

FROM GAS DYNAMICS TO PRESSURELESS GAS DYNAMICS

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ABSTRACT. This paper is devoted to the convergence of solutions of the compressible Euler equations towards solutions of the pressureless gas dynamics system, when the pressure tends to 0. The goal is to prove accurate uniform bounds for particular solutions of the Euler equations.

1. INTRODUCTION

The purpose of this paper is to prove the convergence of solutions of the compressible Euler equations towards solutions of the pressureless gas dynamics system, when the pressure tends to zero, and to establish rates of convergence. More precisely, we consider Euler equations in the whole space \mathbb{R}^d :

$$(1.1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \varepsilon \nabla p(\rho) = 0, \end{cases}$$

where the pressure law p has the following form:

$$(1.2) \quad p(\rho) = \rho^\gamma, \quad 1 < \gamma \leq 1 + \frac{2}{d}.$$

The parameter ε in (1.1) belongs to $]0, 1]$. We also consider the pressureless gas dynamics system in \mathbb{R}^d :

$$(1.3) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = 0. \end{cases}$$

This system is a simplified model for the dynamics of galaxies, see [6]. It seems rather natural to expect that solutions of (1.3) can be obtained as limits of solutions of (1.1). This question has been positively answered by T. Goudon and S. Junca, see [1], for smooth solutions, and in one space dimension. Here, we are still interested in the convergence of smooth solutions, but in the multidimensional case. In order to simplify the analysis, we shall consider the same initial data for (1.1) and (1.3).

There are few examples of global smooth solutions to the multidimensional compressible Euler equations; some are given in [2, 5]. In this paper, we shall consider some initial data for which M. Grassin has shown in [2] that the corresponding solution is global and enjoys suitable decay properties.

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Let us remark that an initial density of order $O(1)$ for the system (1.1) corresponds to an initial density of order $O(\varepsilon^{1/(\gamma-1)})$ for the usual Euler equations (that is, when $\varepsilon = 1$) that were considered in [2]. Consequently, we can apply the result of [2] and obtain a global smooth solution $(\rho^\varepsilon, \mathbf{u}^\varepsilon)$ of (1.1), provided the initial velocity field satisfies the assumptions of [2]. This is because the existence result of [2] relies on a smallness assumption for the initial density. Moreover, the analysis of [2] yields L^∞ bounds for the solution. In particular, these bounds show the convergence of \mathbf{u}^ε towards the velocity field $\bar{\mathbf{u}}$ solution to (1.3). But the rate of convergence is not satisfactory. We shall show how to improve this rate of convergence. We shall also obtain a rate of convergence of the density ρ^ε towards the density solution to (1.3). To prove these estimates, we shall use a different strategy from the one that was used in [2] (where the L^∞ bounds rely on H^s energy estimates and Sobolev imbeddings).

The paper is organized as follows. In section 2, we introduce a symmetrization of (1.1) and (1.3), that is due to T. Makino, S. Ukai and S. Kawashima [3], and that is convenient for the derivation of energy estimates and uniform bounds (this symmetrization motivates the expression (1.2) for the pressure law). We also describe the initial conditions (ρ_0, \mathbf{u}_0) that we consider throughout this work, and we give some preliminary estimates for the solutions of (1.1) and (1.3). In section 3, we give the proof of our main result.

Throughout the paper, $|\cdot|_\infty$ denotes the norm in $L^\infty(\mathbb{R}^d)$, and $Df(t, x)$ denotes the derivative of a function f with respect to the space variables $x \in \mathbb{R}^d$.

2. STATEMENT OF THE RESULT

Following [3], we define

$$\pi := \frac{2\sqrt{\gamma}}{\gamma-1} \rho^{(\gamma-1)/2}.$$

Then (1.1) reads

$$(2.1) \quad \begin{cases} \partial_t \pi + \mathbf{u} \cdot \nabla \pi + \frac{\gamma-1}{2} \pi \operatorname{div} \mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \varepsilon \frac{\gamma-1}{2} \pi \nabla \pi = 0. \end{cases}$$

More precisely, (1.1) is not exactly equivalent to (2.1) when a vacuum occurs. But multiplying the second equation of (2.1) by ρ , one recovers (1.1). It is thus sufficient to construct smooth solutions to (2.1) to obtain solutions to (1.1) with \mathbf{u} and $\rho^{(\gamma-1)/2}$ smooth.

The above reduction still holds when $\varepsilon = 0$, and we obtain that (1.3) reads

$$(2.2) \quad \begin{cases} \partial_t \pi + \mathbf{u} \cdot \nabla \pi + \frac{\gamma-1}{2} \pi \operatorname{div} \mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = 0. \end{cases}$$

In what follows, we consider both systems (2.1) and (2.2), supplemented with the same initial data (π_0, \mathbf{u}_0) . These initial data should satisfy the following assumptions: we consider an integer $m > 1 + d/2$, and we assume that

- H1)** $\pi_0 \in H^m(\mathbb{R}^d)$,
- H2)** $D\mathbf{u}_0 \in L^\infty(\mathbb{R}^d)$ and $D^2\mathbf{u}_0 \in H^{m-1}(\mathbb{R}^d)$,

H3) there exists $\delta > 0$ such that

$$\forall x \in \mathbb{R}^d, \quad \text{dist}(\text{Sp } D\mathbf{u}_0(x), \mathbb{R}_-) \geq \delta.$$

In **H3)**, $\text{Sp } M$ denotes the spectrum of a square matrix M , dist stands for the distance between two subsets of the complex plane, and \mathbb{R}_- is the set of nonpositive real numbers. For later use, we define

$$N(\mathbf{u}_0) := |D\mathbf{u}_0|_\infty + \|D^2\mathbf{u}_0\|_{H^{m-1}(\mathbb{R}^d)}.$$

Of course, the solution $\bar{\mathbf{u}}$ of the quasilinear equation

$$\begin{cases} \partial_t \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} = 0, \\ \bar{\mathbf{u}}(t = 0) = \mathbf{u}_0 \end{cases}$$

is constant along each characteristic line

$$(2.3) \quad X(t, x_0) := x_0 + t \mathbf{u}_0(x_0),$$

and it is proved in [2] that these characteristics do not intersect when **H2)** and **H3)** hold. The solution $\bar{\mathbf{u}}$ is thus global, and it satisfies the following estimates:

Proposition 2.1 (Grassin [2]). *Under assumptions **H2)** and **H3)**, we have*

$$\begin{aligned} |DX(t, \cdot)^{-1}|_\infty &\leq \frac{C}{1+t}, \\ D\bar{\mathbf{u}}(t, x) &= \frac{1}{1+t} Id + \frac{1}{(1+t)^2} J(t, x), \quad \|J\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)} \leq C, \\ |D^2\bar{\mathbf{u}}(t, \cdot)|_\infty &\leq \frac{C}{(1+t)^3}, \end{aligned}$$

where the constant C only depends on $m, d, \delta, N(\mathbf{u}_0)$.

Note that for the *multi-d Burgers equation* satisfied by $\bar{\mathbf{u}}$, F. Poupaud has derived some weaker conditions on the initial data that still ensure global existence, see [4].

To construct the solution of the pressureless system (2.2), it only remains to find the solution $\bar{\pi}$ to the linear transport equation

$$\begin{cases} \partial_t \bar{\pi} + \bar{\mathbf{u}} \cdot \nabla \bar{\pi} + \frac{\gamma-1}{2} \bar{\pi} \text{div } \bar{\mathbf{u}} = 0, \\ \bar{\pi}(t = 0) = \pi_0. \end{cases}$$

The solution $\bar{\pi}$ is obtained by integrating along the characteristics defined by (2.3). We obtain

$$(2.4) \quad \bar{\pi}(t, X(t, x_0)) = \pi_0(x_0) \exp \left[\frac{1-\gamma}{2} \int_0^t (\text{div } \bar{\mathbf{u}})(s, X(s, x_0)) ds \right].$$

In particular, using Proposition 2.1, we can show

$$\bar{\pi}(t, X(t, x_0)) = \frac{\pi_0(x_0)}{(1+t)^{(\gamma-1)d/2}} b(t, x_0), \quad \|b\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)} \leq C.$$

We now discuss the properties of the solution $(\pi^\varepsilon, \mathbf{u}^\varepsilon)$ to (2.1), with initial data (π_0, \mathbf{u}_0) . The following result is proved in [2]:

Theorem 2.2 (Grassin [2]). *Assume that **H1)**, **H2)**, **H3)** hold. Then there exists $\beta > 0$ such that, as long as $\|\pi_0\|_{H^m(\mathbb{R}^d)} \leq \beta$, (2.1) admits a unique solution $(\pi^\varepsilon, \mathbf{u}^\varepsilon)$, with initial data (π_0, \mathbf{u}_0) , satisfying*

$$(\pi^\varepsilon, \mathbf{u}^\varepsilon - \bar{\mathbf{u}}) \in C(\mathbb{R}_+, H^m(\mathbb{R}^d)) \cap C^1(\mathbb{R}_+, H^{m-1}(\mathbb{R}^d)).$$

In addition, this solution satisfies the following estimates:

$$\begin{aligned} \sqrt{\varepsilon} |\pi^\varepsilon(t, \cdot)|_\infty + |\mathbf{u}^\varepsilon(t, \cdot) - \bar{\mathbf{u}}(t, \cdot)|_\infty &\leq \frac{C_0 \sqrt{\varepsilon} \|\pi_0\|_{H^m(\mathbb{R}^d)}}{(1+t)^{a-1}}, \\ \sqrt{\varepsilon} |D\pi^\varepsilon(t, \cdot)|_\infty + |D\mathbf{u}^\varepsilon(t, \cdot) - D\bar{\mathbf{u}}(t, \cdot)|_\infty &\leq \frac{C_0 \sqrt{\varepsilon} \|\pi_0\|_{H^m(\mathbb{R}^d)}}{(1+t)^a}. \end{aligned}$$

The number a is given by

$$a := 1 + \frac{(\gamma - 1)d}{2} \in]1, 2].$$

The constants β and C_0 above depend on m, d, γ, δ , and $N(\mathbf{u}_0)$.

Theorem 2.2 already gives an estimate of the distance between the velocity fields \mathbf{u}^ε and $\bar{\mathbf{u}}$. In particular, it shows the convergence of \mathbf{u}^ε towards $\bar{\mathbf{u}}$ in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, as ε goes to 0. However, nothing can be said about the convergence of π^ε towards $\bar{\pi}$. Furthermore, we shall show that the estimate for the distance between the velocities can be improved, both in terms of powers of ε and in terms of (decreasing) powers of t . Our result is stated as follows:

Theorem 2.3. *Assume that **H1**), **H2**), **H3**) hold, and that $\|\pi_0\|_{H^m(\mathbb{R}^d)} \leq \beta$, with β given by Theorem 2.2. Then there exists a constant $C_1 > 0$ such that for all $\varepsilon \in]0, 1]$ and all $t \geq 0$, one has:*

$$|\mathbf{u}^\varepsilon(t, \cdot) - \bar{\mathbf{u}}(t, \cdot)|_\infty \leq C_1 \varepsilon f(t),$$

where

$$f(t) := \begin{cases} 1/(1+t)^{2a-2} & \text{if } 1 < \gamma < 1 + 1/d, \\ (1 + \ln(1+t))/(1+t) & \text{if } \gamma = 1 + 1/d \text{ and } a = 3/2, \\ 1/(1+t) & \text{if } 1 + 1/d < \gamma \leq 1 + 2/d. \end{cases}$$

Moreover, the following estimate holds for all $\varepsilon \in]0, 1]$ and all $t \geq 0$:

$$|\pi^\varepsilon(t, \cdot) - \bar{\pi}(t, \cdot)|_\infty \leq \frac{C_1 \sqrt{\varepsilon}}{(1+t)^{a-1}}.$$

The constant C_1 depends only on $m, d, \delta, N(\mathbf{u}_0)$.

The proof is given in the next section. Let us note that, as mentioned above, the estimate for $|\mathbf{u}^\varepsilon(t, \cdot) - \bar{\mathbf{u}}(t, \cdot)|_\infty$ is better than the one given by Theorem 2.2. The estimate for $|\pi^\varepsilon(t, \cdot) - \bar{\pi}(t, \cdot)|_\infty$ is not as good as one might expect. (One might hope to obtain a linear estimate with respect to ε .)

3. PROOF OF THEOREM 2.3

We first define the characteristics associated with \mathbf{u}^ε . Using Proposition 2.1 and Theorem 2.2, we have

$$|D\mathbf{u}^\varepsilon(t, \cdot)|_\infty \leq |D\bar{\mathbf{u}}(t, \cdot)|_\infty + |D\mathbf{u}^\varepsilon(t, \cdot) - D\bar{\mathbf{u}}(t, \cdot)|_\infty \leq \frac{C}{1+t} + \frac{C \sqrt{\varepsilon}}{(1+t)^a} \leq C'.$$

Moreover, \mathbf{u}^ε belongs to $C^1(\mathbb{R}_+ \times \mathbb{R}^d)$ by Sobolev imbedding. From standard ODE theory, there exists a unique $X^\varepsilon \in C^1(\mathbb{R}_+ \times \mathbb{R}^d)$ satisfying

$$(3.1) \quad \begin{cases} \frac{\partial X^\varepsilon}{\partial t}(t, x) = \mathbf{u}^\varepsilon(t, X^\varepsilon(t, x)), \\ X^\varepsilon(0, x) = x. \end{cases}$$

For all $t \geq 0$, the mapping $X^\varepsilon(t, \cdot)$ is one-to-one and onto from \mathbb{R}^d to \mathbb{R}^d . We are therefore going to estimate the quantities

$$|\mathbf{u}^\varepsilon(t, X^\varepsilon(t, x)) - \bar{\mathbf{u}}(t, X^\varepsilon(t, x))| \quad \text{and} \quad |\pi^\varepsilon(t, X^\varepsilon(t, x)) - \bar{\pi}(t, X^\varepsilon(t, x))|$$

in order to prove Theorem 2.3.

3.1. Estimates for the velocity. We fix an integer $i \in \{1, \dots, d\}$, and compute

$$\partial_t(\mathbf{u}_i^\varepsilon - \bar{\mathbf{u}}_i) + \mathbf{u}^\varepsilon \cdot \nabla(\mathbf{u}_i^\varepsilon - \bar{\mathbf{u}}_i) = \frac{1-\gamma}{2} \varepsilon \pi^\varepsilon \partial_{x_i} \pi^\varepsilon - (\mathbf{u}^\varepsilon - \bar{\mathbf{u}}) \cdot \nabla \bar{\mathbf{u}}_i.$$

Using Proposition 2.1, we get

$$\nabla \bar{\mathbf{u}}_i(t, x) = \frac{1}{1+t} e_i + \frac{b(t, x)}{(1+t)^2}, \quad \|b\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)} \leq C,$$

where e_i is the i -th vector of the canonical basis. Defining

$$v_i(t, x) := (\mathbf{u}_i^\varepsilon - \bar{\mathbf{u}}_i)(t, X^\varepsilon(t, x)),$$

we obtain the differential equation

$$\frac{dv_i}{dt} + \frac{v_i}{1+t} = \frac{v \cdot b}{(1+t)^2} + \frac{1-\gamma}{2} \varepsilon (\pi^\varepsilon \partial_{x_i} \pi^\varepsilon)(t, X^\varepsilon(t, x)).$$

Above, and in all that follows, we denote by b any scalar or vector valued bounded function of t and x . The notation v stands for the vector (v_1, \dots, v_d) . Using Theorem 2.2, we write

$$\pi^\varepsilon(t, X^\varepsilon(t, x)) = \frac{b(t, x)}{(1+t)^{a-1}}, \quad \partial_{x_i} \pi^\varepsilon(t, X^\varepsilon(t, x)) = \frac{b(t, x)}{(1+t)^a}.$$

We thus obtain

$$(3.2) \quad \frac{d}{dt}[(1+t)v_i] = \frac{v \cdot b}{1+t} + \frac{\varepsilon b}{(1+t)^{2a-2}},$$

with the initial condition $v_i(t=0) = 0$. Integrating (3.2) from 0 to t , and summing over the indices i yields the inequality

$$(3.3) \quad (1+t)|v(t, x)| \leq C \int_0^t \frac{|v(s, x)|}{1+s} ds + C \varepsilon \int_0^t \frac{ds}{(1+s)^{2a-2}}.$$

Recall that $2a - 2$ belongs to $]0, 2]$. The end of the proof simply follows from Gronwall's lemma. The estimate we obtain heavily depends on the convergence of the integral

$$\int_0^{+\infty} \frac{ds}{(1+s)^{2a-2}},$$

and one has to distinguish three cases, as stated in Theorem 2.3.

3.2. Estimates for the density. From (2.1) and (2.2), we compute

$$\partial_t(\pi^\varepsilon - \bar{\pi}) + \mathbf{u}^\varepsilon \cdot \nabla(\pi^\varepsilon - \bar{\pi}) + \frac{\gamma-1}{2} (\pi^\varepsilon - \bar{\pi}) \operatorname{div} \mathbf{u}^\varepsilon = (\bar{\mathbf{u}} - \mathbf{u}^\varepsilon) \cdot \nabla \bar{\pi} + \frac{\gamma-1}{2} \bar{\pi} \operatorname{div} (\bar{\mathbf{u}} - \mathbf{u}^\varepsilon),$$

so defining

$$p(t, x) := (\pi^\varepsilon - \bar{\pi})(t, X^\varepsilon(t, x)),$$

we get

$$(3.4) \quad \frac{dp}{dt} + \frac{\gamma-1}{2} p(\operatorname{div} \mathbf{u}^\varepsilon)(t, X^\varepsilon) = [(\bar{\mathbf{u}} - \mathbf{u}^\varepsilon) \cdot \nabla \bar{\pi}](t, X^\varepsilon) + \frac{\gamma-1}{2} [\bar{\pi} \operatorname{div} (\bar{\mathbf{u}} - \mathbf{u}^\varepsilon)](t, X^\varepsilon).$$

Using the formula (2.4) and Proposition 2.1, we obtain the bounds

$$|\bar{\pi}(t, \cdot)|_{\infty} \leq \frac{C}{(1+t)^{a-1}}, \quad |\nabla \bar{\pi}(t, \cdot)|_{\infty} \leq \frac{C}{(1+t)^a}.$$

Consequently, with the help of Theorem 2.2 and of our new estimate on the velocities, the terms on the right-hand side of (3.4) have the following form:

$$(\bar{\mathbf{u}} - \mathbf{u}^{\varepsilon}) \cdot \nabla \bar{\pi}(t, X^{\varepsilon}) = \frac{\varepsilon b(t, x) f(t)}{(1+t)^a}, \quad [\bar{\pi} \operatorname{div}(\bar{\mathbf{u}} - \mathbf{u}^{\varepsilon})](t, X^{\varepsilon}) = \frac{\sqrt{\varepsilon} b(t, x)}{(1+t)^{2a-1}}.$$

According to the expression of $f(t)$ given in Theorem 2.3, the first term above is always neglectible compared to the second one. We thus obtain the relation

$$\frac{dp}{dt} + \frac{\gamma-1}{2} p (\operatorname{div} \mathbf{u}^{\varepsilon})(t, X^{\varepsilon}) = \frac{\sqrt{\varepsilon} b(t, x)}{(1+t)^{2a-1}}.$$

Integrating with respect to time yields

$$p(t, x) \exp g^{\varepsilon}(t, x) = \sqrt{\varepsilon} \int_0^t \frac{b(s, x)}{(1+s)^{2a-1}} \exp g^{\varepsilon}(s, x) ds,$$

where we have let

$$g^{\varepsilon}(t, x) := \frac{\gamma-1}{2} \int_0^t \operatorname{div} \mathbf{u}^{\varepsilon}(s, X^{\varepsilon}(s, x)) ds.$$

Thanks to Proposition 2.1 and Theorem 2.2, we obtain

$$|\exp(-g^{\varepsilon}(t, x))| \leq \frac{C}{(1+t)^{a-1}}, \quad |\exp(g^{\varepsilon}(s, x))| \leq C(1+s)^{a-1},$$

and the constant C is uniform with respect to ε . The bound stated in Theorem 2.3 follows and the proof is now complete.

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