b)). Thus, $n_{\pi,2}(u(f)) \leq n_{\pi,2}(f)$. Let S be the support of f. As G has polynomial growth, there exists $N \in \mathbb{N}$ such that $|S^m| = O(m^N)$. Now, a similar argument to that of [3, Lemme 6] will give the result, but since we are in a slightly different setting, we will sketch the proof here for clarity. If $m \in \mathbb{N}$ and $\xi \in H$ with $\|\xi\| \leq 1$, then

$$\begin{aligned}\sum_{t\in S^{m^2-1}} \|\pi(u(mf)(t))\xi\| &\leq \left(\sum_{t\in S^{m^2-1}} \|\pi(u(mf)(t))\xi\|^2\right)^{1/2} |S^{m^2-1}|^{1/2} \\ &\leq n_{\pi,2}(u(mf))|S^{m^2-1}|^{1/2} \leq m\, n_{\pi,2}(f)|S^{m^2-1}|^{1/2} \\ &\leq C_1\, m^{N+1}\end{aligned} \tag{3.3}$$

(C_1 is independent of ξ and m). On the other hand, the same argument as that for [3, Lemme 6] shows that

$$\begin{aligned}\sum_{t\in G\setminus S^{m^2-1}} \|\pi(u(mf)(t))\xi\| &\leq \sum_{t\in G\setminus S^{m^2-1}} \left\|\sum_{k=m^2}^{\infty} \frac{(imf)^k}{k!}(t)\right\| \\ &\leq C_2\, m^{-m^2-1}\, e^{m^2+m}\end{aligned} \tag{3.4}$$

(C_2 is independent of ξ and m). Equations (3.3) and (3.4) imply that $n_{\pi,1}(u(mf)) = O(m^{N+1})$. Finally, let $[\lambda]$ be the integral part of λ. Then we have

$$n_{\pi,1}(e^{i\lambda f}) = n_{\pi,1}(e^{i(\lambda-[\lambda])f} e^{i[\lambda]f}) \leq e^{\|f\|_1}(1 + n_{\pi,1}(u(i[\lambda]f))) = O(|\lambda|^{N+1}).$$

□

Again, using the argument of [3, Lemme 7] and Lemma 1.4(a), we have the following theorem.

Theorem 3.8. *Suppose that* G *is a discrete polynomial growth group. Let* (A, G, α) *be a* C^**-dynamical system and let* (π, u) *be a covariant representation of* (A, G, α)*. Let* $f \in K(G; A)$ *with* $f = f^{\#}$ *and let* φ *be a smooth and integrable complex function on* $\mathbb{R}$*.*

(a) *The Bochner integral* $\varphi_\pi\{f\} := \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\lambda)\epsilon_\pi(e^{i\lambda f})\, d\lambda$ *exists in* $l^1_\pi(G; A)^\sim$ *and* $\varphi_\pi\{f\} \in l^1_\pi(G; A)$ *if* $\varphi(0) = 0$*.*

(b) *Suppose that* $u \star \pi$ *extends to a faithful representation of* $A \times_\alpha G$*. Then one can regard* $\mu_{\pi,u}$ *as a* $*$*-homomorphism from* $l^1_\pi(G; A)^\sim$ *to* $A \times_\alpha G$*. Under this identification, for any covariant representation* (ν, v) *of* (A, G, α)*, we have* $(v \star \nu)(\mu_{\pi,u}(\varphi_\pi\{f\})) = \varphi((v \star \nu)(f))$*.*

(c) *Suppose that* $u \star \pi$ *is faithful on* $A \times_\alpha G$*. Then* $l^1_\pi(G; A)$ *is* $*$*-regular.*

Remark 3.9. (a) Suppose that $u \star \pi$ extends to a faithful representation of $A \times_\alpha G$. By Theorem 3.8(b), any non-degenerate $*$-representation of $A \times_\alpha G$ induces (through the map $\mu_{\pi,u}$) a non-degenerate $*$-representation of $l^1_\pi(G; A)$. On the other hand, any non-degenerate $*$-representation of $l^1_\pi(G; A)$ induces (through the map ϵ_π in Proposition 3.5) a non-degenerate $*$-representation of $l^1(G; A)$. Since $\mu_{\pi,u} \circ \epsilon_\pi$ is the canonical embedding of $l^1(G; A)$ in $A \times_\alpha G$, we see that the enveloping C^*-algebra of $l^1_\pi(G; A)$ is again $A \times_\alpha G$.

(b) By Proposition 3.3, Theorem 3.8 can be regarded as a partial generalization of Theorem 2.4.

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