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STABILITY OF WAVELET FRAMES WITH MATRIX DILATIONS

OLE CHRISTENSEN AND WENCHANG SUN

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ABSTRACT. Under certain assumptions we show that a wavelet frame

$$\{\tau(A_j, b_{j,k})\psi\}_{j,k\in\mathbb{Z}} := \{|\det A_j|^{-1/2}\psi(A_j^{-1}(x - b_{j,k}))\}_{j,k\in\mathbb{Z}}$$

in $L^2(\mathbb{R}^d)$ remains a frame when the dilation matrices A_j and the translation parameters $b_{j,k}$ are perturbed. As a special case of our result, we obtain that if $\{\tau(A^j, A^jBn)\psi\}_{j\in\mathbb{Z},n\in\mathbb{Z}^d}$ is a frame for an expansive matrix A and an invertible matrix B, then $\{\tau(A'_j, A^jB\lambda_n)\psi\}_{j\in\mathbb{Z},n\in\mathbb{Z}^d}$ is a frame if $\|A^{-j}A'_j - I\|_2 \le \varepsilon$ and $\|\lambda_n - n\|_\infty \le \eta$ for sufficiently small $\varepsilon, \eta > 0$.

1. Introduction and main results

Wavelet frames are widely used in modern time-frequency analysis. Necessary and sufficient conditions for wavelet systems with dilation by powers of two to be frames are well known [6, 8, 9, 11, 17]. The matrix dilated multivariate case is studied, e.g., in [14], and for irregular matrix dilations we refer to [1, 22].

The stability of frames is required in applications and this problem is well studied for the case of univariate wavelet frames. It was shown in [7, 10, 18, 19, 20] that a wavelet frame in $L^2(\mathbb{R})$ with a nice generating function remains a frame when the translation and dilation parameters are perturbed. Explicit stability bounds are given for some cases.

For the multivariate case, especially for wavelet frames with matrix dilations, very few results are known. One of the main difficulties is that matrix dilations are quite different from scalar dilations and therefore many methods for dealing with univariate wavelet frames do not apply for the multivariate case. As far as we know, the only published result in this aspect appears in [22, Theorem 2.4]; at the end of this section we compare our findings with that theorem.

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We now go more into detail with the content of the present paper. For a $d \times d$ real matrix A and a vector $b \in \mathbb{R}^d$, define the operator τ by

$$(\tau(A, b)\psi)(x) = |\det A|^{-1/2}\psi(A^{-1}(x - b)).$$

Our goal is to study the stability of wavelet frames of the form $\{\tau(A_j, b_{j,k})\psi : j, k \in \mathbb{Z}\}$, where $\{b_{j,k} : k \in \mathbb{Z}\}$ is an arbitrary sequence of vectors in $\mathbb{R}^d$ and $\{A_j : j \in \mathbb{Z}\}$ is a sequence of matrices satisfying

$$\|A_{j+n_0}^T \omega\|_2 \geq \rho \|A_j^T \omega\|_2, \qquad \forall j \in \mathbb{Z}, \omega \in \mathbb{R}^d, \tag{1.1}$$

for some integer $n_0 > 0$ and constant $\rho > 1$. Here A^T denotes the transpose of A.

We also need the following notation. The norm of a matrix A is defined by $\|A\|_2 = \sup_{\|x\|_2=1} \|Ax\|_2$. Obviously, $\|A\|_2 = \|A^T\|_2 = \sup_{\|x\|_2=\|y\|_2=1} |\langle Ax, y\rangle|$.

We call a matrix A expansive if there is some $\rho > 1$ such that

$$\|A^T \omega\|_2 \geq \rho \|\omega\|_2, \qquad \forall \omega \in \mathbb{R}^d.$$

Let $\delta > 0$ be a constant. Define the Wiener space as

$$W(\mathbb{R}^d) = \Big\{ f : f \text{ is continuous and } \|f\|_{W,\delta} = \sum_{n \in \mathbb{Z}^d} \|f \cdot \chi_{E_n}\|_\infty < +\infty \Big\},$$

where $E_n = \prod_{\ell=1}^d [n_\ell \delta, (n_\ell + 1)\delta)$, $n = (n_1, \cdots, n_d)$. It is easy to see that $W(\mathbb{R}^d)$ is independent of the choice of δ and different choices give equivalent norms [6].

A sequence $\{b_n : n \in \mathbb{Z}\}$ is said to be δ-uniformly discrete if $\|b_n - b_{n'}\|_\infty \geq \delta > 0$. We call a sequence relatively uniformly discrete if it is a finite union of uniformly discrete sequences.

We use the following set of multi-index: $\alpha = (\alpha_1, \cdots, \alpha_d)$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $(X^\alpha f)(x) = x^\alpha f(x)$, and $(D^\alpha f)(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f(x)$.

Recall that a family of functions $\{\varphi_k : k \in \mathbb{Z}\}$ in $L^2(\mathbb{R}^d)$ is a frame for $L^2(\mathbb{R}^d)$ if there are two positive numbers A and B such that for any $f \in L^2(\mathbb{R}^d)$,

$$A\|f\|^2 \leq \sum_{k \in \mathbb{Z}} |\langle f, \varphi_k \rangle|^2 \leq B\|f\|^2.$$

A and B are called lower and upper frame bounds, respectively. The reason for the interest in frames is that they generalize (and give more freedom than) orthonormal bases; on the other hand, a frame $\{\varphi_k : k \in \mathbb{Z}\}$ for $L^2(\mathbb{R}^d)$ also has the property that each function in $L^2(\mathbb{R}^d)$ can be expanded as an infinite linear combination of the frame elements. We refer to [23, 9, 6] for an overview of frame theory.

We are now ready to state the main result.

Theorem 1.1. *Let $\{\tau(A_j, b_{j,k})\psi : j, k \in \mathbb{Z}\}$ be a frame for $L^2(\mathbb{R}^d)$. Suppose that* (1.1) *is satisfied for some constants n_0, ρ. Assume that $\hat{\psi}$ is continuous and that*

$$|\hat{\psi}(\omega)| \leq C\|\omega\|_2^\nu (1 + \|\omega\|_2)^{-\gamma}, \qquad \forall \omega \in \mathbb{R}^d, \tag{1.2}$$

where $C, \nu, \gamma > 0$ are constants and $\gamma - \nu > d + 1$. Then there are some $\varepsilon, \eta > 0$ such that for any matrices A'_j and vectors $b'_{j,k}$ satisfying

$$\|A_j^{-1} A'_j - I\|_2 \leq \varepsilon \quad \text{and} \quad \|A_j^{-1}(b_{j,k} - b'_{j,k})\|_\infty \leq \eta, \qquad \forall j, k \in \mathbb{Z}, \tag{1.3}$$

$\{\tau(A'_j, b'_{j,k})\psi : j, k \in \mathbb{Z}\}$ is also a frame.

For a regular wavelet system of the form

$$\tau(A^j, A^j Bn)\psi(x) = |\det A|^{-j/2}\psi(A^{-j}x - Bn),\ j \in \mathbb{Z}, n \in \mathbb{Z}^d,$$

where A, B are given matrices, we have the following.

Corollary 1.2. *Let $\{\tau(A^j, A^j Bn)\psi : j \in \mathbb{Z}, n \in \mathbb{Z}^d\}$ be a frame for $L^2(\mathbb{R}^d)$, for which A is expansive and B is invertible. Suppose that $\hat{\psi}$ is continuous and satisfies* (1.2).

(i) *There are some $\varepsilon, \eta > 0$ such that for matrices A_j' and vectors λ_n satisfying*

$$\|A^{-j}A_j' - I\|_2 \le \varepsilon \quad \textit{and} \quad \|\lambda_n - n\|_\infty \le \eta,$$

$\{\tau(A_j', A^j B\lambda_n)\psi : j \in \mathbb{Z}, n \in \mathbb{Z}^d\}$ is also a frame.

(ii) *There is some $\delta > 0$ such that for any matrix P satisfying $AP = PA$ and $\|I - P\| < \delta$, $\{\tau(A^j, A^j P^{-1}Bn)\psi : j \in \mathbb{Z}, n \in \mathbb{Z}^d\}$ is also a frame for $L^2(\mathbb{R}^d)$.*

Remarks. 1. If we consider only perturbation of the translation parameters, i.e., letting $A_j' = A_j$, then our proof shows that the η appearing in (1.3) can be determined explicitly by the following inequalities:

$$(1.4) \qquad \begin{cases} \eta^2 n_0 r d C \frac{(1+\delta)^{2d}}{\delta^d}\left(\frac{1}{1-\rho^{-\nu}} + \frac{1}{1-\rho^{-(\gamma-\nu)}}\right)\sum\limits_{|\alpha|=1}\|X^\alpha\hat{\psi}\|_{W,1} < m, \\ \eta < \delta/3, \end{cases}$$

where m is the lower frame bound of $\{\tau(A_j, b_{j,k})\psi : j, k \in \mathbb{Z}\}$ and we assume that $\{A_j^{-1}b_{j,k} : k \in \mathbb{Z}\}$ is a union of r δ-uniformly discrete sequences for all $j \in \mathbb{Z}$.

In particular, for the regular case, i.e., $\{(A_j, b_{j,k}) : j, k \in \mathbb{Z}\} = \{(A^j, A^j Bn) : j \in \mathbb{Z}, n \in \mathbb{Z}^d\}$ for some expansive matrix A and invertible matrix B, we have $n_0 = 1$, $\rho = \|A^{-1}\|_2^{-1}$, $r = 1$, and $\delta = (d^{1/2}\|B^{-1}\|_2)^{-1}$. Hence (1.4) turns out to be

$$(1.5) \quad \begin{cases} \eta^2 dC \frac{(1+d^{1/2}\|B^{-1}\|_2)^{2d}}{(d^{1/2}\|B^{-1}\|_2)^d}\left(\frac{1}{1-\|A^{-1}\|_2^{\nu}} + \frac{1}{1-\|A^{-1}\|_2^{\gamma-\nu}}\right)\sum\limits_{|\alpha|=1}\|X^\alpha\hat{\psi}\|_{W,1} < m, \\ \eta < \delta/3, \end{cases}$$

2. Corollary 1.2 (ii) generalizes a stability result for 1-dimension [2], which states that for certain ψ, $\{\tau(a^j, a^j bk)\psi : j, k \in \mathbb{Z}\}$ remains a frame when b is perturbed.

In the rest of this section we compare our results with the main result in [22], which reads as follows.

Proposition 1.3 ([22, Theorem 2.4])**.** *Let $\{A_j : j \in \mathbb{Z}\}$ be a sequence of invertible matrices such that $\sup_{\omega\in\mathbb{R}^d} \#\left\{j : a < \|A_j^{-T}\omega\|_2 < b\right\} < +\infty, \forall\ 0 < a < b < +\infty$. Suppose that $\hat{\psi}$ is continuous, $\inf_{\omega\in\mathbb{R}^d\setminus\{0\}}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}(A_j^{-T}\omega)\right|^2 > 0$, and there exist constants $C > 0, \alpha > 0$ and $\beta > d$ such that*

$$(1.6) \qquad \left|\hat{\psi}(\omega)\right| \le C\min\{|\omega|^\alpha, |\omega|^{-\beta}\}, \quad \forall \omega \in \mathbb{R}^d \setminus \{0\}.$$

Then for any γ with $0 < \gamma < \min\{2, \beta\}$ there exist positive constants δ_i $(i = 1, 2, 3)$ such that $\{|\det S_j|^{1/2}\psi(S_j x - B\lambda_k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is a frame for $L^2(\mathbb{R}^d)$ for any real nonsingular $d \times d$ matrix sequence $\{S_j\}_{j\in\mathbb{Z}}$ satisfying

$$(1.7) \qquad \left\|(S_j^{-T} - A_j^{-T})\omega\right\|_2 \le \delta_1\|A_j^{-T}\omega\|_2, \quad \forall\omega \in \mathbb{R}^d, j \in \mathbb{Z},$$

any nonsingular matrix B with $\|B\| \le \delta_2$, and any sequence $\{\lambda_k\}_{k\in\mathbb{Z}^d} \subset \mathbb{R}^d$ with $\sum_{k\in\mathbb{Z}^n}\|k - \lambda_k\|_2^\gamma < \delta_3/(|\det B| \cdot \|B\|^\gamma)$.

In order to compare the results, we first note that Theorem 1.1 always allows simultaneous perturbation of A_j and $b_{j,k}$; cf. the condition (1.3). This is not the case with Proposition 1.3. In fact, for a given frame, the conditions in Proposition 1.3 do not imply that the frame property of $\{|\det S_j|^{1/2}\psi(S_jx - B\lambda_k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ holds for other choices of S_j than the obvious choice $S_j = A_j$. This observation follows from the proof of Proposition 1.3 in [22], which shows that δ_2 depends on δ_1; that is, it can happen that the given matrix B does not satisfy the condition $||B|| \le \delta_2$ for any $\delta_1 > 0$. In this case we are forced to use $S_j = A_j$.

When Proposition 1.3 is considered as a stability result, it says that

If $\{\tau(A_j, A_jBk)\psi : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is a frame, ψ and A_j satisfy certain conditions, then one can find some $\eta > 0$ such that $\{\tau(A_j, A_jB\lambda_k)\psi : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is a frame whenever $\|B\|_2$ is small enough and $\sum_{k\in\mathbb{Z}^d} \|k-\lambda_k\|_2^\gamma < \eta$, where $0 < \gamma < 2$ is a constant.

Obviously, the condition $\sum_{k\in\mathbb{Z}^d} \|k - \lambda_k\|_2^\gamma < \eta$ forces the difference between k and λ_k to tend to zero as k tends to infinity. But the result only proves the existence of a stability bound η. On the other hand, Theorem 1.1 gives an explicit stability bound for η for this type of perturbation with similar assumptions on ψ and A_j; cf. (1.4) and (1.5). Looking at the condition (1.3) we also note that Theorem 1.1 has two more important features: we can perturb each translation parameter with the same amount, and we do not require $b_{j,k}$ to have a special form.

Finally, we note that our approach is quite different from that in other papers, e.g., [22]. In particular, Hölder's inequality on a special form plays a key role in our approach; it is not used in [22].

2. Proofs

Let $\psi, f \in L^2(\mathbb{R}^d)$. The wavelet transform of f with respect to ψ is defined by

$$(W_\psi f)(A, b) = \langle f, \tau(A, b)\psi\rangle,$$

where A is an expansive matrix and $b \in \mathbb{R}^d$.

First, we state a result on uniform continuity of the wavelet transform.

Lemma 2.1. *Suppose that $\psi, f \in L^2(\mathbb{R}^d)$. For any $\varepsilon > 0$, there is some $\delta > 0$, depending only on ψ and ε, such that*

$$|(W_\psi f)(A, t + b) - (W_\psi f)(A', t + b')| < \varepsilon\|f\|_2$$

for any matrices A, A' and vectors b, b', t satisfying

$$\|A^{-1}A' - I\|_2 < \delta \quad \text{and} \quad \|A^{-1}(b - b')\|_2 < \delta.$$

Proof. It is easy to check that

$$\begin{aligned}
& |(W_\psi f)(A, t + b) - (W_\psi f)(A', t + b')| \\
= \ & |\langle \tau(I, -t)f, \tau(A, b)\psi - \tau(A', b')\psi\rangle| \\
\le \ & \|f\|_2 \cdot \|\tau(A, b)\psi - \tau(A', b')\psi\|_2 \\
\le \ & \|f\|_2 \cdot (\|\tau(A, b)\psi - \tau(A, b')\psi\|_2 + \|\tau(A, b')\psi - \tau(A', b')\psi\|_2) \\
= \ & \|f\|_2 \cdot (\|\tau(A, b)\psi - \tau(A, b')\psi\|_2 + \|\tau(A, 0)\psi - \tau(A', 0)\psi\|_2) \\
= \ & \|f\|_2 \cdot \left(\left\|\psi - \tau(I, A^{-1}(b - b'))\psi\right\|_2 + \left\|\psi - \tau(A^{-1}A', 0)\psi\right\|_2\right).
\end{aligned}$$

If $\psi \in C_c(\mathbb{R}^d)$, the space of functions which are continuous and compactly supported, then we can make $\|\psi - \tau(1, A^{-1}(b-b'))\psi\|_2$ and $\|\psi - \tau(A^{-1}A', 0)\psi\|_2$ arbitrarily small by choosing δ sufficiently small. Since $C_c(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, the conclusion holds for any $f, \psi \in L^2(\mathbb{R}^d)$. This completes the proof. □

Lemma 2.2. *Let $\{\tau(A_j, b_{j,k})\psi : j, k \in \mathbb{Z}\}$ be a Bessel sequence in $L^2(\mathbb{R}^d)$. Then there are some constants $r, \delta > 0$ such that for any $j \in \mathbb{Z}$, $\{A_j^{-1}b_{j,k} : k \in \mathbb{Z}\}$ is the union of at most r δ-uniformly discrete sequences.*

Proof. Let $f_j = \tau(A_j, 0)\psi$. By Lemma 2.1, we can find a constant $\delta > 0$, which is independent of j, such that

$$|\langle f_j, \tau(A_j, b)\psi\rangle - \langle f_j, \tau(A_j, 0)\psi\rangle| \le \frac{1}{2}\|\psi\|_2^2,$$

whenever $\|A_j^{-1}b\|_\infty \le \delta$. Hence, in that case,

$$|\langle f_j, \tau(A_j, b)\psi\rangle| \ge |\langle f_j, \tau(A_j, 0)\rangle\psi| - \frac{1}{2}\|\psi\|_2^2 = \frac{1}{2}\|\psi\|_2^2. \tag{2.1}$$

Let M be the upper frame bound for $\{\tau(A_j, b_{j,k})\psi : j, k \in \mathbb{Z}\}$. For any $n \in \mathbb{Z}^d$, take some $x_{j,n} \in \mathbb{R}^d$ such that $A_j^{-1}x_{j,n} = -n\delta$. Then we have

$$\begin{aligned} M\|\psi\|_2^2 &= M\|f_j(\cdot + x_{j,n})\|_2^2 \ge \sum_{k\in\mathbb{Z}} |\langle f_j(\cdot + x_{j,n}), \tau(A_j, b_{j,k})\psi\rangle|^2 \\ &= \sum_{k\in\mathbb{Z}} |\langle f_j, \tau(A_j, x_{j,n} + b_{j,k})\psi\rangle|^2. \end{aligned}$$

It follows from (2.1) that

$$\#\left\{k : \left\|A_j^{-1}b_{j,k} - n\delta\right\|_\infty \le \frac{\delta}{2}\right\} = \#\left\{k : \left\|A_j^{-1}(x_{j,n} + b_{j,k})\right\|_\infty \le \frac{\delta}{2}\right\} \le \frac{4M}{\|\psi\|_2^2}.$$

Hence we can split $\{A_j^{-1}b_{j,k} : k \in \mathbb{Z}\}$ into at most $(\lceil 8M/\|\psi\|_2^2\rceil + 1)^d$ δ-uniformly discrete subsequences. This completes the proof. □

Our perturbation result is based on the following general statement from [4].

Proposition 2.3. *Suppose that $\{g_n : n \in I\}$ is a frame for a Hilbert space $\mathcal{H}$ and M_1 and M_2 are lower and upper frame bounds, respectively. Let $\{h_n : n \in I\}$ be a sequence of functions in $\mathcal{H}$. If*

$$\sum_{n\in I} |\langle f, g_n - h_n\rangle|^2 \le \Delta\|f\|_2^2, \qquad \forall f \in \mathcal{H},$$

for some constant $\Delta < M_1$, then $\{h_n : n \in I\}$ is a frame for $\mathcal{H}$ with frame bounds $(M_1^{1/2} - \Delta^{1/2})^2$ and $(M_2^{1/2} + \Delta^{1/2})^2$.

The univariate version of the following result appeared in [3, Lemma 39]. Here we give a multivariate version, which can be proved similarly.

Lemma 2.4. *Let $\delta > 0$ be a constant. Then, for any δ-uniformly discrete sequence $\{b_k : k \in \mathbb{Z}\} \subset \mathbb{R}^d$ and cube E with side length 1, $\{e^{i2\pi\langle\cdot, b_k\rangle} : k \in \mathbb{Z}\}$ is a Bessel sequence for $L^2(E)$ with upper bound $M_\delta = \delta^{-d}(1+\delta)^{2d}$.*

Lemma 2.5. *Let $\psi \in L^2(\mathbb{R}^d)$ be such that*

$$|\hat{\psi}(\omega)| \le C\|\omega\|_2^{\nu}(1+\|\omega\|_2)^{-\gamma}, \tag{2.2}$$

where $\gamma - d > \nu > 0$ are constants. Suppose that $\{A_j : j \in \mathbb{Z}\}$ is a sequence of invertible matrices and there is some $\delta > 0$ such that $\{A_j^{-1} b_{j,k} : k \in \mathbb{Z}\}$ is δ-uniformly discrete for all $j \in \mathbb{Z}$.

(i) *If there is a constant $\rho > 1$ such that*

$$\|A_{j+1}^T \omega\|_2 \ge \rho \|A_j^T \omega\|_2, \qquad \forall j \in \mathbb{Z}, \omega \in \mathbb{R}^d, \tag{2.3}$$

then $\{\tau(A_j, b_{j,k})\psi : j,k \in \mathbb{Z}\}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$ with upper bound $C\|\hat{\psi}\|_{W,1} M_\delta \cdot \left(\frac{1}{1-\rho^{-\nu}} + \frac{1}{1-\rho^{-(\gamma-\nu)}}\right)$.

(ii) *Let $\{A_j' : j \in \mathbb{Z}\}$ be a sequence of matrices such that*

$$\|A_j^{-1} A_j' - I\|_2 \le \varepsilon, \tag{2.4}$$

where $0 < \varepsilon < 1/2$. Then $\{\tau(A_j, b_{j,k})\psi - \tau(A_j', b_{j,k})\psi : j,k \in \mathbb{Z}\}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$ with upper bound $M(\psi, \delta, \rho, \nu, \gamma, C, \varepsilon)$ satisfying $\lim_{\varepsilon \to 0} M(\psi, \delta, \rho, \nu, \gamma, C, \varepsilon) = 0$.

Proof. (i) By (2.2), $\hat{\psi} \in W(\mathbb{R}^d)$. For any $n = (n_1, \cdots, n_d) \in \mathbb{Z}^d$, let

$$\begin{aligned} E_n &= \prod_{\ell=1}^{d} [n_\ell, n_\ell + 1), \\ p_n &= \left\| \hat{\psi} \cdot \chi_{E_n} \right\|_\infty + 2^{-(|n_1| + \cdots + |n_d|)} \eta, \end{aligned} \tag{2.5}$$

where $\eta > 0$ is a constant. Then we have

$$\sum_{n \in \mathbb{Z}} p_n \le C_1 := \|\hat{\psi}\|_{W,1} + 3^d \eta.$$

For any $f \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\sum_{j,k \in \mathbb{Z}} |\langle f, \tau(A_j, b_{j,k})\psi \rangle|^2 \\
= &\sum_{j,k \in \mathbb{Z}} \left| \int_{\mathbb{R}^d} \hat{f}(\omega) |\det A_j|^{1/2} \overline{\hat{\psi}(A_j^T \omega)} e^{i2\pi\langle \omega, b_{j,k} \rangle} d\omega \right|^2 \\
= &\sum_{j,k \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}^d} \int_{E_n} \hat{f}(A_j^{-T}\omega) |\det A_j|^{-1/2} \overline{\hat{\psi}(\omega)} e^{i2\pi\langle \omega, A_j^{-1} b_{j,k} \rangle} d\omega \right|^2 \\
\le &\sum_{j,k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}^d} \frac{1}{p_n} \left| \int_{E_n} \hat{f}(A_j^{-T}\omega) |\det A_j|^{-1/2} \overline{\hat{\psi}(A_j^T \omega)} e^{i2\pi\langle \omega, A_j^{-1} b_{j,k} \rangle} d\omega \right|^2 \cdot \sum_{n \in \mathbb{Z}^d} p_n \\
\le &\ C_1 \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}^d} \frac{1}{p_n} \sum_{k \in \mathbb{Z}} \left| \int_{E_n} \hat{f}(A_j^{-T}\omega) |\det A_j|^{-1/2} \overline{\hat{\psi}(\omega)} e^{i2\pi\langle \omega, A_j^{-1} b_{j,k} \rangle} d\omega \right|^2 \\
\le &\ C_1 \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}^d} \frac{1}{p_n} M_\delta \int_{E_n} \left| \hat{f}(A_j^{-T}\omega) \right|^2 \cdot |\det A_j|^{-1} \left| \hat{\psi}(\omega) \right|^2 d\omega \\
&\quad \text{(using Lemma 2.4)}
\end{aligned}$$

$$\begin{aligned} &\le C_1 M_\delta \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}^d} \int_{A_j^{-T}E_n} |\hat{f}(\omega)|^2 \cdot \left|\hat{\psi}(A_j^T\omega)\right| d\omega \\ &= C_1 M_\delta \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 \cdot \left|\hat{\psi}(A_j^T\omega)\right| d\omega. \end{aligned}$$

For any $\omega \neq 0$, we see from (2.3) that there is some $j_0 \in \mathbb{Z}$ such that

$$\|A_{j_0}^T\omega\|_2 \le 1 < \|A_{j_0+1}^T\omega\|_2. \tag{2.6}$$

Consequently,

$$\|A_j^T\omega\| \le \rho^{j-j_0}, \qquad \forall j \le j_0,$$

and

$$\|A_j^T\omega\| \ge \rho^{j-j_0-1}, \qquad \forall j \ge j_0+1,$$

Therefore,

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}(A_j^T\omega)\right| &= \sum_{j\le j_0} \left|\hat{\psi}(A_j^T\omega)\right| + \sum_{j\ge j_0+1} \left|\hat{\psi}(A_j^T\omega)\right| \\ &\le \sum_{j\le j_0} C(\rho^{j-j_0})^\nu + \sum_{j\ge j_0+1} C(\rho^{j-j_0-1})^{-(\gamma-\nu)} \\ &\le \frac{C}{1-\rho^{-\nu}} + \frac{C}{1-\rho^{-(\gamma-\nu)}}. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{j,k\in\mathbb{Z}} |\langle f, \tau(A_j, b_{j,k})\psi\rangle|^2 &\le C_1 M_\delta \left(\frac{C}{1-\rho^{-\nu}} + \frac{C}{1-\rho^{-(\gamma-\nu)}}\right) \|f\|_2^2 \\ &= C(\|\hat{\psi}\|_{W,1} + 3^d\eta) M_\delta \left(\frac{1}{1-\rho^{-\nu}} + \frac{1}{1-\rho^{-(\gamma-\nu)}}\right) \|f\|_2^2. \end{aligned}$$

By letting η tend to 0, we get (i).

(ii) Let E_n be defined as in (2.5) and

$$p_{j,n} = \|\hat{\psi}\cdot\chi_{E_n}\|_\infty + |\det A_j^{-1}A_j'|^{1/2} \cdot \|\hat{\psi}({A_j'}^T A_j^{-T}\omega)\cdot\chi_{E_n}(\omega)\|_\infty + 2^{|n_1|+\cdots+|n_d|}\eta.$$

By (2.4), $\|A_j^{-1}A_j'\|_2 \le 1+\varepsilon \le 2$. Hence $|\det A_j^{-1}A_j'| \le \|A_j^{-1}A_j'\|_2^d \le 2^d$. On the other hand, since $\|A_j^{-1}A_j' - I\|_2 \le \varepsilon$, we have $\|A_j^{-1}A_j'\omega\|_2 \ge (1-\varepsilon)\|\omega\|_2$ for any $\omega \in \mathbb{R}^d$. Hence $\|{A_j'}^{-1}A_j\omega\|_2 \le \frac{\|\omega\|_2}{1-\varepsilon}$. Therefore, $\|A_j^T {A_j'}^{-T}\|_2 = \|{A_j'}^{-1}A_j\|_2 \le \frac{1}{1-\varepsilon}$.

Let $n, n' \in \mathbb{Z}^d$ be such that $E_m \cap {A_j'}^T A_j^{-T} E_n, E_m \cap {A_j'}^T A_j^{-T} E_{n'} \neq \emptyset$. Then we can find some $y_n \in E_n$ and $y_{n'} \in E_{n'}$ such that ${A_j'}^T A_j^{-T} y_n, {A_j'}^T A_j^{-T} y_{n'} \in E_m$. Hence,

$$\begin{aligned} 1 &\ge \left\|{A_j'}^T A_j^{-T}(y_n - y_{n'})\right\|_\infty \\ &\ge \frac{1}{\sqrt{d}} \left\|{A_j'}^T A_j^{-T}(y_n - y_{n'})\right\|_2 \\ &\ge \frac{1}{\sqrt{d}} \left\|A_j^T {A_j'}^{-T}\right\|_2^{-1} \cdot \|y_n - y_{n'}\|_2 \\ &\ge \frac{1}{\sqrt{d}} \cdot (1-\varepsilon)\,\|y_n - y_{n'}\|_\infty \\ &\ge \frac{1}{\sqrt{d}} \cdot (1-\varepsilon)(\|n-n'\|_\infty - 1). \end{aligned}$$

Therefore,

$$\|n-n'\|_\infty \le \frac{d^{1/2}}{1-\varepsilon}+1 \le 2d^{1/2}+1.$$

Consequently,

$$\#\Big\{n:\, E_m \cap {A'_j}^T A_j^{-T} E_n \ne \emptyset\Big\} \le 2^d (d^{1/2}+1)^d.$$

It follows that

$$\begin{aligned}
& \sum_{n\in\mathbb{Z}^d} |\det A_j^{-1}A'_j|^{1/2}\cdot \|\hat\psi({A'_j}^T A_j^{-T}\omega)\cdot \chi_{E_n}(\omega)\|_\infty \\
\le\ & \sum_{n\in\mathbb{Z}^d} \sum_{\substack{m\in\mathbb{Z}^d \\ E_m\cap {A'_j}^T A_j^{-T}E_n\ne\emptyset}} 2^{d/2}\|\hat\psi\cdot\chi_{E_m}\|_\infty \\
=\ & \sum_{m\in\mathbb{Z}^d} \sum_{\substack{n\in\mathbb{Z}^d \\ E_m\cap {A'_j}^T A_j^{-T}E_n\ne\emptyset}} 2^{d/2}\|\hat\psi\cdot\chi_{E_m}\|_\infty \\
\le\ & 2^{3d/2}(d^{1/2}+1)^d \sum_{m\in\mathbb{Z}^d}\|\hat\psi\cdot\chi_{E_m}\|_\infty \\
=\ & 2^{3d/2}(d^{1/2}+1)^d \|\hat\psi\|_{W,1}.
\end{aligned}$$

Hence

$$\sum_{n\in\mathbb{Z}^d} p_{j,n} \le C'_1 = \Big(2^{3d/2}(d^{1/2}+1)^d+1\Big)\,\|\hat\psi\|_{W,1}+3^d\eta.$$

Similar to the first part we can prove that

$$\begin{aligned}
(2.7)\quad & \sum_{j,k\in\mathbb{Z}} \big|\langle f, \tau(A_j,b_{j,k})-\tau(A'_j,b_{j,k})\psi\rangle\big|^2 \\
=\ & \sum_{j,k\in\mathbb{Z}} \left|\int_{\mathbb{R}^d} \hat f(\omega)\left(|\det A_j|^{1/2}\overline{\hat\psi(A_j^T\omega)} - |\det A'_j|^{1/2}\overline{\hat\psi({A'_j}^T\omega)}\ \right) e^{i2\pi\langle\omega,b_{j,k}\rangle}d\omega\right|^2 \\
=\ & \sum_{j,k\in\mathbb{Z}} \Bigg| \sum_{n\in\mathbb{Z}^d}\int_{E_n} \hat f(A_j^{-T}\omega)\Big(|\det A_j|^{-1/2}\overline{\hat\psi(\omega)} \\
& \qquad -|\det A_j|^{-1}|\det A'_j|^{1/2}\overline{\hat\psi({A'_j}^T A_j^{-T}\omega)}\ \Big)e^{i2\pi\langle\omega,A_j^{-1}b_{j,k}\rangle}d\omega\Bigg|^2 \\
\le\ & \sum_{j,k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\frac{1}{p_{j,n}}\Bigg|\int_{E_n}\hat f(A_j^{-T}\omega)\Big(|\det A_j|^{-1/2}\overline{\hat\psi(\omega)} \\
& \qquad -|\det A_j|^{-1}|\det A'_j|^{1/2}\overline{\hat\psi({A'_j}^T A_j^{-T}\omega)}\ \Big)e^{i2\pi\langle\omega,A_j^{-1}b_{j,k}\rangle}d\omega\Bigg|^2\cdot\sum_{n\in\mathbb{Z}^d}p_{j,n} \\
\le\ & C'_1\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\frac{1}{p_{j,n}}M_\delta\int_{E_n}\left|\hat f(A_j^{-T}\omega)\right|^2\cdot\Big||\det A_j|^{-1/2}\hat\psi(\omega) \\
& \qquad -|\det A_j|^{-1}|\det A'_j|^{1/2}\hat\psi({A'_j}^T A_j^{-T}\omega)\Big|^2 d\omega \\
& \quad \text{(using Lemma 2.4)}
\end{aligned}$$

$$\begin{aligned}
&= C_1' \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}^d} \frac{1}{p_{j,n}} M_\delta \int_{A_j^{-T}E_n} \left|\hat{f}(\omega)\right|^2 \cdot \left|\hat{\psi}(A_j^T\omega) - |\det A_j^{-1}A_j'|^{1/2}\hat{\psi}({A_j'}^T\omega)\right|^2 d\omega \\
&\le C_1' \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}^d} M_\delta \int_{A_j^{-T}E_n} \left|\hat{f}(\omega)\right|^2 \cdot \left|\hat{\psi}(A_j^T\omega) - |\det A_j^{-1}A_j'|^{1/2}\hat{\psi}({A_j'}^T\omega)\right| d\omega \\
&= C_1' \sum_{j\in\mathbb{Z}} M_\delta \int_{\mathbb{R}^d} \left|\hat{f}(\omega)\right|^2 \cdot \left|\hat{\psi}(A_j^T\omega) - |\det A_j^{-1}A_j'|^{1/2}\hat{\psi}({A_j'}^T\omega)\right| d\omega.
\end{aligned}$$

Let $\Delta > 0$ be a constant. Fix some $\omega \neq 0$. Note that

$\|{A_j'}^T\omega\|_2 = \|{A_j'}^T A_j^{-T} A_j^T\omega\|_2 \le \|{A_j'}^T A_j^{-T}\|_2 \cdot \|A_j^T\omega\|_2 \le (1+\varepsilon)\|A_j^T\omega\|_2 \le 2\|A_j^T\omega\|_2.$

We have

$$\sum_{j:\, \|A_j^T\omega\|_2 \le 1/N} |\det A_j^{-1}A_j'|^{1/2} \cdot \left|\hat{\psi}({A_j'}^T\omega)\right| \le 2^{d/2} C \sum_{j\le 0} \rho^{j\nu}\left(\frac{2}{N}\right)^\nu \le 2^{d/2} C \frac{(2/N)^v}{1-\rho^{-\nu}}.$$

Similarly, we can get that

$$\begin{aligned}
\sum_{j:\, \|A_j^T\omega\|_2 \ge N} |\det A_j^{-1}A_j'|^{1/2} \cdot \left|\hat{\psi}({A_j'}^T\omega)\right| &\le 2^{d/2} C \sum_{j\ge 0} \rho^{-j(\gamma-\nu)}\left(\frac{N}{2}\right)^{-(\gamma-\nu)} \\
&\le 2^{d/2} C \frac{(N/2)^{-(\gamma-\nu)}}{1-\rho^{-(\gamma-\nu)}}.
\end{aligned}$$

Hence we can choose N large enough such that

$$\sum_{j:\, \|A_j^T\omega\|_2 \notin [1/N, N]} |\det A_j^{-1}A_j'|^{1/2} \cdot \left|\hat{\psi}({A_j'}^T\omega)\right| < \frac{\Delta}{4}. \tag{2.8}$$

By setting $A_j' = A_j$ we get

$$\sum_{j:\, \|A_j^T\omega\|_2 \notin [1/N, N]} \left|\hat{\psi}({A_j}^T\omega)\right| < \frac{\Delta}{4}. \tag{2.9}$$

On the other hand, since $\|A_{j+1}^T\omega\|_2 \ge \rho\|A_j^T\omega\|_2$, we have

$$\#\{j :\, N^{-1} \le \|A_j^T\omega\| \le N\} \le \frac{2\ln N}{\ln \rho} + 1.$$

Note that

$$\begin{aligned}
\|{A_j'}^T\omega - A_j^T\omega\|_2 &= \left\|({A_j'}^T A_j^{-T} - I)A_j^T\omega\right\|_2 \le \left\|{A_j'}^T A_j^{-T} - I\right\|_2 \cdot \|A_j^T\omega\|_2 \\
&= \left\|A_j^{-1}A_j' - I\right\|_2 \cdot \|A_j^T\omega\|_2.
\end{aligned}$$

Using the uniform continuity of $\hat{\psi}$, we can choose ε small enough (depending only on $\hat{\psi}$ and N) such that whenever $\|A_j^{-1}A_j' - I\|_2 \le \varepsilon$,

$$\sum_{j:\, 1/N \le \|A_j^T\omega\|_2 \le N} \left|\hat{\psi}(A_j^T\omega) - |\det A_j^{-1}A_j'|^{1/2}\hat{\psi}({A_j'}^T\omega)\right| < \frac{\Delta}{2}. \tag{2.10}$$

Putting (2.8), (2.9) and (2.10) together, we get

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}(A_j^T\omega) - |\det A_j^{-1}A_j'|^{1/2}\hat{\psi}({A_j'}^T\omega)\right| < \Delta, \qquad \forall\omega.$$

Now we see from (2.7) that

$$\sum_{j,k\in\mathbb{Z}} \left|\langle f, \tau(A_j, b_{j,k}) - \tau(A'_j, b_{j,k})\psi\rangle\right|^2 \le C'_1 M_\delta \Delta.$$

Since Δ is arbitrary, the conclusion follows. □

Proof of Theorem 1.1. First, we have

$$\begin{aligned}
&\left(\tau(A, b')\psi - \tau(A, b)\psi\right)(x)\\
= \;& |\det A|^{-1/2}\left(\psi(A^{-1}x - A^{-1}b') - \psi(A^{-1}x - A^{-1}b)\right)\\
= \;& \int_0^1 |\det A|^{-1/2}\frac{\partial}{\partial_t}\psi\Big(A^{-1}x - A^{-1}(b + t(b' - b))\Big)dt\\
= \;& \sum_{|\alpha|=1}\left(A^{-1}(b - b')\right)^\alpha \int_0^1 |\det A|^{-1/2}(D^\alpha\psi)\Big(A^{-1}x - A^{-1}(b + t(b' - b))\Big)dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\left\langle f, \tau(A_j, b'_{j,k})\psi - \tau(A_j, b_{j,k})\psi\right\rangle\right|^2\\
= \;& \left|\sum_{|\alpha|=1}\left(A_j^{-1}(b_{j,k} - b'_{j,k})\right)^\alpha \int_0^1 \left\langle f, \tau\left(A_j, A_j^{-1}(b_{j,k} + t(b'_{j,k} - b_{j,k}))\right)(D^\alpha\psi)\right\rangle dt\right|^2\\
\le \;& \left\|A_j^{-1}(b_{j,k} - b'_{j,k})\right\|_2^2 \cdot \sum_{|\alpha|=1}\left|\int_0^1 \left\langle f, \tau\left(A_j, A_j^{-1}(b_{j,k} + t(b'_{j,k} - b_{j,k}))\right)(D^\alpha\psi)\right\rangle dt\right|^2\\
\le \;& \eta^2 d \cdot \sum_{|\alpha|=1}\int_0^1 \left|\left\langle f, \tau\left(A_j, A_j^{-1}(b_{j,k} + t(b'_{j,k} - b_{j,k}))\right)(D^\alpha\psi)\right\rangle\right|^2 dt.
\end{aligned}$$

By Lemma 2.2, we can find an integer $r > 0$, a constant $\delta > 0$ and partitions of $\mathbb{Z}$ such that

$$\{A_j^{-1}b_{j,k} : k \in \mathbb{Z}\} = \bigcup_{\ell=1}^{r}\{A_j^{-1}b_{j,k} : k \in \mathbb{Z}_{j,\ell}\},$$

where $\{A_j^{-1}b_{j,k} : k \in \mathbb{Z}_{j,\ell}\}$ is δ-uniformly discrete. Assume that $0 < \eta < \delta/3$. Then $\{A_j^{-1}(b_{j,k} + t(b'_{j,k} - b_{j,k})) : k \in \mathbb{Z}_{j,\ell}\}$ is $\delta/3$-uniformly discrete. Denote $\mathbb{Z}_p = \{jn_0 + p : j \in \mathbb{Z}\}$. By Lemma 2.5, we can find a constant M, depending only on $C, r, n_0, \delta, \rho, \nu, \gamma$, such that

$$\begin{aligned}
&\sum_{j,k\in\mathbb{Z}}\left|\left\langle f, \tau\left(A_j, A_j^{-1}(b_{j,k} + t(b'_{j,k} - b_{j,k}))\right)(D^\alpha\psi)\right\rangle\right|^2\\
= \;& \sum_{p=1}^{n_0}\sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}_p, k\in\mathbb{Z}_{j,\ell}}\left|\left\langle f, \tau\left(A_j, A_j^{-1}(b_{j,k} + t(b'_{j,k} - b_{j,k}))\right)(D^\alpha\psi)\right\rangle\right|^2\\
\le \;& n_0 r\|X^\alpha\hat{\psi}\|_{W,1}M.
\end{aligned}$$

Hence

$$\sum_{j,k\in\mathbb{Z}}\left|\left\langle f, \tau(A_j, b'_{j,k})\psi - \tau(A_j, b_{j,k})\psi\right\rangle\right|^2 \le \eta^2 n_0 r d \sum_{|\alpha|=1}\|X^\alpha\hat{\psi}\|_{W,1}M\|f\|_2^2. \tag{2.11}$$

On the other hand, since $\{A_j b'_{j,k} : k \in \mathbb{Z}\}$ is a union of r $\delta/3$-uniformly discrete sequences, by Lemma 2.5, we can make $\frac{1}{\|f\|_2^2} \sum_{j,k\in\mathbb{Z}} \left| \left\langle f, \tau(A_j, b'_{j,k})\psi - \tau(A'_j, b'_{j,k})\psi \right\rangle \right|^2$ arbitrarily small (independently of f) by choosing ε small enough. Therefore, we can choose η, ε small enough such that $\frac{1}{\|f\|_2^2} \sum_{j,k\in\mathbb{Z}} \left| \left\langle f, \tau(A_j, b_{j,k})\psi - \tau(A'_j, b'_{j,k})\psi \right\rangle \right|^2 \leq \mu$ for some μ which is smaller than the lower frame bound of $\{\tau(A_j, b_{j,k})\psi : j, k \in \mathbb{Z}\}$. By Proposition 2.3, $\{\tau(A'_j, b'_{j,k})\psi : j, k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R}^d)$. □

Proof of Corollary 1.2. The first part is obvious. Let us prove the second part. By Theorem 1.1, we can find some $\delta > 0$ such that for any matrix P with $\|I - P\|_2 \leq \delta$, $\{\tau(A^j P, A^j Bn)\psi : j \in \mathbb{Z}, n \in \mathbb{Z}^d\}$ is also a frame. Or equivalently, we can find constants $M_1, M_2 > 0$ such that for any $f \in L^2(\mathbb{R}^d)$,

$$M_1\|f\|_2^2 \leq \sum_{j\in\mathbb{Z}, n\in\mathbb{Z}^d} \left| \int_{\mathbb{R}^d} f(x) |\det(A^{-j}P^{-1})|^{1/2} \overline{\psi(P^{-1}A^{-j}x - P^{-1}Bn)} dx \right|^2 \leq M_2\|f\|_2^2.$$

Since $AP = PA$, by an exchange of the variable of the form $x \to Px$, we get

$$M_1\|f\|_2^2 \leq \sum_{j\in\mathbb{Z}, n\in\mathbb{Z}^d} \left| \int_{\mathbb{R}^d} f(Px) |\det(A^{-j}P)|^{1/2} \overline{\psi(A^{-j}x - P^{-1}Bn)} dx \right|^2 \leq M_2\|f\|_2^2.$$

Hence $\{\tau(A^j, A^j P^{-1}Bn)\psi : j \in \mathbb{Z}, n \in \mathbb{Z}^d\}$ is a frame for $L^2(\mathbb{R}^d)$. □

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References

[1] A. Aldroubi, C. Cabrelli, and U. Molter, Wavelets on irregular grids with arbitrary dilation matrices, and frame atoms for $L^2(\mathbb{R}^d)$, *Appl. Comput. Harmon. Anal.* **17**(2004), 119–140. MR2082156

[2] R. Balan, Stability theorems for Fourier frames and wavelet Riesz bases, *J. Four. Anal. Appl.*, **3**(1997), 499–504. MR1491930 (99f:42060)

[3] J.J. Benedetto, Irregular sampling and frames, *in* "Wavelets: A Tutorial in Theory and Applications (C.K. Chui, Ed.), 445–507, Academic Press, Boston, 1992. MR1161260 (93c:42030)

[4] O. Christensen, A Paley-Wiener theorem for frames, *Proc. Amer. Math. Soc.*, **123**(1995), 2199-2202. MR1246520 (95i:46027)

[5] O. Christensen, Frames, Riesz bases, and discrete Gabor/wavelet expansions, *Bulletin (New series) of Amer. Math. Soc.*, **38**(2001), 273–291. MR1824891 (2002c:42040)

[6] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003. MR1946982 (2003k:42001)

[7] O. Christensen, S. Favier and F. Zó, Irregular wavelet frames and Gabor frames, *Approx. Theory Appl.*, **17**(2001), 90–101. MR1885530 (2003c:42033)

[8] C.K. Chui and X.L. Shi, Orthonormal wavelets and tight frames with arbitrary real dilations, *Appl. Comput. Harmon. Anal.*, **9**(2000), 243–264. MR1793418 (2002a:42025)

[9] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992. MR1162107 (93e:42045)

[10] S. Favier and R. Zalik, On the stability of frames and Riesz bases, *Appl. Comp. Harm. Anal.*, **2**(1995), 160–173. MR1325538 (96e:42030)

[11] M. Frazier, G. Garrigos, K. Wang, and G. Weiss, *A characterization of functions that generate wavelet and related expansion.* J. Fourier Anal. Appl. **3** (1997), 883–906. MR1600215 (99c:42058)

[12] K. Gröchenig, Irregular sampling of wavelet and short-time Fourier transforms, *Constr. Approx.*, **9**(1993), 283–297. MR1215773 (94m:42077)
[13] C. Heil and D. Walnut, Continuous and discrete wavelet transforms, *SIAM Review*, **31**(1989), 628–666. MR1025485 (91c:42032)
[14] E. Hernandez, D. Labate, and G. Weiss, A unified characterization of reproducing systems generated by a finite family. II, J. Geom. Anal. **12** no. 4 (2002), 615–662. MR1916862 (2003j:42036)
[15] R. S. Laugesen, Completeness of orthonormal wavelet systems for arbitrary real dilations, *Appl. Comput. Harmon. Anal.*, **11**(2001), 455–473. MR1866351 (2002h:42073)
[16] P. Olsen and K. Seip, A note on irregular discrete wavelet transforms, *IEEE Trans. Inform. Theory*, **38**(1992), 861–863. MR1162222
[17] A. Ron and Z. Shen, Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator, *J. Funct. Anal.*, **148**(1997), 408–447. MR1469348 (99g:42043)
[18] W. Sun and X. Zhou, Irregular wavelet frames, *Science in China, series A*, **43**(2000), 122–127. MR1763090 (2000m:42035)
[19] W. Sun and X. Zhou, Irregular wavelet/Gabor frames, *Appl. Comput. Harmon. Anal.*, **13**(2002), 63–76. MR1930176 (2003i:42057)
[20] W. Sun and X. Zhou, Density and stability of wavelet frames, *Appl. Comput. Harmon. Anal.*, **15**(2003), 117–133. MR2007054 (2004h:42045)
[21] W. Sun and X. Zhou, Density of irregular wavelet frames, *Proc. Amer. Math. Soc.*, **132**(2004), 2377-2387. MR2052416
[22] D. Yang and X. Zhou, Wavelet frames with irregular matrix dilations and their stability, *J. Math. Anal. Appl.*, **295**(2004), 97–106. MR2064410
[23] R. Young, *An Introduction to Non-Harmonic Fourier Series,* Academic Press, New York, 1980. MR0591684 (81m:42027)
[24] X. Zhou and Y. Li, A class of irregular wavelet frames, *Chinese Science Bulletin,* **42**(1997), 1420–1424. MR1488511

Department of Mathematics, Technical University of Denmark, Building 303, 2800 Lyngby, Denmark
E-mail address: Ole.Christensen@mat.dtu.dk

Department of Mathematics and LPMC, Nankai University, Tianjin 300071, People's Republic of China – and – NUHAG, Department of Mathematics, University of Vienna, Nordbergstrasse 15, A-1090 Vienna, Austria
E-mail address: sunwch@nankai.edu.cn