MAPS INTO COMPLEX SPACE

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(Communicated by Jozef Dodziuk)

ABSTRACT. If the dimension of $M$ is denoted by $2k - 1$ or $2k$, then a generic map $F: M \to C^k$ satisfies $dF_1 \wedge \ldots \wedge dF_k \neq 0$, while in certain cases there is no map $F: M \to C^{k+1}$ that satisfies $dF_1 \wedge \ldots \wedge dF_{k+1} \neq 0$.

1. Statement of results

Let $F: M \to R^2$ be any map. The components of $F$ may be approximated by Morse functions. From this, we see that the rank of the Jacobian of a generic map of $M \to R^2$ is everywhere greater than zero. This can be reformulated as: A generic map $F: M \to C^1$ has, at all points, $dF \neq 0$. This suggests the question: For a given value of $n$, what is the largest value of $r$ for which every manifold of dimension $n$ has a map $F = (F_1, \ldots, F_r)$ into $C^r$ with $dF_1 \wedge \ldots \wedge dF_r$ never zero?

Given $n$, set $r = \left\lceil \frac{n}{2} \right\rceil$, the smallest integer greater than or equal to $\frac{n}{2}$.

Theorem 1.1. For every manifold $M^n$ of dimension $n$ there is a generic set of maps $F: M^n \to C^r$ with $dF_1 \wedge \ldots \wedge dF_r$ never equal to zero.

By a generic set of functions or maps on a compact manifold we mean a set that is open in the $C^1$ topology and dense in the $C^\infty$ topology; on a non-compact manifold we mean the countable intersection of such sets. Note that the theorem states that $M^n$ can be mapped into $R^k$, $k \leq n$ for $n$ even and $k \leq n + 1$ for $n$ odd, in a special way.

Corollary 1. Every manifold $M^n$ of dimension $n$ admits an involutive sub-bundle of rank $k$ for each $k$, $k \geq \left\lceil \frac{n}{2} \right\rceil$.

A bundle $V \subset CTM$ is involutive if the Lie bracket condition $[V, V] \subset V$ holds for all local sections. This is an interesting restriction, motivated by partial differential equations and generalizing the Frobenius condition for sub-bundles of $TM$. Here $\left\lceil \frac{n}{2} \right\rceil$ is the largest integer less than or equal to $\frac{n}{2}$ and so $n = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor$. The corollary follows from the theorem by taking the annihilator of $\{dF_1, \ldots, dF_r\}$.

The value for $r$ is sharp (for $n = 4k$) if we insist that the same $r$ works for all manifolds of a fixed dimension. The basic idea is that every complex bundle of rank $N$ over $M^n$, $N \geq \frac{n}{2}$, admits $N - \left\lfloor \frac{n}{2} \right\rfloor$ independent global sections. The rank of $CT^*M^n$ is $n$ and $dF_1, \ldots, dF_r$ provide $r$ independent global sections. Thus for $r > n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$, we have a restriction on the bundle. Here is one simple way to formulate this restriction. Note that we use only the existence of the global
section and ignore the more refined information that each section is given by a closed one-form.

**Theorem 1.2.** Let \( F : M^{4k} \to C^{2k+1} \) satisfy \( dF_1 \land \ldots \land dF_{2k+1} \) never equal to zero. Then the top Pontryagin class of \( M \) is zero.

Since there exist manifolds \( M^{4k} \) with \( P_k \neq 0 \), \( r = 2k \) is the largest \( r \) that works for all manifolds of dimension \( 4k \).

**Examples.** There exists a map \( F : CP^2 \to C^2 \) with \( dF_1 \land dF_2 \) never equal to zero; any map \( F : CP^2 \to C^3 \) has \( dF_1 \land dF_2 \land dF_3 \) somewhere equal to zero. Whether there exists a map of \( CP^1 \to C^2 \) with \( dF_1 \land dF_2 \neq 0 \) is left open by these theorems. Note that the existence of such a map implies the (true) statement that \( CTS^2 \) is trivial. There does exist such a map into \( C^3 \), and it can be obtained by projecting \( S^2 \) into any plane. Also note that the case \( n = 2 \) is related to symplectic structures. Namely, identify \( R^4 \) with \( C^2 \) by using \( z_1 = x_1 - i y_1 \) and \( z_2 = x_2 + i y_2 \).

Then \( dz_1 \land dz_2 = \omega_1 + \omega_2 \) for symplectic forms \( \omega_1 \) and \( \omega_2 \), positive with respect to the orientation given by \( (x_1, x_2, y_1, y_2) \). No symplectic structure on \( R^4 \) has a compact symplectic submanifold. So if \( F : S^2 \to C^2 \), then \( F^* \omega_1 \) and \( F^* \omega_2 \) must have zeroes, and we are asking if \( F \) can be chosen such that they have no common zeroes.

2. Proof of Theorem 1.1

We use notation and results from \cite{1}, Chapter 2. Let \( X^{(1)} \) be the space of one jets of maps \( F : M^n \to C^r \) written locally as \( \{ (p, F(p), dF_1(p), \ldots, dF_r(p)) \} \) and let \( S \) be the subset given by \( \{ (p, c, \theta) : \theta_1 \land \ldots \land \theta_r = 0 \} \).

**Lemma 2.1.** \( S \) is a stratified subset of \( X^{(1)} \) of codimension \( 2(n+1-r) \).

To prove this, we need to show

(1) \[ S = \bigcup_{i=0}^{N} S_i, \]

where each \( S_i \) is a locally closed sub-manifold of \( X^{(1)} \) satisfying

\[ \tilde{S}_j = \bigcup_{i=j}^{N} S_i, \]

and

(2) \( \text{codim} S_0 = 2(n+1-r) \).

**Proof.** Let \( S_i \) be the subset of \( S \) defined by

\[ \{ (p, c, \theta) : \theta_1 \land \ldots \land \theta_{r-i} = 0 \text{ and } \theta_{r-i+1} \land \ldots \land \theta_{r-1} \neq 0 \}. \]

So \( N = r-1 \) and \( S_N = \{ p, c, \theta : \theta_1 = 0 \} \). Then the first condition is obvious. For the second, we count the equations that define \( S_0 \) near one of its points, \( (p, c, \theta^0) \). We have that \( (p, c, \theta) \in S_0 \) provided \( \theta_0 \in \text{span}\{\theta_1, \ldots, \theta_{r-1}\} \). Expressing \( \theta_r \) as a linear combination with complex coefficients of basis elements \( \{\theta_1, \ldots, \theta_{r-1}, e_1, \ldots, e_{n-r+1}\} \), we see that there are \( n-r+1 \) independent equations. This establishes the second condition.
We now require $2(n + 1 - r) > n$. Thus the codimension of $S$ in $X^{(1)}$ is greater than the dimension of $M$. It follows from a basic result of differential topology that any map of $M$ into $X^{(1)}$ may be perturbed so as to not intersect $S$. The (simplest case of) the Thom Transversality Theorem is more precise. Any map of $M \to C^r$ may be perturbed to obtain a map of $M \to C^r$ whose one jet does not intersect $S$. Further, maps of this latter kind are generic. Finally, note that $2(n + 1 - r) > n$ is equivalent to $r \leq \left\lfloor \frac{n}{2} \right\rfloor$.

3. Proof of Theorem 1.2

Consider a map $F : M^n \to C^r$ with $dF_1 \wedge \ldots \wedge dF_r$ never equal to zero. Let $V \subset CTM$ be the bundle whose fiber at $p$ is given by

$$V = \{ v : dF_j(v) = 0 \text{ at } p, \ j = 1, \ldots, r \}$$

and let $\Omega \subset CT^*M$ be the span of $\{dF_1, \ldots, dF_r\}$. Use any hermitian metric to get a decomposition

$$CTM = V \oplus Q.$$

**Lemma 3.1.** $\Omega \simeq Q^*$.  

**Proof.** Recall that $Q^*$ is the set of linear functionals on the fibers of $Q$. Restricting an element $\theta \in \Omega$ to act on $Q$ produces an element $l_\theta$ of $Q^*$. The map $\theta \to l_\theta$ is injective and, since $\Omega$ and $Q^*$ have the same dimension, it is also surjective.

Each $dF_j$ is a global, non-zero section of $\Omega$. Thus $\Omega$, $Q^*$, and $Q$ are all trivial bundles. Thus, for the Chern class of $CTM$ we have, as in [2], page 164,

$$c(CTM) = c(V \oplus Q) = c(V) \wedge c(Q) = c(V).$$

Now let $P_k(TM) \in H^{4k}(M, Z)$ be the top Pontryagin class of $M$. We have from [2], page 174,

$$P_k(TM) = (-1)^k c_{2k}(CTM) = (-1)^k c_{2k}(V).$$

In the case of Theorem 1.2 $\text{rank} V = 4k - (2k + 1) = 2k - 1 < 2k$ and so $P_k(TM) = 0$.

**References**


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