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MAPS INTO COMPLEX SPACE

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Abstract. If the dimension of M is denoted by $2k-1$ or $2k$, then a generic map $F : M \to C^k$ satisfies $dF_1 \wedge \ldots \wedge dF_k \neq 0$, while in certain cases there is no map $F : M \to C^{k+1}$ that satisfies $dF_1 \wedge \ldots \wedge dF_{k+1} \neq 0$.

1. Statement of results

Let $F : M \to R^2$ be any map. The components of F may be approximated by Morse functions. From this, we see that the rank of the Jacobian of a generic map of $M \to R^2$ is everywhere greater than zero. This can be reformulated as: A generic map $F : M^n \to C^1$ has, at all points, $dF \neq 0$. This suggests the question: For a given value of n, what is the largest value of r for which every manifold of dimension n has a map $F = (F_1, \ldots F_r)$ into C^r with $dF_1 \wedge \ldots \wedge dF_r$ never zero?

Given n, set $r = [[\frac{n}{2}]]$, the smallest integer greater than or equal to $\frac{n}{2}$.

Theorem 1.1. *For every manifold M^n of dimension n there is a generic set of maps $F : M^n \to C^r$ with $dF_1 \wedge \ldots \wedge dF_r$ never equal to zero.*

By a generic set of functions or maps on a compact manifold we mean a set that is open in the C^1 topology and dense in the C^∞ topology; on a non-compact manifold we mean the countable intersection of such sets. Note that the theorem states that M^n can be mapped into R^k, $k \leq n$ for n even and $k \leq n+1$ for n odd, in a special way.

Corollary 1. *Every manifold M^n of dimension n admits an involutive sub-bundle of rank k for each k, $k \geq [\frac{n}{2}]$.*

A bundle $V \subset CTM$ is involutive if the Lie bracket condition $[V, V] \subset V$ holds for all local sections. This is an interesting restriction, motivated by partial differential equations and generalizing the Frobenius condition for sub-bundles of TM. Here $[\frac{n}{2}]$ is the largest integer less than or equal to $\frac{n}{2}$ and so $n = [\frac{n}{2}] + [[\frac{n}{2}]]$. The corollary follows from the theorem by taking the annihilator of $\{dF_1, \ldots, dF_r\}$.

The value for r is sharp (for $n = 4k$) if we insist that the same r works for all manifolds of a fixed dimension. The basic idea is that every complex bundle of rank N over M^n, $N > \frac{n}{2}$, admits $N - [\frac{n}{2}]$ independent global sections. The rank of CT^*M^n is n and $dF_1, \ldots, dF_r$ provide r independent global sections. Thus for $r > n - [\frac{n}{2}] = [[\frac{n}{2}]]$, we have a restriction on the bundle. Here is one simple way to formulate this restriction. Note that we use only the existence of the global

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section and ignore the more refined information that each section is given by a closed one-form.

Theorem 1.2. *Let* $F : M^{4k} \to C^{2k+1}$ *satisfy* $dF_1 \wedge \ldots \wedge dF_{2k+1}$ *never equal to zero. Then the top Pontryagin class of* M *is zero.*

Since there exist manifolds M^{4k} with $P_k \neq 0$, $r = 2k$ is the largest r that works for all manifolds of dimension $4k$.

Examples. There exists a map $F : CP^2 \to C^2$ with $dF_1 \wedge dF_2$ never equal to zero; any map $F : CP^2 \to C^3$ has $dF_1 \wedge dF_2 \wedge dF_3$ somewhere equal to zero. Whether there exists a map of $CP^1 \to C^2$ with $dF_1 \wedge dF_2 \neq 0$ is left open by these theorems. Note that the existence of such a map implies the (true) statement that CTS^2 is trivial. There does exist such a map into C^1, and it can be obtained by projecting S^2 into any plane. Also note that the case $n = 2$ is related to symplectic structures. Namely, identify R^4 with C^2 by using $z_1 = x_1 - iy_1$ and $z_2 = x_2 + iy_2$. Then $dz_1 \wedge dz_2 = \omega_1 + \omega_2$ for symplectic forms ω_1 and ω_2, positive with respect to the orientation given by (x_1, x_2, y_1, y_2). No symplectic structure on R^4 has a compact symplectic submanifold. So if $F : S^2 \to C^2$, then $F^*\omega_1$ and $F^*\omega_2$ must have zeroes, and we are asking if F can be chosen such that they have no common zeroes.

2. Proof of Theorem 1.1

We use notation and results from [1], Chapter 2. Let $X^{(1)}$ be the space of one jets of maps $F : M^n \to C^r$ written locally as $\{(p, F(p), dF_1(p), \ldots, dF_r(p))\}$ and let S be the subset given by $\{(p, c, \theta) : \theta_1 \wedge \ldots \wedge \theta_r = 0\}$.

Lemma 2.1. *S is a stratified subset of $X^{(1)}$ of codimension* $2(n + 1 - r)$.

To prove this, we need to show

(1)
$$S = \bigcup_{i=0}^{N} S_i,$$
where each S_i is a locally closed sub-manifold of $X^{(1)}$ satisfying
$$\bar{S_j} = \bigcup_{i=j}^{N} S_i,$$
and

(2) $codim S_0 = 2(n + 1 - r)$.

Proof. Let S_i be the subset of S defined by
$$\{(p, c, \theta) : \theta_1 \wedge \ldots \wedge \theta_{r-i} = 0 \text{ and } \theta_1 \wedge \ldots \wedge \theta_{r-i-1} \neq 0\}.$$
So $N = r - 1$ and $S_N = \{p, c, \theta : \theta_1 = 0\}$. Then the first condition is obvious. For the second, we count the equations that define S_0 near one of its points, (p, c, θ^0). We have that $(p, c, \theta) \in S_0$ provided $\theta_r \in span\{\theta_1, \ldots, \theta_{r-1}\}$. Expressing θ_r as a linear combination with complex coefficients of basis elements $\{\theta_1, \ldots, \theta_{r-1}, e_1, \ldots, e_{n-r+1}\}$, we see that there are $n-r+1$ independent equations. This establishes the second condition.

We now require $2(n+1-r) > n$. Thus the codimension of S in $X^{(1)}$ is greater than the dimension of M. It follows from a basic result of differential topology that any map of M into $X^{(1)}$ may be perturbed so as to not intersect S. The (simplest case of) the Thom Transversality Theorem is more precise. Any map of $M \to C^r$ may be perturbed to obtain a map of $M \to C^r$ whose one jet does not intersect S. Further, maps of this latter kind are generic. Finally, note that $2(n+1-r) > n$ is equivalent to $r \leq [[\frac{n}{2}]]$.

3. Proof of Theorem 1.2

Consider a map $F : M^n \to C^r$ with $dF_1 \wedge \ldots \wedge dF_r$ never equal to zero. Let $V \subset CTM$ be the bundle whose fiber at p is given by

$$V = \{v : dF_j(v) = 0 \text{ at } p, \quad j = 1, \ldots, r\}$$

and let $\Omega \subset CT^*M$ be the span of $\{dF_1, \ldots, dF_r\}$. Use any hermitian metric to get a decomposition

$$CTM = V \oplus Q.$$

Lemma 3.1. $\Omega \simeq Q^*$.

Proof. Recall that Q^* is the set of linear functionals on the fibers of Q. Restricting an element $\theta \in \Omega$ to act on Q produces an element l_θ of Q^*. The map $\theta \to l_\theta$ is injective and, since Ω and Q^* have the same dimension, it is also surjective.

Each dF_j is a global, non-zero section of Ω. Thus Ω, Q^*, and Q are all trivial bundles. Thus, for the Chern class of CTM we have, as in [2], page 164,

$$c(CTM) = c(V \oplus Q) = c(V) \wedge c(Q) = c(V).$$

Now let $P_k(TM) \in H^{4k}(M, Z)$ be the top Pontryagin class of M. We have from [2], page 174,

$$P_k(TM) = (-1)^k c_{2k}(CTM) = (-1)^k c_{2k}(V).$$

In the case of Theorem 1.2, $rank V = 4k-(2k+1) = 2k-1 < 2k$ and so $P_k(TM) = 0$.

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