LIFTS OF $C_{\infty}$- AND $L_{\infty}$-MORPHISMS TO $G_{\infty}$-MORPHISMS

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Abstract. Let $g_2$ be the Hochschild complex of cochains on $C^{\infty}(\mathbb{R}^n)$ and let $g_1$ be the space of multivector fields on $\mathbb{R}^n$. In this paper we prove that given any $G_{\infty}$-structure (i.e. Gerstenhaber algebra up to homotopy structure) on $g_2$, and any $C_{\infty}$-morphism $\varphi$ (i.e. morphism of a commutative, associative algebra up to homotopy) between $g_1$ and $g_2$, there exists a $G_{\infty}$-morphism $\Phi$ between $g_1$ and $g_2$ that restricts to $\varphi$. We also show that any $L_{\infty}$-morphism (i.e. morphism of a Lie algebra up to homotopy), in particular the one constructed by Kontsevich, can be deformed into a $G_{\infty}$-morphism, using Tamarkin’s method for any $G_{\infty}$-structure on $g_2$. We also show that any two of such $G_{\infty}$-morphisms are homotopic.

0. Introduction

Let $M$ be a differential manifold and let $g_2 = (C^*(A, A), b)$ be the Hochschild cochain complex on $A = C^\infty(M)$. The classical Hochschild-Kostant-Rosenberg theorem states that the cohomology of $g_2$ is the graded Lie algebra $g_1 = \Gamma(M, \Lambda^*TM)$ of multivector fields on $M$. There is also a graded Lie algebra structure on $g_2$ given by the Gerstenhaber bracket. In particular $g_1$ and $g_2$ are also Lie algebras up to homotopy ($L_{\infty}$-algebra for short). In the case $M = \mathbb{R}^n$, using different methods, Kontsevich ([Ko1] and [Ko2]) and Tamarkin ([Ta]) have proved the existence of Lie homomorphisms “up to homotopy” ($L_{\infty}$-morphisms) from $g_1$ to $g_2$. Kontsevich’s proof uses graph complex and is related to multizeta functions, whereas Tamarkin’s construction uses the existence of Drinfeld’s associators. In fact Tamarkin’s $L_{\infty}$-morphism comes from the restriction of a Gerstenhaber algebra up to homotopy homomorphism ($G_{\infty}$-morphism) from $g_1$ to $g_2$. The $G_{\infty}$-algebra structure on $g_1$ is induced by its classical Gerstenhaber algebra structure and a far less trivial $G_{\infty}$-structure on $g_2$ was proved to exist by Tamarkin [Ta] and relies on Drinfeld’s associator. Tamarkin’s $G_{\infty}$-morphism also restricts into a commutative, associative up to homotopy morphism ($C_{\infty}$-morphism for short). The $C_{\infty}$-structure on $g_2$ (given by restriction of the $G_{\infty}$-one) highly depends on Drinfeld’s associator, and any two choices of a Drinfeld associator yields a priori different $C_{\infty}$-structures. When $M$ is a Poisson manifold, Kontsevich and Tamarkin homomorphisms imply the existence of a star-product (see [BFFLS1] and [BFFLS2] for a definition). A connection between the two approaches has been given in [KS] but the morphisms

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given by Kontsevich and Tamarkin are not the same. The aim of this paper is to show that, given any $G_\infty$-structure on $\mathfrak{g}_2$ and any $C_\infty$-morphism $\varphi$ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$, there exists a $G_\infty$-morphism $\Phi$ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ that restricts to $\varphi$. We also show that any $L_\infty$-morphism can be deformed into a $G_\infty$-one.

In the first section, we fix notation and recall the definitions of $L_\infty$- and $G_\infty$-structures. In the second section we state and prove the main theorem. In the last section we show that any two $G_\infty$-morphisms given by Tamarkin’s method are homotopic.

**Remark.** In the sequel, unless otherwise stated, the manifold $M$ is $\mathbb{R}^n$ for some $n \geq 1$. Most results could be generalized to other manifolds using techniques of Kontsevich [Ko1] (see also [TS], [CFT]).

1. **$C_\infty$, $L_\infty$- and $G_\infty$-structures**

For any graded vector space $\mathfrak{g}$, we choose the following degree on $\wedge^\bullet \mathfrak{g}$: if $X_1, \ldots, X_k$ are homogeneous elements of respective degree $|X_1|, \ldots, |X_k|$, then

$$|X_1 \wedge \cdots \wedge X_k| = |X_1| + \cdots + |X_k| - k.$$

In particular the component $\mathfrak{g} = \wedge^1 \mathfrak{g} \subset \wedge^\bullet \mathfrak{g}$ is the same as the space $\mathfrak{g}$ with degree shifted by one. The space $\wedge^\bullet \mathfrak{g}$ with the deconcatenation cobracket is the cofree cocommutative coalgebra on $\mathfrak{g}$ with degree shifted by one (see [LS], Section 2). Any degree one map $d^k : \wedge^k \mathfrak{g} \to \mathfrak{g}$ ($k \geq 1$) extends into a derivation $d^k : \wedge^\bullet \mathfrak{g} \to \wedge^\bullet \mathfrak{g}$ of the coalgebra $\wedge^\bullet \mathfrak{g}$ by cofreeness property.

**Definition 1.1.** A vector space $\mathfrak{g}$ is endowed with a $L_\infty$-algebra (Lie algebras “up to homotopy”) structure if there are degree one linear maps $m^1, \ldots, m^n$ such that if we extend them to maps $\wedge^1 \mathfrak{g} \to \wedge^1 \mathfrak{g}$, then $d \circ d = 0$ where $d$ is the derivation

$$d = m^1 + m^{1,1} + \cdots + m^{1,\cdots,1} + \cdots.$$

For more details on $L_\infty$-structures, see [LS]. It follows from the definition that a $L_\infty$-algebra structure induces a differential coalgebra structure on $\wedge^\bullet \mathfrak{g}$ and that the map $m^1 : \mathfrak{g} \to \mathfrak{g}$ is a differential. If $m^1, \ldots, m^n$ are 0 for $k \geq 3$, we get the usual definition of (differential if $m^1 \neq 0$) graded Lie algebras.

For any graded vector space $\mathfrak{g}$, we denote $\mathfrak{g}^{\otimes n}$ to be the quotient of $\mathfrak{g}^{\otimes n}$ by the image of all shuffles of length $n$ (see [GK] or [GH] for details). The graded vector space $\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$ is a quotient coalgebra of the tensor coalgebra $\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$. It is well known that this coalgebra $\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$ is the cofree Lie coalgebra on the vector space $\mathfrak{g}$ (with degree shifted by minus one).

**Definition 1.2.** A $C_\infty$-algebra (commutative and associative “up to homotopy”) structure on a vector space $\mathfrak{g}$ is given by a collection of degree one linear maps $m^k : \mathfrak{g}^{\otimes k} \to \mathfrak{g}$ such that if we extend them to maps $\bigoplus \mathfrak{g}^{\otimes 1} \to \bigoplus \mathfrak{g}^{\otimes 1}$, then $d \circ d = 0$ where $d$ is the derivation

$$d = m^1 + m^2 + m^3 + \cdots.$$

In particular a $C_\infty$-algebra is an $A_\infty$-algebra.

For any space $\mathfrak{g}$, we denote by $\wedge^\bullet \mathfrak{g}^{\otimes \bullet}$ the graded space

$$\wedge^\bullet \mathfrak{g}^{\otimes \bullet} = \bigoplus_{m \geq 1, \; p_1 + \cdots + p_n = m} \mathfrak{g}^{\otimes p_1} \wedge \cdots \wedge \mathfrak{g}^{\otimes p_n}.$$
We use the following grading on \( \Lambda^\bullet \mathfrak{g}^{\otimes \bullet} \): for \( x_1, \ldots, x_n \in \mathfrak{g} \), we define
\[
| x_1 \otimes \cdots \otimes x_1 \wedge \cdots \wedge x_n \otimes \cdots \otimes x_n | = \sum_{i_1}^{p_1} |x_1^{i_1}| + \cdots + \sum_{i_n}^{p_n} |x_n^{i_n}| - n.
\]

Note that the induced grading on \( \Lambda^\bullet \mathfrak{g} \subset \Lambda^\bullet \mathfrak{g}^{\otimes \bullet} \) is the same as the one introduced above. The cobracket on \( \oplus \mathfrak{g}^{\otimes \bullet} \) and the coproduct on \( \Lambda^\bullet \mathfrak{g}^{\otimes \bullet} \) extend to a cobracket and a coproduct on \( \Lambda^\bullet \mathfrak{g}^{\otimes \bullet} \) which yield a Gerstenhaber coalgebra structure on \( \Lambda^\bullet \mathfrak{g}^{\otimes \bullet} \).

It is well known that this coalgebra structure is cofree (see [Gi], Section 3, for example).

**Definition 1.3.** A \( G_\infty \)-algebra (Gerstenhaber algebra “up to homotopy”) structure on a graded vector space \( \mathfrak{g} \) is given by a collection of degree one maps
\[
m^{p_1, \ldots, p_n} : \mathfrak{g}^{\otimes p_1} \wedge \cdots \wedge \mathfrak{g}^{\otimes p_n} \to \mathfrak{g}
\]
indexed by \( p_1, \ldots, p_n \geq 1 \) such that their canonical extension \( \Lambda^\bullet \mathfrak{g}^{\otimes \bullet} \to \Lambda^\bullet \mathfrak{g}^{\otimes \bullet} \) satisfies \( d \circ d = 0 \), where
\[
d = \sum_{m \geq 1, \ p_1 + \cdots + p_n = m} m^{p_1, \ldots, p_n}.
\]

Again, as the coalgebra structure of \( \Lambda^\bullet \mathfrak{g}^{\otimes \bullet} \) is cofree, the map \( d \) makes \( \Lambda^\bullet \mathfrak{g}^{\otimes \bullet} \) into a differential coalgebra. If the maps \( m^{p_1, \ldots, p_n} \) are 0 for \( (p_1, p_2, \ldots) \neq (1, 0, \ldots), \ (1, 1, 0, \ldots) \) or \( (2, 0, \ldots) \), we get the usual definition of (differential if \( m^1 \neq 0 \)) Gerstenhaber algebra.

The space of multivector fields \( \mathfrak{g}_1 \) is endowed with a graded Lie bracket \( [-, -]_S \) called the Schouten bracket (see [Kos]). This Lie algebra can be extended into a Gerstenhaber algebra, with commutative structure given by the exterior product \( \circ \). Setting \( D_1 = m_1^{1,1} + m_2^{1,1} \), where \( m_1^{1,1} : \wedge^2 \mathfrak{g}_1 \to \mathfrak{g}_1 \), and \( m_2^{1,1} : \mathfrak{g}_1^{\otimes 2} \to \mathfrak{g}_1 \) are the extension of the Schouten bracket and the exterior product, we find that \( (\mathfrak{g}_1, D_1) \) is a \( G_\infty \)-algebra.

In the same way, one can define a differential Lie algebra structure on the vector space \( \mathfrak{g}_2 = C(A, A) = \bigoplus_{k \geq 0} C^k(A, A) \), the space of Hochschild cochains (generated by differential \( k \)-linear maps from \( A^k \) to \( A \)), where \( A = C^\infty(M) \) is the algebra of smooth differential functions over \( M \). Its bracket \( [-, -]_G \), called the Gerstenhaber bracket, is defined, for \( D, E \in \mathfrak{g}_2 \), by
\[
[D, E]_G = \{D|E\} - (-1)^{|E||D|}\{E|D\},
\]
where
\[
\{D|E\}(x_1, \ldots, x_{d+e-1}) = \sum_{i \geq 0} (-1)^{|E||i|} D(x_1, \ldots, x_i, E(x_{i+1}, \ldots, x_{i+e}), \ldots).
\]

The space \( \mathfrak{g}_2 \) has a grading defined by \( |D| = k \iff D \in C^{k+1}(A, A) \) and its differential is \( b = [m, -]_G \), where \( m \in C^2(A, A) \) is the commutative multiplication on \( A \).

Tamarkin (see [Ta] or also [GH]) stated the existence of a \( G_\infty \)-structure on \( \mathfrak{g}_2 \) (depending on a choice of a Drinfeld associator) given by a differential \( d_2 = m_1^{2,1} + m_2^{2,1} + \cdots + m_2^{p_1,\ldots,p_n} + \cdots \), on \( \Lambda^\bullet \mathfrak{g}_2^{\otimes \bullet} \) satisfying \( d_2 \circ d_2 = 0 \). Although
this structure is non-explicit, it satisfies the following three properties:

(a) \( m^1_2 \) is the extension of the differential \( b \).

(b) \( m^{1,1}_2 \) is the extension of the Gerstenhaber bracket \([−, −]_G\)

\[
(1.1) \quad m^{1,1}_2 = 0.
\]

(c) \( m^2 \) induces the exterior product in cohomology and the

collection of the \( (m^k)_{k \geq 1} \) defines a \( C_\infty \)-structure on \( \mathfrak{g}_2 \).

**Definition 1.4.** A \( L_\infty \)-morphism between two \( L_\infty \)-algebras \((\mathfrak{g}_1, d_1 = m^1_1 + \cdots)\)
and \((\mathfrak{g}_2, d_2 = m^1_2 + \cdots)\) is a morphism of differential coalgebras

\[
\varphi : (\bigwedge \mathfrak{g}_1, d_1) \to (\bigwedge \mathfrak{g}_2, d_2).
\]

Such a map \( \varphi \) is uniquely determined by a collection of maps \( \varphi^n : \bigwedge^n \mathfrak{g}_1 \to \mathfrak{g}_2 \) (again by cofreeness properties). In the case \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) are respectively the graded Lie algebra \((\Gamma(M, \Lambda T^*M), [−, −]_S)\)
and the differential graded Lie algebra \((C(A, A), [−, −]_G)\), the formality theorems of Kontsevich and Tamarkin state the
existence of a \( L_\infty \)-morphism between \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) such that \( \varphi^1 \) is the Hochschild-
Kostant-Rosenberg quasi-isomorphism.

**Definition 1.5.** A morphism of \( C_\infty \)-algebras between two \( C_\infty \)-algebras \((\mathfrak{g}_1, d_1)\) and
\((\mathfrak{g}_2, d_2)\) is a \( \mathfrak{g} \)-map \( \phi : (\bigwedge \mathfrak{g}_1, d_1) \to (\bigwedge \mathfrak{g}_2, d_2) \) of codifferential coalgebras.

A \( C_\infty \)-morphism is in particular a morphism of \( A_\infty \)-algebras and is uniquely
determined by maps \( \partial^k : \mathfrak{g} \to \mathfrak{g} \).

**Definition 1.6.** A morphism of \( G_\infty \)-algebras between two \( G_\infty \)-algebras \((\mathfrak{g}_1, d_1)\) and
\((\mathfrak{g}_2, d_2)\) is a \( \mathfrak{g} \)-map \( \phi : (\bigwedge \mathfrak{g}_1, d_1) \to (\bigwedge \mathfrak{g}_2, d_2) \) of codifferential coalgebras.

There are coalgebra inclusions \( \bigwedge \mathfrak{g} \to \bigwedge \mathfrak{g} \to \bigwedge \mathfrak{g} \), \( \bigwedge \mathfrak{g} \to \bigwedge \mathfrak{g} \), and it is easy
to check that any \( G_\infty \)-morphism between two \( G_\infty \)-algebras \((\mathfrak{g}, \sum m^{p_1 \cdots p_n})\) and
\((\mathfrak{g}', \sum m^{p_1 \cdots p_n})\) restricts to a \( L_\infty \)-morphism \((\bigwedge \mathfrak{g}, \sum m^{1 \cdots 1}) \to (\bigwedge \mathfrak{g}', \sum m^{1 \cdots 1})\)
and a \( C_\infty \)-morphism \((\bigwedge \mathfrak{g}, \sum m^k) \to (\bigwedge \mathfrak{g}, \sum m^k)\).

2. Main theorem

We keep the notations of the previous section, in particular \( \mathfrak{g}_2 \) is the Hochschild
complex of cochains on \( C_\infty(M) \) and \( \mathfrak{g}_1 \) its cohomology. Here is our main theorem.

**Theorem 2.1.** Given any \( G_\infty \)-structure \( d_2 \) on \( \mathfrak{g}_2 \) satisfying the three properties
of \((1.1)\), and any \( C_\infty \)-morphism \( \varphi \) between \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) such that \( \varphi^1 \) is the Hochschild-
Kostant-Rosenberg map, there exists a \( G_\infty \)-morphism \( \Phi : (\mathfrak{g}_1, d_1) \to (\mathfrak{g}_2, d_2) \) that
restricts to \( \varphi \).

Also, given any \( L_\infty \)-morphism \( \gamma \) between \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) such that \( \gamma^1 \) is the Hochschild-
Kostant-Rosenberg map, there exists a \( G_\infty \)-structure \((\mathfrak{g}_1, d_1')\) on \( \mathfrak{g}_1 \) and \( G_\infty \)-mor-
phism \( \Gamma : (\mathfrak{g}_1, d_1') \to (\mathfrak{g}_2, d_2) \) that restricts to \( \gamma \). Moreover there exists a \( G_\infty \)-
morphism \( \Gamma' : (\mathfrak{g}_1, d_1) \to (\mathfrak{g}_1, d_1) \).
In particular, Theorem 2.1 applies to the formality map of Kontsevich and also to any \( C_\infty \)-map derived (see [Ta], [GH]) from any \( B_\infty \)-structure on \( g_2 \) lifting the Gerstenhaber structure of \( g_1 \).

Let us first recall the proof of Tamarkin’s formality theorem (see [GH] for more details):

1. First one proves there exists a \( G_\infty \)-structure on \( g_2 \), with differential \( d_2 \), as in (1.1).
2. Then, one constructs a \( G_\infty \)-structure on \( g_1 \) given by a differential \( d'_1 \) together with a \( G_\infty \)-morphism \( \Phi \) between \( (g_1, d_1) \) and \((g_2, d_2)\).
3. Finally, one constructs a \( G_\infty \)-morphism \( \Phi' \) between \( (g_1, d_1) \) and \((g_1, d'_1)\).

The composition \( \Phi \circ \Phi' \) is then a \( G_\infty \)-morphism between \((g_1, d_1) \) and \((g_2, d_2)\), thus it restricts to a \( L_\infty \)-morphism between the differential graded Lie algebras \( g_1 \) and \( g_2 \).

We suppose now that, in the first step, we take any \( G_\infty \)-structure on \( g_2 \) given by a differential \( d_2 \). We also suppose we are given a \( C_\infty \)-morphism \( \varphi \) and a \( L_\infty \)-morphism \( \gamma \) between \( g_1 \) and \( g_2 \) satisfying \( \gamma^1 = \varphi^1 = \varphi_{\text{HKR}} \), the Hochschild-Kostant-Rosenberg quasi-isomorphism.

**Proof of Theorem 2.1.** The theorem will follow if we prove that steps 2 and 3 of Tamarkin’s construction are still true with the extra conditions that the restriction of the \( G_\infty \)-morphism \( \Phi \) (resp. \( \Phi' \)) on the \( C_\infty \)-structures is the \( C_\infty \)-morphism \( \varphi : g_1 \to g_2 \) (resp. id).

Let us recall (see [GH]) that the constructions of \( \Phi \) and \( d'_1 \) can be made by induction. For \( i = 1,2 \) and \( n \geq 0 \), let us set

\[
V_i^{[n]} = \bigoplus_{p_1 + \cdots + p_k = n} g_1^{\otimes p_1} \wedge \cdots \wedge g_1^{\otimes p_k}
\]

and \( V_i^{[\leq n]} = \sum_{k \leq n} V_i^{[k]} \). Let \( d_2^{[n]} \) and \( d_2^{[\leq n]} \) be the sums

\[
d_2^{[n]} = \sum_{p_1 + \cdots + p_k = n} d_2^{p_1, \ldots, p_k} \quad \text{and} \quad d_2^{[\leq n]} = \sum_{p \leq n} d_2^{[p]}.
\]

Clearly, \( d_2 = \sum_{n \geq 1} d_2^{[n]} \). In the same way, we denote \( d'_1 = \sum_{n \geq 1} d'_1^{[n]} \) by

\[
d'_1^{[n]} = \sum_{p_1 + \cdots + p_k = n} d'_1^{p_1, \ldots, p_k} \quad \text{and} \quad d'_1^{[\leq n]} = \sum_{1 \leq k \leq n} d'_1^{[k]}.
\]

We know from Section 1 that a morphism \( \Phi : (\wedge g_1^{\mathbf{*}}, d'_1) \to (\wedge g_2^{\mathbf{*}}, d_2) \) is uniquely determined by its components \( \Phi^{p_1, \ldots, p_k} : g_1^{\otimes p_1} \wedge \cdots \wedge g_1^{\otimes p_k} \to g_2 \). Again, we have

\[
\Phi = \sum_{n \geq 1} \Phi^{[n]} = \sum_{p_1 + \cdots + p_k = n} \Phi^{p_1, \ldots, p_k} \quad \text{and} \quad \Phi^{[\leq n]} = \sum_{1 \leq k \leq n} \Phi^{[k]}.
\]

We want to construct the maps \( d'_1^{[n]} \) and \( \Phi^{[n]} \) by induction with the initial condition

\[
d'_1^{[1]} = 0 \quad \text{and} \quad \Phi^{[1]} = \varphi_{\text{HKR}},
\]

where \( \varphi_{\text{HKR}} : (g_1,0) \to (g_2,b) \) is the Hochschild-Kostant-Rosenberg quasi-isomorphism (see [HKR]) defined, for \( \alpha \in g_1, f_1, \ldots, f_n \in A \), by

\[
\varphi_{\text{HKR}} : \alpha \mapsto \big((f_1, \ldots, f_n) \mapsto (\alpha, df_1 \wedge \cdots \wedge df_n)\big).
\]
Moreover, we want the following extra conditions to be true:

$$\Phi^{k \geq 2} = \varphi^k, \quad d'^2_1 = d''_1, \quad d'^{k \geq 3} = 0.$$  

(2.3)

Now suppose the construction is done for $n - 1$ ($n \geq 2$), i.e., we have built maps $(d'^{[i]}_1)_{i \leq n-1}$ and $(\Phi^{[i]})_{i \leq n-1}$ satisfying conditions (2.3) and

$$\Phi^{[i \leq n-1]} \circ d'^{[i \leq n-1]}_1 = d''_1 \circ \Phi^{[i \leq n-1]} \quad \text{on } V^{[i \leq n-1]}_1$$

and

$$d'^{[i \leq n-1]}_1 \circ d''_1 = 0 \quad \text{on } V^{[i \leq n]}_1.$$  

(2.4)

In [GH], we prove that for any such $(d'^{[i]}_1)_{i \leq n-1}$ and $(\Phi^{[i]})_{i \leq n-1}$, one can construct $d'^{[i]}_1$ and $\Phi^{[i]}$ such that condition (2.4) is true for $n$ instead of $n - 1$. To complete the proof of Theorem 2.1 (step 2), we have to show that $d'^{[n]}_1$ and $\Phi^{[n]}$ can be chosen to satisfy conditions (2.3). In equation (2.4), the terms $d''_1$ and $\Phi^k$ only act on $V^{[n]}_1$. So one can replace $\Phi^n$ with $\varphi^n$, $d''_1$ with $d''_1$ (or $d'_{1, i \geq 3}$ with 0) provided conditions (2.4) are still satisfied on $V^n_1$. The other terms acting on $V^{[n]}_1$ in equation (2.4) only involve terms $\Phi^m = \varphi^m$ and $d'^m_1$. Then conditions (2.4) on $V^{[1 \ldots n-1]}_1$ are the equations that should be satisfied by a $C_{\infty}$-morphism between the $C_{\infty}$-algebras $(g_1, d''_1 = d''_1)$ and $(g_1, \sum_{k \geq 1} d''_1)$ restricted to $V^n_1$. Hence by hypothesis on $\varphi$ the conditions hold.

Similarly the construction of $\Phi'$ can be made by induction. Let us recall the proof given in [GH]. Again a morphism $\Phi' : (\wedge g^{\otimes^*}, d'_1) \to (\wedge g^{\otimes^*}, d''_1)$ is uniquely determined by its components $\Phi'^{p_1 \ldots p_k} : g_1^{\otimes p_1} \wedge \ldots \wedge g_1^{\otimes p_k} \to g_1$. We write $\Phi' = \sum_{n \geq 1} \Phi'^{[n]}$ with

$$\Phi'^{[n]} = \sum_{p_1 + \ldots + p_k = n} \Phi'^{p_1 \ldots p_k} \quad \text{and} \quad \Phi'^{[i \leq n]} = \sum_{1 \leq k \leq n} \Phi'^{[k]}.$$  

We construct the maps $\Phi'^{[n]}$ by induction with the initial condition $\Phi'^{[1]} = \text{id}$. Moreover, we want the following extra conditions to be true:

$$\Phi'^n = 0 \quad \text{for } n \geq 2.$$  

(2.5)

Now suppose the construction is done for $n - 1$ ($n \geq 2$), i.e., we have built maps $(\Phi'^{[i]})_{i \leq n-1}$ satisfying conditions (2.5) and

$$\Phi'^{[i \leq n-1]} \circ d'^{[i \leq n-1]}_1 = d''_1 \circ \Phi'^{[i \leq n-1]} \quad \text{on } V^{[i \leq n-1]}_1.$$  

(2.6)

In [GH], we prove that for any such $(\Phi'^{[i]})_{i \leq n-1}$, one can construct $\Phi'^{[n]}$ such that condition (2.6) is true for $n$ instead of $n - 1$ in the following way: making the equation $\Phi' d'_1 = d''_1 \Phi'$ on $V^{[n+1]}_1$ explicit, we get

$$\Phi'^{[i \leq n+1]} d'_1 = d''_1 \Phi'^{[n+1]}.\Phi'^{[i \leq n]}.$$  

(2.7)

If we now take into account that $d^{[i]}_1 = 0$ for $i \neq 2$, $d'_1^{[1]} = 0$ and that on $V^{[n+1]}_1$ we have $\Phi'^{[k]} d''_1 = d''_1 \Phi'^{[k]} = 0$ for $k + l > n + 2$, the identity (2.7) becomes

$$\Phi'^{[i \leq n+1]} d'_1 = \sum_{k=2}^{n+1} d'^{[k]}_1 \Phi'^{[i \leq n-k+2]}.$$
Since $d_1'[2] = d_1[2]$, (2.7) is equivalent to
\[ d_1[2] \delta[\leq n] - \delta[\leq n] d_1[2] = \left[ d_1[2], \Gamma[\leq n] \right] - \sum_{k=3}^{n+1} d_k' \Gamma[\leq n-k+2]. \]

Note that $d_1'[2] = m_{1,1} + m_2^2$. Then the construction will be possible when the term
\[ \sum_{k=3}^{n+1} d_k' \Gamma[\leq n-k+2], \]
consisting of maps which restrict to zero on $\bigoplus_{n \geq 2} g_1 \otimes \ldots \otimes g_1$, is a coboundary in the subcomplex of $(\text{End}(\wedge g_1 \otimes), [d_1'[2], -])$ consisting of maps which restrict to zero on $\bigoplus_{n \geq 2} g_1 \otimes \ldots \otimes g_1$. It is always a cocycle by straightforward computation (see [GH]), and the subcomplex is acyclic because both $(\text{End}(\wedge g_1 \otimes), [d_1'[2], -])$ and the Harrison cohomology of $g_1$ are trivial according to Tamarkin [13] (see also [GH], Proposition 5.1, and [HI], 5.4).

In the case of the $L_\infty$-morphism $\gamma$, the first step is similar: the fact that $\gamma$ is a $L_\infty$-map enables us to build a $G_\infty$-structure $(g_1', d_1')$ on $g_1$ and a $G_\infty$-morphism $\Gamma : (g_1, d_1) \to (g_2, d_2)$ such that:
\[ \Gamma_{1, \ldots, 1} = \gamma_{1, \ldots, 1}, \quad d_1'^{1,1} = d_1^{1,1}, \quad d_1'^{1,1, \ldots, 1} = 0. \]

For the second step, we have to build a map $\Gamma'$ satisfying the equation
\[ d_1[2] \Gamma'[\leq n] - \Gamma'[\leq n] d_1[2] = \left[ d_1[2], \Gamma'[\leq n] \right] - \sum_{k=3}^{n+1} d_k' \Gamma'[\leq n-k+2] \]
on $V_1^{[n+1]}$ for any $n \geq 1$. Again, because Tamarkin has proved that the complex $(\text{End}(\wedge g_1 \otimes), [d_1'[2], -])$ is acyclic (we are in the case $M = \mathbb{R}^n$), the result follows from the fact that $\sum_{k=3}^{n+1} d_k' \Gamma'[\leq n-k+2]$ is a cocycle. The difference with the $C_\infty$-case is that the $\Gamma^{1,1, \ldots, 1}$ could be non-zero.

\[ \square \]

3. The difference between two $G_\infty$-maps

In this section we investigate the difference between two different $G_\infty$-formality maps.

We fix once and for all a $G_\infty$-structure on $g_2$ (given by a differential $d_2$) satisfying the conditions (1.1) and a morphism of $G_\infty$-algebras $T : (g_1, d_1) \to (g_2, d_2)$ such that $T^1 : g_1 \to g_2$ is $\varphi_{\text{HHR}}$. Let $K : (g_1, d_1) \to (g_2, d_2)$ be any other $G_\infty$-morphism with $K^1 = \varphi_{\text{HHR}}$ (for example any lift of a Kontsevich formality map or any $G_\infty$-maps lifting another $C_\infty$-morphism).

**Theorem 3.1.** There exists a map $h : \wedge g_1 \otimes \to \wedge g_2 \otimes$ such that
\[ T - K = h \circ d_1 + d_2 \circ h. \]

In other words the formality $G_\infty$-morphisms $K$ and $T$ are homotopic.

The maps $T$ and $K$ are elements of the cochain complex $\left( \text{Hom}(\wedge g_1 \otimes, \wedge g_2 \otimes), \delta \right)$ with differential given, for all $f \in \text{Hom}(\wedge g_1 \otimes, \wedge g_2 \otimes)$, $|f| = k$, by
\[ \delta(f) = d_2 \circ f - (-1)^k f \circ d_1. \]

We first compare this cochain complex with the complexes $\left( \text{End}(\wedge g_1 \otimes), [d_1; -] \right)$ and $\left( \text{End}(\wedge g_2 \otimes), [d_2; -] \right)$, where $[-; -]$ is the graded commutator of morphisms. There are morphisms
\[ T_* : \text{End}(\wedge g_1 \otimes) \to \text{Hom}(\wedge g_1 \otimes, \wedge g_2 \otimes), \quad T^* : \text{End}(\wedge g_2 \otimes) \to \text{Hom}(\wedge g_1 \otimes, \wedge g_2 \otimes). \]
defined, for \( f \in \text{End}(\wedge g_2^\bullet) \) and \( g \in \text{Hom}(\wedge g_1^\bullet, \wedge g_2^\bullet) \), by
\[
T_*(f) = T \circ f, \quad T^*(g) = g \circ T.
\]

Lemma 3.2. The morphisms
\[
T_* : \left( \text{End}(\wedge g_1^\bullet), [d_1; -] \right) \to \left( \text{Hom}(\wedge g_1^\bullet, \wedge g_2^\bullet), \delta \right)
\]
\[
\left( \text{End}(\wedge g_2^\bullet), [d_2; -] \right) : T^*
\]
of cochain complexes are quasi-isomorphisms.

Remark. This lemma holds for every manifold \( M \) and any \( G_{\infty} \)-morphism \( T : (g_1, d_1) \to (g_2, d_2) \).

Proof. First we show that \( T_* \) is a morphism of complexes. Let \( f \in \text{End}(\wedge g_2^\bullet) \)
with \(|f| = k\); then
\[
T_*([d_1; f]) = T \circ d_1 \circ f - (-1)^k T \circ f \circ d_1
\]
\[
d_2 \circ (T \circ f) - (-1)^k (T \circ f) \circ d_1 = \delta(T_*(f)).
\]

Let us now prove that \( T_* \) is a quasi-isomorphism. For any graded vector space \( g \), the
space \( \wedge g^\bullet \) has the structure of a filtered space where the \( m \)-level of the filtration
is \( F^m(\wedge g^\bullet) = \bigoplus_{p_1 + \cdots + p_n = m} \wedge g_{[p_1]} \wedge \cdots \wedge g_{[p_n]} \). Clearly the differential \( d_1 \) and
\( d_2 \) are compatible with the filtrations on \( \wedge g_1^\bullet \) and \( \wedge g_2^\bullet \), hence \( \text{End}(\wedge g_1^\bullet) \) and
\( \text{Hom}(\wedge g_1^\bullet, \wedge g_2^\bullet) \) are filtered cochain complex. This yields two spectral sequences
(lying in the first quadrant) \( E_*^\bullet \) and \( \tilde{E}_*^\bullet \) which converge respectively toward the
cohomology \( H^\bullet(\text{End}(\wedge g_1^\bullet)) \) and \( H^\bullet(\text{Hom}(\wedge g_1^\bullet, \wedge g_2^\bullet)) \). By standard spectral
sequence techniques it is enough to prove that the map \( T_*^0 : E_0^\bullet \to \tilde{E}_0^\bullet \) induced by \( T_* \) on the associated graded is a quasi-isomorphism.

The induced differentials on \( E_0^\bullet \) and \( \tilde{E}_0^\bullet \) are respectively \([d_1; -] = 0 \) and
\( d_1 \circ (-) - (-) \circ d_1 = b \circ (-) \), where \( b \) is the Hochschild coboundary. By cofreeness
property we have the following two isomorphisms:
\[
E_0^\bullet \cong \text{End}(g_1), \quad \tilde{E}_0^\bullet \cong \text{Hom}(g_1, g_2).
\]
The map \( T_*^0 : E_0^\bullet \to \tilde{E}_0^\bullet \) induced by \( T_* \) is \( \varphi_{\text{HHR}} \circ (-) \). Let \( p : g_2 \to g_1 \) be the
projection onto the cohomology, i.e. \( p \circ \varphi_{\text{HHR}} = \text{id} \). Let \( u : g_1 \to g_2 \) be any map
satisfying \( b(u) = 0 \) and set \( v = p \circ u \in \text{End}(g_1) \). One can choose a map \( w : g_1 \to g_2 \)
which satisfies for any \( x \in g_1 \) the following identity:
\[
\varphi_{\text{HHR}} \circ p \circ u(x) - u(x) = b \circ w(x).
\]
It follows that \( \varphi_{\text{HHR}}(v) \) has the same class of homology as \( u \) which proves the
surjectivity of \( T_*^0 \) in cohomology. The identity \( p \circ \varphi_{\text{HHR}} = \text{id} \) easily implies that \( T_*^0 \)
is also injective in cohomology which finishes the proof of the lemma for \( T_* \).

The proof that \( T^* \) is also a quasi-isomorphism is analogous.

\( \square \)

Proof of Theorem 3.1. It is easy to check that \( T - K \) is a cocycle in
\[
\left( \text{Hom}(\wedge g_1^\bullet, \wedge g_2^\bullet), \delta \right).
\]
The complex of cochain \( \text{End}(\wedge g_1^{\bullet \bullet}, [d_1, -]) \cong \text{Hom}(\wedge g_1^{\bullet \bullet}, g_1, [d_1, -]) \) is tri-graded with \(|x|_1\) being the degree coming from the graduation of \( g_1 \) and any element \( x \) lying in \( g_1^{p_1} \wedge \cdots \wedge g_1^{p_q} \) satisfies \(|x|_2 = q - 1, |x|_3 = p_1 + \cdots + p_q - q \). In the case \( M = \mathbb{R}^n \), the cohomology \( H^*(\text{End}(\wedge g_1^{\bullet \bullet}, [d_1, -])) \) is concentrated in bidegree \((|x|_2, |x|_3) = (0, 0)\) (see \([La], [Hi]\)). By Lemma 3.2, this is also the case for the cochain complex \( \text{Hom}(\wedge g_1^{\bullet \bullet}, g_1, [d_1, -])\). Thus, its cohomology classes are determined by complex morphisms \((g_1, 0) \to (g_2, d_2)\), and it is enough to prove that \( T \) and \( K \) determine the same complex morphism \((g_1, 0) \to (g_2, d_2 = b)\) which is clear because \( T^1 \) and \( K^1 \) are both equal to the Hochschild-Kostant-Rosenberg map. \( \square \)

**Remark 3.3.** It is possible to have an explicit formula for the map \( h \) in Theorem 3.1. In fact the quasi-isomorphism coming from Lemma 3.2 can be made explicit using explicit homotopy formulae for the Hochschild-Kostant-Rosenberg map (see \([La]\) for example) and deformation retract techniques (instead of spectral sequences) as in \([Ka]\). The same techniques also apply to give explicit formulae for the quasi-isomorphism giving the acyclicity of \( \text{End}(\wedge g_1^{\bullet \bullet}, [d_1, -]) \) in the proof of Theorem 3.1 (see \([GH]\) for example).

**References**


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