LIFTS OF $C_\infty$- AND $L_\infty$-MORPHISMS TO $G_\infty$-MORPHISMS

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Abstract. Let $g_2$ be the Hochschild complex of cochains on $C^\infty(R^n)$ and let $g_1$ be the space of multivector fields on $R^n$. In this paper we prove that given any $G_\infty$-structure (i.e. Gerstenhaber algebra up to homotopy structure) on $g_2$, and any $C_\infty$-morphism $\varphi$ (i.e. morphism of a commutative, associative algebra up to homotopy) between $g_1$ and $g_2$, there exists a $G_\infty$-morphism $\Phi$ between $g_1$ and $g_2$ that restricts to $\varphi$. We also show that any $L_\infty$-morphism (i.e. morphism of a Lie algebra up to homotopy), in particular the one constructed by Kontsevich, can be deformed into a $G_\infty$-morphism, using Tamarkin’s method for any $G_\infty$-structure on $g_2$. We also show that any two of such $G_\infty$-morphisms are homotopic.

0. Introduction

Let $M$ be a differential manifold and let $g_2 = (C^\bullet(A, A), b)$ be the Hochschild cochain complex on $A = C^\infty(M)$. The classical Hochschild-Kostant-Rosenberg theorem states that the cohomology of $g_2$ is the graded Lie algebra $g_1 = \Gamma(M, \wedge T^*M)$ of multivector fields on $M$. There is also a graded Lie algebra structure on $g_2$ given by the Gerstenhaber bracket. In particular $g_1$ and $g_2$ are also Lie algebras up to homotopy ($L_\infty$-algebra for short). In the case $M = R^n$, using different methods, Kontsevich ([Ko1] and [Ko2]) and Tamarkin ([Ta]) have proved the existence of Lie homomorphisms “up to homotopy” ($L_\infty$-morphisms) from $g_1$ to $g_2$. Kontsevich’s proof uses graph complex and is related to multizeta functions, whereas Tamarkin’s construction uses the existence of Drinfeld’s associators. In fact Tamarkin’s $L_\infty$-morphism comes from the restriction of a Gerstenhaber algebra up to homotopy homomorphism ($G_\infty$-morphism) from $g_1$ to $g_2$. The $G_\infty$-algebra structure on $g_1$ is induced by its classical Gerstenhaber algebra structure and a far less trivial $G_\infty$-structure on $g_2$ was proved to exist by Tamarkin [Ta] and relies on Drinfeld’s associator. Tamarkin’s $G_\infty$-morphism also restricts into a commutative, associative up to homotopy morphism ($C_\infty$-morphism for short). The $C_\infty$-structure on $g_2$ (given by restriction of the $G_\infty$-one) highly depends on Drinfeld’s associator, and any two choices of a Drinfeld associator yields a priori different $C_\infty$-structures. When $M$ is a Poisson manifold, Kontsevich and Tamarkin homomorphisms imply the existence of a star-product (see [BFFLS1] and [BFFLS2] for a definition). A connection between the two approaches has been given in [KS] but the morphisms...
given by Kontsevich and Tamarkin are not the same. The aim of this paper is to show that, given any $G_\infty$-structure on $\mathfrak{g}_2$ and any $C_\infty$-morphism $\varphi$ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$, there exists a $G_\infty$-morphism $\Phi$ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ that restricts to $\varphi$. We also show that any $L_\infty$-morphism can be deformed into a $G_\infty$-one.

In the first section, we fix notation and recall the definitions of $L_\infty$- and $G_\infty$-structures. In the second section we state and prove the main theorem. In the last section we show that any two $G_\infty$-morphisms given by Tamarkin’s method are homotopic.

Remark. In the sequel, unless otherwise stated, the manifold $M$ is $\mathbb{R}^n$ for some $n \geq 1$. Most results could be generalized to other manifolds using techniques of Kontsevich [Ko1] (see also [TS], [CFT]).

1. $C_\infty$, $L_\infty$ and $G_\infty$-structures

For any graded vector space $\mathfrak{g}$, we choose the following degree on $\bigwedge^\bullet \mathfrak{g}$: if $X_1, \ldots, X_k$ are homogeneous elements of respective degree $|X_1|, \ldots, |X_k|$, then

$$|X_1 \wedge \cdots \wedge X_k| = |X_1| + \cdots + |X_k| - k.$$ 

In particular the component $\mathfrak{g} = \bigwedge^1 \mathfrak{g} \subset \bigwedge^\bullet \mathfrak{g}$ is the same as the space $\mathfrak{g}$ with degree shifted by one. The space $\bigwedge^\bullet \mathfrak{g}$ with the deconcatenation cobracket is the cofree cocommutative coalgebra on $\mathfrak{g}$ with degree shifted by one (see [LS], Section 2). Any degree one map $d^k : \bigwedge^k \mathfrak{g} \to \mathfrak{g}$ ($k \geq 1$) extends into a derivation $d^k : \bigwedge^\bullet \mathfrak{g} \to \bigwedge^\bullet \mathfrak{g}$ of the coalgebra $\bigwedge^\bullet \mathfrak{g}$ by cofreeness property.

Definition 1.1. A vector space $\mathfrak{g}$ is endowed with a $L_\infty$-algebra (Lie algebras “up to homotopy”) structure if there are degree one linear maps $m^1, \ldots, m^3, m^\infty : \bigwedge^k \mathfrak{g} \to \mathfrak{g}$ such that if we extend them to maps $\bigwedge^\bullet \mathfrak{g} \to \bigwedge^\bullet \mathfrak{g}$, then $d \circ d = 0$ where $d$ is the derivation

$$d = m^1 + m^{1,1} + \cdots + m^{1,\cdots,1} + \cdots.$$ 

For more details on $L_\infty$-structures, see [LS]. It follows from the definition that a $L_\infty$-algebra structure induces a differential coalgebra structure on $\bigwedge^\bullet \mathfrak{g}$ and that the map $m^1 : \mathfrak{g} \to \mathfrak{g}$ is a differential. If $m^{1,\cdots,1} = \bigwedge^k \mathfrak{g} \to \mathfrak{g}$ are 0 for $k \geq 3$, we get the usual definition of (differential if $m^1 \neq 0$) graded Lie algebras.

For any graded vector space $\mathfrak{g}$, we denote $\mathfrak{g}^\otimes n$ to be the quotient of $\mathfrak{g}^\otimes n$ by the image of all shuffles of length $n$ (see [GK] or [GH] for details). The graded vector space $\bigoplus_{n \geq 0} \mathfrak{g}^\otimes n$ is a quotient coalgebra of the tensor coalgebra $\bigoplus_{n \geq 0} \mathfrak{g}^\otimes n$. It is well known that this coalgebra $\bigoplus_{n \geq 0} \mathfrak{g}^\otimes n$ is the cofree Lie coalgebra on the vector space $\mathfrak{g}$ (with degree shifted by minus one).

Definition 1.2. A $C_\infty$-algebra (commutative and associative “up to homotopy”) structure on a vector space $\mathfrak{g}$ is given by a collection of degree one linear maps $m^k : \mathfrak{g}^\otimes k \to \mathfrak{g}$ such that if we extend them to maps $\bigoplus \mathfrak{g}^\otimes \bullet \to \bigoplus \mathfrak{g}^\otimes \bullet$, then $d \circ d = 0$ where $d$ is the derivation

$$d = m^1 + m^2 + m^3 + \cdots.$$ 

In particular a $C_\infty$-algebra is an $A_\infty$-algebra.

For any space $\mathfrak{g}$, we denote by $\bigwedge^\bullet \mathfrak{g}^{\otimes \bullet}$ the graded space

$$\bigwedge^\bullet \mathfrak{g}^{\otimes \bullet} = \bigoplus_{m \geq 1, p_1 + \cdots + p_n = m} \mathfrak{g}^{\otimes p_1} \wedge \cdots \wedge \mathfrak{g}^{\otimes p_n}.$$ 

We use the following grading on $\Lambda^\bullet \mathfrak{g}^{\otimes \bullet}$: for $x_1^1, \ldots, x_n^p \in \mathfrak{g}$, we define

$$|x_1^1 \otimes \cdots \otimes x_1^{p_1} \wedge \cdots \wedge x_n^1 \otimes \cdots \otimes x_n^{p_n}| = \sum_{i_1}^{p_1} |x_1^{i_1}| + \cdots + \sum_{i_n}^{p_n} |x_n^{i_n}| - n.$$

Note that the induced grading on $\Lambda^\bullet \mathfrak{g} \subset \Lambda^\bullet \mathfrak{g}^{\otimes \bullet}$ is the same as the one introduced above. The cobracket on $\mathfrak{g}^{\otimes \bullet}$ and the coproduct on $\Lambda^\bullet \mathfrak{g}$ extend to a cobracket and a coproduct on $\Lambda^\bullet \mathfrak{g}^{\otimes \bullet}$ which yield a Gerstenhaber coalgebra structure on $\Lambda^\bullet \mathfrak{g}^{\otimes \bullet}$. It is well known that this coalgebra structure is cofree (see [Gi], Section 3, for example).

**Definition 1.3.** A $G_\infty$-algebra (Gerstenhaber algebra “up to homotopy”) structure on a graded vector space $\mathfrak{g}$ is given by a collection of degree one maps

$$m^{p_1, \ldots, p_n} : \mathfrak{g}^{\otimes p_1} \wedge \cdots \wedge \mathfrak{g}^{\otimes p_n} \to \mathfrak{g}$$

indexed by $p_1, \ldots, p_n \geq 1$ such that their canonical extension $\Lambda^\bullet \mathfrak{g}^{\otimes \bullet} \to \Lambda^\bullet \mathfrak{g}^{\otimes \bullet}$ satisfies $d \circ d = 0$, where

$$d = \sum_{m \geq 1, p_1 + \cdots + p_n = m} m^{p_1, \ldots, p_n}.$$

Again, as the coalgebra structure of $\Lambda^\bullet \mathfrak{g}^{\otimes \bullet}$ is cofree, the map $d$ makes $\Lambda^\bullet \mathfrak{g}^{\otimes \bullet}$ into a differential coalgebra. If the maps $m^{p_1, \ldots, p_n}$ are 0 for $(p_1, p_2, \ldots) \neq (1, 0, \ldots)$, $(1, 1, 0, \ldots)$ or $(2, 0, \ldots)$, we get the usual definition of (differential if $m^1 \neq 0$) Gerstenhaber algebra.

The space of multivector fields $\mathfrak{g}_1$ is endowed with a graded Lie bracket $[-,-]_S$ called the Schouten bracket (see [Kos]). This Lie algebra can be extended into a Gerstenhaber algebra, with commutative structure given by the exterior product $(\alpha, \beta) \mapsto \alpha \wedge \beta$.

Setting $d_1 = m_1^{1,1} + m_1^2$, where $m_1^{1,1} : \Lambda^2 \mathfrak{g}_1 \to \mathfrak{g}_1$, and $m_1^2 : \mathfrak{g}_1^{\otimes 2} \to \mathfrak{g}_1$ are the extension of the Schouten bracket and the exterior product, we find that $(\mathfrak{g}_1, d_1)$ is a $G_\infty$-algebra.

In the same way, one can define a differential Lie algebra structure on the vector space $\mathfrak{g}_2 = C(A, A) = \bigoplus_{k \geq 0} C^k(A, A)$, the space of Hochschild cochains (generated by differential $k$-linear maps from $A^k$ to $A$), where $A = C^\infty(M)$ is the algebra of smooth differential functions over $M$. Its bracket $[-,-]_G$, called the Gerstenhaber bracket, is defined, for $D, E \in \mathfrak{g}_2$, by

$$[D, E]_G = \{D|E\} - (-1)^{|E||D|}\{E|D\},$$

where

$$\{D|E\}(x_1, \ldots, x_{d+e-1}) = \sum_{i \geq 0} (-1)^{|E||i|} D(x_1, \ldots, x_i, E(x_{i+1}, \ldots, x_{i+e}), \ldots).$$

The space $\mathfrak{g}_2$ has a grading defined by $|D| = k \iff D \in C^{k+1}(A, A)$ and its differential is $b = [m, -]_G$, where $m \in C^2(A, A)$ is the commutative multiplication on $A$.

Tamarkin (see [Ta] or also [GH]) stated the existence of a $G_\infty$-structure on $\mathfrak{g}_2$ (depending on a choice of a Drinfeld associator) given by a differential $d_2 = m_2^{1,1} + m_2^2 + \cdots + m_2^{p_1, \ldots, p_n} + \cdots$, on $\Lambda^\bullet \mathfrak{g}^{\otimes \bullet}_2$ satisfying $d_2 \circ d_2 = 0$. Although
this structure is non-explicit, it satisfies the following three properties:

(a) $m^1_2$ is the extension of the differential $b$.

(b) $m^{1,1}_2$ is the extension of the Gerstenhaber bracket $[-,-]_G$

\begin{equation}
(1.1)
\end{equation}

and $m^{1,1}_{2} = 0$.

(c) $m^2_2$ induces the exterior product in cohomology and the

collection of the $(m^k)_{k \geq 1}$ defines a $C_\infty$-structure on $\mathfrak{g}_2$.

**Definition 1.4.** A $L_\infty$-morphism between two $L_\infty$-algebras $(\mathfrak{g}_1, d_1 = m_1 + \cdots)$ and $(\mathfrak{g}_2, d_2 = m_2 + \cdots)$ is a morphism of differential coalgebras

\begin{equation}
(1.2)
\varphi : (\bigwedge^\bullet \mathfrak{g}_1, d_1) \to (\bigwedge^\bullet \mathfrak{g}_2, d_2).
\end{equation}

Such a map $\varphi$ is uniquely determined by a collection of maps $\varphi^n : \bigwedge^n \mathfrak{g}_1 \to \mathfrak{g}_2$ (again by cofreeness properties). In the case $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are respectively the graded Lie algebra $(\Gamma(M, \Lambda TM), [-,-]_S)$ and the differential graded Lie algebra $(C(A,A), [-,-]_G)$, the formality theorems of Kontsevich and Tamarkin state the existence of a $L_\infty$-morphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that $\varphi^1$ is the Hochschild-Kostant-Rosenberg map, there exists a $L_\infty$-morphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that $\varphi^1$ is the Hochschild-Kostant-Rosenberg quasi-isomorphism.

**Definition 1.5.** A morphism of $C_\infty$-algebras between two $C_\infty$-algebras $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$ is a map $\phi : (\bigoplus \bigwedge^* \mathfrak{g}_1, d_1) \to (\bigoplus \bigwedge^* \mathfrak{g}_2, d_2)$ of codifferential coalgebras.

A $C_\infty$-morphism is in particular a morphism of $A_\infty$-algebras and is uniquely determined by maps $\partial^k : \mathfrak{g}^\otimes k \to \mathfrak{g}$.

**Definition 1.6.** A morphism of $G_\infty$-algebras between two $G_\infty$-algebras $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$ is a map $\phi : (\bigwedge^* \mathfrak{g}_1, d_1) \to (\bigwedge^* \mathfrak{g}_2, d_2)$ of codifferential coalgebras.

There are coalgebra inclusions $\bigwedge^* \mathfrak{g} \to \bigwedge^* \mathfrak{g} \otimes \bigwedge^* \mathfrak{g} \otimes \cdots \otimes \bigwedge^* \mathfrak{g}$, and it is easy to check that any $G_\infty$-morphism between two $G_\infty$-algebras $(\mathfrak{g}, \sum m^{p_1} \cdots m^{p_n})$ and $(\mathfrak{g}', \sum m'^{p_1} \cdots m'^{p_n})$ restricts to a $L_\infty$-morphism $\left(\bigwedge^* \mathfrak{g}, \sum m^{1} \cdots \cdots \right) \to \left(\bigwedge^* \mathfrak{g}', \sum m'^{1} \cdots \right)$ and a $C_\infty$-morphism $\left(\bigwedge^* \mathfrak{g} \otimes \cdots \otimes \bigwedge^* \mathfrak{g}, \sum m^k \right) \to \left(\bigwedge^* \mathfrak{g}' \otimes \cdots \otimes \bigwedge^* \mathfrak{g}', \sum m'^k \right)$. In the case where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are as above, Tamarkin’s theorem states that there exists a $G_\infty$-morphism between the two $G_\infty$-algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ (with the $G_\infty$ structure he built) that restricts to a $C_\infty$- and a $L_\infty$-morphism.

2. Main theorem

We keep the notations of the previous section, in particular $\mathfrak{g}_2$ is the Hochschild complex of cochains on $C_\infty(M)$ and $\mathfrak{g}_1$ its cohomology. Here is our main theorem.

**Theorem 2.1.** Given any $G_\infty$-structure $d_2$ on $\mathfrak{g}_2$ satisfying the three properties of (1.1), and any $C_\infty$-morphism $\varphi$ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that $\varphi^1$ is the Hochschild-Kostant-Rosenberg map, there exists a $G_\infty$-morphism $\Phi : (\mathfrak{g}_1, d_1) \to (\mathfrak{g}_2, d_2)$ that restricts to $\varphi$.

Also, given any $L_\infty$-morphism $\gamma$ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that $\gamma^1$ is the Hochschild-Kostant-Rosenberg map, there exists a $G_\infty$-structure $(\mathfrak{g}_1, d'_1)$ on $\mathfrak{g}_1$ and $G_\infty$-morphism $\Gamma : (\mathfrak{g}_1, d'_1) \to (\mathfrak{g}_2, d_2)$ that restricts to $\gamma$. Moreover there exists a $G_\infty$-morphism $\Gamma' : (\mathfrak{g}_1, d_1) \to (\mathfrak{g}_1, d'_1)$.
In particular, Theorem 2.1 applies to the formality map of Kontsevich and also to any $C_\infty$-map derived (see [Ta], [GH]) from any $B_\infty$-structure on $g_2$ lifting the Gerstenhaber structure of $g_1$.

Let us first recall the proof of Tamarkin’s formality theorem (see [GH] for more details):

1. First one proves there exists a $G_\infty$-structure on $g_2$, with differential $d_2$, as in (1.1).
2. Then, one constructs a $G_\infty$-structure on $g_1$ given by a differential $d_1'$ together with a $G_\infty$-morphism $\Phi$ between $(g_1, d_1')$ and $(g_2, d_2)$.
3. Finally, one constructs a $G_\infty$-morphism $\Phi'$ between $(g_1, d_1)$ and $(g_1, d_1')$.

The composition $\Phi \circ \Phi'$ is then a $G_\infty$-morphism between $(g_1, d_1)$ and $(g_2, d_2)$, thus it restricts to a $L_\infty$-morphism between the differential graded Lie algebras $g_1$ and $g_2$.

We suppose now that, in the first step, we take any $G_\infty$-structure on $g_2$ given by a differential $d_2$. We also suppose we are given a $C_\infty$-morphism $\varphi$ and a $L_\infty$-morphism $\gamma$ between $g_1$ and $g_2$ satisfying $\gamma^1 = \varphi^1 = \varphi_{HKR}$, the Hochschild-Kostant-Rosenberg quasi-isomorphism.

**Proof of Theorem 2.1.** The theorem will follow if we prove that steps 2 and 3 of Tamarkin’s construction are still true with the extra conditions that the restriction of the $G_\infty$-morphism $\Phi$ (resp. $\Phi'$) on the $C_\infty$-structures is the $C_\infty$-morphism $\varphi : g_1 \to g_2$ (resp. id).

Let us recall (see [GH]) that the constructions of $\Phi$ and $d_1'$ can be made by induction. For $i = 1, 2$ and $n \geq 0$, let us set

$$V_i^{[n]} = \bigoplus_{p_1 + \cdots + p_k = n} g_1^{\otimes p_1} \wedge \cdots \wedge g_1^{\otimes p_k}$$

and $V_i^{[\leq n]} = \sum_{k \leq n} V_i^{[k]}$. Let $d_2^{[n]}$ and $d_2^{[\leq n]}$ be the sums

$$d_2^{[n]} = \sum_{p_1 + \cdots + p_k = n} g_1^{\otimes p_1} \wedge \cdots \wedge g_1^{\otimes p_k} \quad \text{and} \quad d_2^{[\leq n]} = \sum_{p \leq n} d_2^{[p]}.$$

Clearly, $d_2 = \sum_{n \geq 1} d_2^{[n]}$. In the same way, we denote $d_1' = \sum_{n \geq 1} d_1'^{[n]}$ by

$$d_1'^{[n]} = \sum_{p_1 + \cdots + p_k = n} g_1^{\otimes p_1} \wedge \cdots \wedge g_1^{\otimes p_k} \quad \text{and} \quad d_1'^{[\leq n]} = \sum_{1 \leq k \leq n} d_1'^{[k]}.$$ 

We know from Section 1 that a morphism $\Phi : (\wedge g_1^{\otimes \bullet} \dot{\otimes}, d_1') \to (\wedge g_2^{\otimes \bullet} \dot{\otimes}, d_2)$ is uniquely determined by its components $\Phi^{p_1, \cdots, p_k} : g_1^{\otimes p_1} \wedge \cdots \wedge g_1^{\otimes p_k} \to g_2$. Again, we have

$$\Phi^{[n]} = \sum_{p_1 + \cdots + p_k = n} \Phi^{p_1, \cdots, p_k} \quad \text{and} \quad \Phi^{[\leq n]} = \sum_{1 \leq k \leq n} \Phi^{[k]}.$$ 

We want to construct the maps $d_1'^{[n]}$ and $\Phi^{[n]}$ by induction with the initial condition

$$d_1'^{[1]} = 0 \quad \text{and} \quad \Phi^{[1]} = \varphi_{HKR},$$

where $\varphi_{HKR} : (g_1, 0) \to (g_2, b)$ is the Hochschild-Kostant-Rosenberg quasi-isomorphism (see [HKR]) defined, for $\alpha \in g_1$, $f_1, \cdots, f_n \in A$, by

$$\varphi_{HKR} : \alpha \mapsto \left( (f_1, \cdots, f_n) \mapsto \langle \alpha, df_1 \wedge \cdots \wedge df_n \rangle \right).$$
Moreover, we want the following extra conditions to be true:

\[(2.3) \quad \Phi^k \geq 2 = \varphi^k, \quad d_1^2 = d_1^2, \quad d_1^{k \geq 3} = 0.\]

Now suppose the construction is done for \(n - 1\) \((n \geq 2)\), i.e., we have built maps \((d_{1[i]}^{[i]})_{i \leq n-1}\) and \((\Phi^{[i]})_{i \leq n-1}\) satisfying conditions \((2.3)\) and

\[
\Phi^{[i]} \leq n-1| \circ d_1^{[i]} \leq n-1| = d_2^{[i]} \leq n-1| \circ \Phi^{[i]} \leq n-1| \quad \text{on } V_1^{[i]} \leq n-1| \quad \text{and}
\]

\[
d_1^{[i]} \leq n-1| \circ d_1^{[i]} \leq n-1| = 0 \quad \text{on } V_1^{[i]} \leq n-1|.
\]

In [GH], we prove that for any such \((d_{1[i]}^{[i]})_{i \leq n-1}\) and \((\Phi^{[i]})_{i \leq n-1}\), one can construct \(d_{1[n]}^{[n]}\) and \(\Phi^{[n]}\) such that condition \((2.4)\) is true for \(n\) instead of \(n-1\). To complete the proof of Theorem 2.1 (step 2), we have to show that \(d_{1[n]}^{[n]}\) and \(\Phi^{[n]}\) can be chosen to satisfy conditions \((2.3)\). In equation \((2.4)\), the terms \(d_{1[n]}^{[n]}\) and \(\Phi^{[n]}\) only act on \(V_1^n\). So one can replace \(\Phi^n\) with \(\varphi^n\), \(d_{1[n]}^{[n]}\) with \(d_1^n\) (or \(d_{1[i]}^{[i]}, i \geq 3\) with 0) provided conditions \((2.4)\) are still satisfied on \(V_1^n\). The other terms acting on \(V_1^n\) in equation \((2.4)\) only involve terms \(\Phi^m = \varphi^m\) and \(d_{1[m]}^{[m]}\). Then conditions \((2.4)\) on \(V_1^{n1,...,1}\) are the equations that should be satisfied by a \(C_\infty\)-morphism between the \(C_\infty\)-algebras \((g_1, d_{1[i]}^{[i]} = d_{1[i]}^{[i]})\) and \((g_2, \sum_{k \geq 1} d_{2[k]}^{[k]})\) restricted to \(V_1^n\). Hence by hypothesis on \(\varphi\) the conditions hold.

Similarly the construction of \(\Phi'\) can be made by induction. Let us recall the proof given in [GH]. Again a morphism \(\Phi' : (\wedge g_1^\otimes \cdot \cdot \cdot d_1) \rightarrow (\wedge g_2^\otimes \cdot \cdot \cdot d_1)\) is uniquely determined by its components \(\Phi'^{p_1,...,p_k} : g_1^\otimes p_1 \wedge \cdot \cdot \cdot \wedge g_1^\otimes p_k \rightarrow g_1\). We write \(\Phi' = \sum_{n \geq 1} \Phi'^{[n]}\) with

\[
\Phi'^{[n]} = \sum_{p_1 + \cdot \cdot \cdot + p_k = n} \Phi'^{p_1,...,p_k} \quad \text{and} \quad \Phi'^{[i]} \leq n = \sum_{1 \leq k \leq n} \Phi'^{[k]}.
\]

We construct the maps \(\Phi'^{[n]}\) by induction with the initial condition \(\Phi'^{[1]} = \text{id}\).

Moreover, we want the following extra conditions to be true:

\[(2.5) \quad \Phi'^{n} = 0 \quad \text{for } n \geq 2.\]

Now suppose the construction is done for \(n - 1\) \((n \geq 2)\), i.e., we have built maps \((\Phi'^{[i]})_{i \leq n-1}\) satisfying conditions \((2.5)\) and

\[
\Phi'^{[i]} \leq n-1| d_1^{[i]} \leq n = d_{1[n]}^{[i]} \Phi'^{[i]} \leq n-1| \quad \text{on } V_1^{[i]} \leq n.
\]

In [GH], we prove that for any such \((\Phi'^{[i]})_{i \leq n-1}\), one can construct \(\Phi'^{[n]}\) such that condition \((2.6)\) is true for \(n\) instead of \(n-1\) in the following way: making the equation \(\Phi'd_1^{[i]} = d_{1[i]}^{[i]} \Phi'\) on \(V_1^{[i+1]}\) explicit, we get

\[
\Phi'^{[i]} \leq n+1| d_1^{[i]} \leq n+1| = d_{1[i]}^{[i]} \Phi'^{[i]} \leq n|.
\]

If we now take into account that \(d_{1[i]}^{[i]} = 0\) for \(i \neq 2\), \(d_{1[i]}^{[1]} = 0\) and that on \(V_1^{[n+1]}\) we have \(\Phi'^{[k]} d_{1[i]}^{[i]} = d_{1[i]}^{[i]} \Phi'^{[k]} = 0\) for \(k + 1 > n + 2\), the identity \((2.7)\) becomes

\[
\Phi'^{[i]} \leq n| d_1^{[i]} d_1^{[i]} = \sum_{k=2}^{n+1} d_{1[i]}^{[k]} \Phi'^{[i]} \leq n-k+2|.
\]
Since $d_1^{[2]} = d_1^{[2]}$, (2.7) is equivalent to

$$d_1^{[2]} \psi^{[\leq n]} - \Phi^{[\leq n]} d_1^{[2]} = \left[ d_1^{[2]}, \Phi^{[\leq n]} \right] = - \sum_{k=3}^{n+1} d_1^{[k]} \Phi^{[\leq n-k+2]}.$$  

Note that $d_1^{[2]} = m_1^{1,1} + m_2^{1}$. Then the construction will be possible when the term $\sum_{k=3}^{n+1} d_1^{[k]} \Phi^{[\leq n-k+2]}$ is a coboundary in the subcomplex of $(\text{End}(\wedge \mathfrak{g}_1^\otimes), [d_1^{[2]}, -])$ consisting of maps which restrict to zero on $\bigoplus_{n \geq 2} \mathfrak{g}_1^\otimes_n$. It is always a cocycle by straightforward computation (see [GH]), and the subcomplex is acyclic because both $(\text{End}(\wedge \mathfrak{g}_1^\otimes), [d_1^{[2]}, -])$ and the Harrison cohomology of $\mathfrak{g}_1$ are trivial according to Tamarkin [13] (see also [GH], Proposition 5.1, and [HI], 5.4).

In the case of the $L_\infty$-map $\gamma$, the first step is similar: the fact that $\gamma$ is a $L_\infty$-map enables us to build a $G_\infty$-structure $(\mathfrak{g}_1, d_1')$ on $\mathfrak{g}_1$, and a $G_\infty$-morphism $\Gamma : (\mathfrak{g}_1, d_1') \to (\mathfrak{g}_2, d_2)$ such that:

$$\Gamma^{[1}] = \gamma^{[1]}, \quad d_1^{[1]} = d_1^{[1]}, \quad d_1^{[1]} = 0.$$  

For the second step, we have to build a map $\Gamma'$ satisfying the equation

$$d_1^{[2]} \Gamma^{[\leq n]} - \Gamma^{[\leq n]} d_1^{[2]} = \left[ d_1^{[2]}, \Gamma^{[\leq n]} \right] = - \sum_{k=3}^{n+1} d_1^{[k]} \Gamma^{[\leq n-k+2]}$$  

on $V^{[n+1]}$ for any $n \geq 1$. Again, because Tamarkin has proved that the complex $(\text{End}(\wedge \mathfrak{g}_1^\otimes), [d_1^{[2]}, -])$ is acyclic (we are in the case $M = \mathbb{R}^n$), the result follows from the fact that $\sum_{k=3}^{n+1} d_1^{[k]} \Gamma^{[\leq n-k+2]}$ is a cocycle. The difference with the $C_\infty$-case is that the $\Gamma^{[1]}$-case could be non-zero. □

3. THE DIFFERENCE BETWEEN TWO $G_\infty$-MAPS

In this section we investigate the difference between two different $G_\infty$-formality maps.

We fix once and for all a $G_\infty$-structure on $\mathfrak{g}_2$ (given by a differential $d_2$) satisfying the conditions (1.1) and a morphism of $G_\infty$-algebras $T : (\mathfrak{g}_1, d_1) \to (\mathfrak{g}_2, d_2)$ such that $T^1 : \mathfrak{g}_1 \to \mathfrak{g}_2$ is $\varphi_{\text{HKR}}$. Let $K : (\mathfrak{g}_1, d_1) \to (\mathfrak{g}_2, d_2)$ be any other $G_\infty$-morphism with $K^1 = \varphi_{\text{HKR}}$ (for example any lift of a Kontsevich formality map or any $G_\infty$-maps lifting another $C_\infty$-morphism).

**Theorem 3.1.** There exists a map $h : \wedge \mathfrak{g}_1^\otimes \to \wedge \mathfrak{g}_2^\otimes$ such that $T - K = h \circ d_1 + d_2 \circ h$.

In other words the formality $G_\infty$-morphisms $K$ and $T$ are homotopic.

The maps $T$ and $K$ are elements of the cochain complex $\left( \text{Hom}(\wedge \mathfrak{g}_1^\otimes, \wedge \mathfrak{g}_2^\otimes), \delta \right)$ with differential given, for all $f \in \text{Hom}(\wedge \mathfrak{g}_1^\otimes, \wedge \mathfrak{g}_2^\otimes)$, by $|f| = k$, by $\delta(f) = d_2 \circ f - (-1)^k f \circ d_1$.

We first compare this cochain complex with the complexes $\left( \text{End}(\wedge \mathfrak{g}_1^\otimes), [d_1; -] \right)$ and $\left( \text{End}(\wedge \mathfrak{g}_2^\otimes), [d_2; -] \right)$, where $[\cdot; -]$ is the graded commutator of morphisms. There are morphisms $T_* : \text{End}(\wedge \mathfrak{g}_1^\otimes) \to \text{Hom}(\wedge \mathfrak{g}_1^\otimes, \wedge \mathfrak{g}_2^\otimes)$, $T^* : \text{End}(\wedge \mathfrak{g}_2^\otimes) \to \text{Hom}(\wedge \mathfrak{g}_1^\otimes, \wedge \mathfrak{g}_2^\otimes)$.
defined, for $f \in \text{End}(\wedge \mathfrak{g}_2^{\bullet\bullet})$ and $g \in \text{Hom}(\wedge \mathfrak{g}_1^{\bullet\bullet}, \wedge \mathfrak{g}_2^{\bullet\bullet})$, by

$$T_*(f) = T \circ f, \ T^*(g) = g \circ T.$$  

**Lemma 3.2.** The morphisms

$$T_* : \left( \text{End}(\wedge \mathfrak{g}_1^{\bullet\bullet}), [d_1; -] \right) \to \left( \text{Hom}(\wedge \mathfrak{g}_1^{\bullet\bullet}, \wedge \mathfrak{g}_2^{\bullet\bullet}), \delta \right)$$

$$\left( \text{End}(\wedge \mathfrak{g}_2^{\bullet\bullet}), [d_2; -] \right) : T^*$$

of cochain complexes are quasi-isomorphisms.

**Remark.** This lemma holds for every manifold $M$ and any $G_\infty$-morphism $T : (g_1, d_1) \to (g_2, d_2)$.

**Proof.** First we show that $T_*$ is a morphism of complexes. Let $f \in \text{End}(\wedge \mathfrak{g}_2^{\bullet\bullet})$ with $|f| = k$; then

$$T_*([d_1; f]) = T \circ d_1 \circ f - (-1)^k T \circ f \circ d_1$$

$$= d_2 \circ (T \circ f) - (-1)^k (T \circ f) \circ d_1$$

$$= \delta(T_*(f)).$$

Let us now prove that $T_*$ is a quasi-isomorphism. For any graded vector space $\mathfrak{g}$, the space $\wedge \mathfrak{g}^{\bullet\bullet}$ has the structure of a filtered space where the $m$-level of the filtration is $F^m(\wedge \mathfrak{g}^{\bullet\bullet}) = \bigoplus_{p_1 + \cdots + p_n = m} \mathfrak{g}^{p_1} \wedge \cdots \wedge \mathfrak{g}^{p_n}$. Clearly the differential $d_1$ and $d_2$ are compatible with the filtrations on $\wedge \mathfrak{g}_1^{\bullet\bullet}$ and $\wedge \mathfrak{g}_2^{\bullet\bullet}$, hence $\text{End}(\wedge \mathfrak{g}_1^{\bullet\bullet})$ and $\text{Hom}(\wedge \mathfrak{g}_1^{\bullet\bullet}, \wedge \mathfrak{g}_2^{\bullet\bullet})$ are filtered cochain complex. This yields two spectral sequences (lying in the first quadrant) $E_n^{p,q}$ and $\tilde{E}_n^{p,q}$ which converge respectively toward the cohomology $H^*(\text{End}(\wedge \mathfrak{g}_1^{\bullet\bullet}))$ and $H^*(\text{Hom}(\wedge \mathfrak{g}_1^{\bullet\bullet}, \wedge \mathfrak{g}_2^{\bullet\bullet}))$. By standard spectral sequence techniques it is enough to prove that the map $T_*^0 : E_0^{p,q} \to \tilde{E}_0^{p,q}$ induced by $T_*$ on the associated graded is a quasi-isomorphism.

The induced differentials on $E_0^{p,q}$ and $\tilde{E}_0^{p,q}$ are respectively $[d_1; -] = 0$ and $d_2 \circ (-) - (-) \circ d_1 = b \circ (-)$, where $b$ is the Hochschild coboundary. By cofreeness property we have the following two isomorphisms:

$$E_0^{p,q} \cong \text{End}(\mathfrak{g}_1), \quad \tilde{E}_0^{p,q} \cong \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_2).$$

The map $T^0_* : E_0^{p,q} \to \tilde{E}_0^{p,q}$ induced by $T_*$ is $\varphi_{\text{HKR}} \circ (-)$. Let $p : \mathfrak{g}_2 \to \mathfrak{g}_1$ be the projection onto the cohomology, i.e. $p \circ \varphi_{\text{HKR}} = \text{id}$. Let $u : \mathfrak{g}_1 \to \mathfrak{g}_2$ be any map satisfying $b(u) = 0$ and set $v = p \circ u \in \text{End}(\mathfrak{g}_1)$. One can choose a map $w : \mathfrak{g}_1 \to \mathfrak{g}_2$ which satisfies for any $x \in \mathfrak{g}_1$ the following identity:

$$\varphi_{\text{HKR}} \circ p \circ u(x) - u(x) = b \circ w(x).$$

It follows that $\varphi_{\text{HKR}}(v)$ has the same class of homology as $u$ which proves the surjectivity of $T^0_*$ in cohomology. The identity $p \circ \varphi_{\text{HKR}} = \text{id}$ easily implies that $T^0_*$ is also injective in cohomology which finishes the proof of the lemma for $T_*$. The proof that $T^*$ is also a quasi-isomorphism is analogous. 

**Proof of Theorem 3.1.** It is easy to check that $T - K$ is a cocycle in

$$\left( \text{Hom}(\wedge \mathfrak{g}_1^{\bullet\bullet}, \wedge \mathfrak{g}_2^{\bullet\bullet}), \delta \right).$$
The complex of cochain \( \left( \text{End}(\wedge g_1^{\bigotimes \bullet}), [d_1, -] \right) \cong \left( \text{Hom}(\wedge g_1^{\bigotimes \bullet}, g_1), [d_1, -] \right) \) is tri-graded with \(|\cdot|\) being the degree coming from the graduation of \(g_1\) and any element \(x\) lying in \(g_1^{\bigotimes p_1} \wedge \cdots \wedge g_1^{\bigotimes p_q}\) satisfies \(|x|_2 = q - 1, |x|_3 = p_1 + \cdots + p_q - q\). In the case \(M = \mathbb{R}^n\), the cohomology \(H^* \left( \text{End}(\wedge g_1^{\bigotimes \bullet}), [d_1, -] \right) \) is concentrated in bidegree \((|\cdot|_2, |\cdot|_3) = (0, 0)\) (see [La, Hi]). By Lemma 3.2, this is also the case for the cochain complex \( \left( \text{Hom}(\wedge g_1^{\bigotimes \bullet}, g_1^{\bigotimes \bullet}), \delta \right) \). Thus, its cohomology classes are determined by complex morphisms \((g_1, 0) \rightarrow (g_2, d_2^1)\), and it is enough to prove that \(T\) and \(K\) determine the same complex morphism \((g_1, 0) \rightarrow (g_2, d_2^1 = b)\) which is clear because \(T^1\) and \(K^1\) are both equal to the Hochschild-Kostant-Rosenberg map. □

**Remark 3.3.** It is possible to have an explicit formula for the map \(h\) in Theorem 3.1. In fact the quasi-isomorphism coming from Lemma 3.2 can be made explicit using explicit homotopy formulae for the Hochschild-Kostant-Rosenberg map (see [Ha] for example) and deformation retract techniques (instead of spectral sequences) as in [Ka]. The same techniques also apply to give explicit formulae for the quasi-isomorphism giving the acyclicity of \( \left( \text{End}(\wedge g_1^{\bigotimes \bullet}), [d_1, -] \right) \) in the proof of Theorem 3.1 (see [GH] for example).

**References**


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