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BUBBLING PHENOMENA FOR FOURTH-ORDER FOUR-DIMENSIONAL PDES WITH EXPONENTIAL GROWTH

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ABSTRACT. We are concerned in this paper with the bubbling phenomenon for nonlinear fourth-order four-dimensional PDE's. The operators in the equations are perturbations of the bi-Laplacian. The nonlinearity is of exponential growth. Such equations arise naturally in statistical physics and geometry. As a consequence of our theorem we get a priori bounds for solutions of our equations.

We are concerned in this paper with understanding the bubbling phenomenon for fourth-order four-dimensional PDE's of exponential growth. Such equations arise naturally in statistical physics and in geometry (see [7] and [9]). In what follows, we let (M, g) be a smooth compact Riemannian 4-manifold without boundary. We also let $(b_\varepsilon)_{\varepsilon>0}$ and $(f_\varepsilon)_{\varepsilon>0}$ be sequences of smooth functions on M, and we let $(A_\varepsilon)_{\varepsilon>0}$ be a sequence of smooth $(2,0)$-symetric tensor fields. We assume that (b_ε), (f_ε) and (A_ε) converge as $\varepsilon \to 0$ in the C^k-topologies, k a positive integer, to limiting objects of the same nature, b_0, f_0 and A_0. Then we consider sequences $(u_\varepsilon)_{\varepsilon>0}$ of solutions of

$$\Delta_g^2 u_\varepsilon + R_\varepsilon(x, du_\varepsilon) = f_\varepsilon(x) e^{u_\varepsilon}, \tag{1}$$

where $\Delta_g = -div_g(\nabla .)$ is the Laplace-Beltrami operator and

$$R_\varepsilon(x, du) = -div_g(A_\varepsilon du) + b_\varepsilon. \tag{2}$$

Following standard terminology, we say that the u_ε's blow up if $u_\varepsilon(x_\varepsilon) \to +\infty$ as $\varepsilon \to 0$ for a sequence (x_ε) of points in M. We let

$$L_0 = \Delta_g^2 u - div_g(A_0 du) \tag{3}$$

be the limit operator in (1). At last, we let G be the Green function of L_0. The Green function is unique up to a constant when the kernel of L_0 consists only of constants. We write G as

$$G(x,y) = \frac{1}{8\pi^2} \ln \frac{1}{d_g(x,y)} + \beta(x,y)$$

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for $(x,y) \in M \times M \setminus D$, with $D = \{(x,x),\, x \in M\}$ the diagonal in $M \times M$, where $\beta \in C^1(M \times M)$. We let φ be the function given by

$$\varphi(x) = \int_M G(x,y)\, b_0(y)\, dv_g(y).$$

For u a function on M we let

$$\bar{u} = \frac{1}{Vol_g(M)} \int_M u\, dv_g$$

be the mean value of u, where $Vol_g(M)$ is the volume of M with respect to g. Our theorem is stated as follows:

Theorem 1. *Let (M,g) be a smooth compact Riemannian manifold of dimension 4 without boundary. Let (u_ε) be a blowing-up sequence of solutions of (1). Assume that the kernel of L_0 consists only of constants and that f_0 is a positive function on M. Then*

$$\int_M b_0\, dv_g = 64\pi^2 N$$

for some $N \in \mathbb{N}^\star$. Moreover there exists a finite subset $S \subset M$, consisting of N points x_i, $i = 1, \dots, N$, such that

$$u_\varepsilon - \bar{u}_\varepsilon \to 64\pi^2 \sum_{i=1}^N G(x_i, .) - \varphi$$

in $C^4_{loc}(M \backslash S)$. At last, we have that

$$64\pi^2 \nabla_y \beta(x_i, x_i) + 64\pi^2 \sum_{j \neq i} \nabla_x G(x_i, x_j) - \nabla\varphi(x_i) = -\frac{\nabla f_0(x_i)}{f_0(x_i)}$$

for all $i = 1, \dots, N$.

The proof of Theorem 1 comes with strong pointwise estimates on the u_ε's and the observation that concentration points are isolated (we refer to section 1 for details). This should be compared to the more intricate situation of Yamabe-type equations for which concentration points are not necessarily isolated (see [3, 4, 5, 6]). Independently, as is easily checked, a priori C^4-bounds on sequences of solutions follow from the above theorem when $\int_M b_0\, dv_g \notin 64\pi^2\mathbb{N}$. This includes compactness of the geometric Paneitz equation with arbitrary prescribed Q-curvature (we refer to the nice surveys [1] and [2] for material on the Q-curvature). Such a priori C^4-bounds should be regarded as a first step towards a Morse theory for the equations we consider in this paper. We refer to [11], where this question was handled in the case of the Yamabe equation.

1. Proof of Theorem 1

Let us assume that we have a sequence (u_ε) of smooth solutions of

$$L_\varepsilon u_\varepsilon + b_\varepsilon(x) = f_\varepsilon(x) e^{u_\varepsilon}, \tag{4}$$

where $L_\varepsilon = \Delta_g^2 - div_g(A_\varepsilon d\,.)$. Since we assumed that $Ker\, L_0 = \{constants\}$, it is clear that $Ker\, L_\varepsilon = \{constants\}$ for all $\varepsilon > 0$ small enough. Thus, if the sequence (u_ε) is bounded from above, it follows from standard elliptic theory that (u_ε) is uniformly bounded in $C^4(M)$ except if $\int_M b_0\, dv_g = 0$. This clarifies the remarks after the theorem. From now on, we assume that the u_ε's blow up, i.e. that

$$\max_M u_\varepsilon \to +\infty \text{ as } \varepsilon \to 0. \tag{5}$$

Before starting the proof of Theorem 1, we note that, integrating equation (4),

$$\int_M f_\varepsilon e^{u_\varepsilon}\, dv_g = \int_M b_\varepsilon\, dv_g = \int_M b_0\, dv_g + o(1). \tag{6}$$

We divide the proof into several steps. The first step is as follows:

STEP 1 - *Assume that* (5) *holds. Then there exist* $N \in \mathbb{N}^\star$ *and* N *sequences* $(x_{i,\varepsilon})$ *of converging points in* M *such that, after passing to a subsequence, the following assertions hold:*

a) $\frac{d_g(x_{i,\varepsilon}, x_{j,\varepsilon})}{\mu_{i,\varepsilon}} \to +\infty$ *as* $\varepsilon \to 0$ *for all* $i, j = 1, \dots, N$, $i \neq j$, *where*

$$f_\varepsilon(x_{i,\varepsilon})\, \mu_{i,\varepsilon}^4 e^{u_\varepsilon(x_{i,\varepsilon})} = 1.$$

b) *We have that*

$$v_{i,\varepsilon}(x) = u_\varepsilon\big(\exp_{x_{i,\varepsilon}}(\mu_{i,\varepsilon} x)\big) - u_\varepsilon(x_{i,\varepsilon}) \to V_0(x) = -4 \ln\left(1 + \frac{|x|^2}{8\sqrt{6}}\right)$$

in $C^4_{loc}(\mathbb{R}^4)$ *as* $\varepsilon \to 0$ *for all* $i = 1, \dots, N$.

c) *For all* $i = 1, \dots, N$, *we have that*

$$\lim_{R \to +\infty} \lim_{\varepsilon \to 0} \int_{B_{x_{i,\varepsilon}}(R\mu_{i,\varepsilon})} f_\varepsilon e^{u_\varepsilon}\, dv_g = 64\pi^2.$$

d) *At last, there exists* $C > 0$ *such that*

$$\big(\inf_{i=1,\dots,N} d_g(x_{i,\varepsilon}, x)^4\big) e^{u_\varepsilon(x)} \le C$$

for all $\varepsilon > 0$ *and all* $x \in M$.

Proof of Step 1. We briefly sketch the proof below and we refer to [10] for the details. We let $x_\varepsilon \in M$ be such that $u_\varepsilon(x_\varepsilon) = \max_M u_\varepsilon$. By (5), $u_\varepsilon(x_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. We let $\mu_\varepsilon > 0$ be defined by

$$f_\varepsilon(x_\varepsilon)\, \mu_\varepsilon^4 e^{u_\varepsilon(x_\varepsilon)} = 1 \tag{7}$$

so that $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. For $x \in B_0(\delta \mu_\varepsilon^{-1})$, the Euclidean ball of center 0 and radius $\delta\mu_\varepsilon^{-1}$, $\delta > 0$ small fixed, we let

$$\begin{aligned} & v_\varepsilon(x) = u_\varepsilon\left(\exp_{x_\varepsilon}(\mu_\varepsilon x)\right) - u_\varepsilon(x_\varepsilon), \\ & g_\varepsilon(x) = \left(\exp_{x_\varepsilon}^\star g\right)(\mu_\varepsilon x), \tilde{A}_\varepsilon(x) = \left(\exp_{x_\varepsilon}^\star A_\varepsilon\right)(\mu_\varepsilon x), \\ & \tilde{b}_\varepsilon(x) = b_\varepsilon\left(\exp_{x_\varepsilon}(\mu_\varepsilon x)\right) \text{ and } \tilde{f}_\varepsilon(x) = f_\varepsilon\left(\exp_{x_\varepsilon}(\mu_\varepsilon x)\right). \end{aligned} \tag{8}$$

We then have that

$$\Delta_{g_\varepsilon}^2 v_\varepsilon - \mu_\varepsilon^2 div_{g_\varepsilon}\left(\tilde{A}_\varepsilon dv_\varepsilon\right) + \mu_\varepsilon^4 \tilde{b}_\varepsilon = \frac{\tilde{f}_\varepsilon}{f_\varepsilon(x_\varepsilon)} e^{v_\varepsilon} \tag{9}$$

in $B_0(\delta\mu_\varepsilon^{-1})$. We write with the Green representation formula that

$$u_\varepsilon(x) - \bar{u}_\varepsilon = \int_M G_\varepsilon(x, y)\, L_\varepsilon u_\varepsilon(y)\, dv_g(y)$$

for all $x \in M$, where G_ε is the Green function of L_ε. Using equation (4) and differentiating the above with respect to x, we obtain for $k = 1, 2, 3$ that

$$\begin{aligned} \left|\nabla^k u_\varepsilon\right|_g(x) &\le \int_M \left|\nabla_x^k G_\varepsilon(x,y)\right|_g \left|f_\varepsilon(y)e^{u_\varepsilon(y)} - b_\varepsilon(y)\right| dv_g(y) \\ &\le \int_M \left|\nabla_x^k G_\varepsilon(x,y)\right|_g f_\varepsilon(y)e^{u_\varepsilon(y)}\, dv_g(y) + O(1) \end{aligned}$$

since $b_\varepsilon \to b_0$ in $C^0(M)$ as $\varepsilon \to 0$. Let $y_\varepsilon \in B_{x_\varepsilon}(R\mu_\varepsilon)$, $R > 0$ fixed. We write that

$$\begin{aligned} &\int_M \left|\nabla_x^k G(y_\varepsilon, y)\right|_g e^{u_\varepsilon(y)}\, dv_g(y) \\ &= O\left(\mu_\varepsilon^{-k} \int_{M\setminus B_{y_\varepsilon}(\mu_\varepsilon)} e^{u_\varepsilon}\, dv_g\right) + O\left(e^{u_\varepsilon(x_\varepsilon)} \int_{B_{y_\varepsilon}(\mu_\varepsilon)} d_g(y_\varepsilon, y)^{-k}\, dv_g(y)\right) \\ &= O\left(\mu_\varepsilon^{-k}\right) \end{aligned}$$

thanks to the fact that $u_\varepsilon \le u_\varepsilon(x_\varepsilon)$, to (7) and to standard estimates on the Green function (which are uniform in ε). Together with the definition (8) of v_ε, this gives that (v_ε) is uniformly bounded in $C^3(K)$ for all compact subsets K of $\mathbb{R}^4$. Standard elliptic theory then gives thanks to equation (9) that

$$\lim_{\varepsilon\to 0} v_\varepsilon = V_0 \text{ in } C^4_{loc}\left(\mathbb{R}^4\right) \tag{10}$$

where V_0 is a solution of

$$\Delta_\xi^2 V_0 = e^{V_0} \tag{11}$$

in $\mathbb{R}^4$ satisfying $V_0(x) \le V_0(0) = 0$ for all $x \in \mathbb{R}^4$. Moreover, since

$$\lim_{\varepsilon\to 0} \int_{B_{x_\varepsilon}(R\mu_\varepsilon)} f_\varepsilon e^{u_\varepsilon}\, dv_g = \int_{B_0(R)} e^{V_0}\, dx,$$

equation (6) implies that $e^{V_0} \in L^1\left(\mathbb{R}^4\right)$. From the classification of the solutions of equation (11) by Lin [8], we get that either

$$V_0(x) = -4\ln\left(1 + \frac{|x|^2}{8\sqrt{6}}\right) \tag{12}$$

or there exists $a > 0$ such that

$$\Delta_\xi V_0 \ge a \tag{13}$$

in $\mathbb{R}^4$. Let us prove that we are in the first situation. For that purpose, we write with the Green representation formula and equation (4) that

$$\begin{aligned} &\int_{B_0(R)} \left|\Delta_{g_\varepsilon} v_\varepsilon\right|_{g_\varepsilon} dv_{g_\varepsilon} = \mu_\varepsilon^{-2} \int_{B_{x_\varepsilon}(R\mu_\varepsilon)} \left|\Delta_g u_\varepsilon\right|_g dv_g \\ &\le C\mu_\varepsilon^{-2} \int_{x\in B_{x_\varepsilon}(R\mu_\varepsilon)} \int_{y\in M} \left|\Delta_{g,x} G_\varepsilon(x,y)\right|_g \left(e^{u_\varepsilon(y)} + 1\right) dv_g(y)\, dv_g(x) \\ &\le C\mu_\varepsilon^{-2} \int_{y\in M} \left(e^{u_\varepsilon(y)} + 1\right) \left(\int_{x\in B_{x_\varepsilon}(R\mu_\varepsilon)} d_g(x,y)^{-2}\, dv_g(x)\right) dv_g(y) \\ &\le CR^2 \end{aligned}$$

thanks to standard estimates on the Green function and to (6), where $C > 0$ denotes some constant independent of R and $\varepsilon > 0$. Letting $\varepsilon \to 0$, we get that

$$\int_{B_0(R)} |\Delta_\xi V_0|_\xi \, dx \le CR^2$$

for all $R > 0$. This clearly eliminates the possibility (13). Then (12) must hold. It is then easily checked that

$$\lim_{R\to+\infty} \lim_{\varepsilon\to 0} \int_{B_{x_\varepsilon}(R\mu_\varepsilon)} f_\varepsilon e^{u_\varepsilon} \, dv_g = \int_{\mathbb{R}^4} e^{V_0} \, dx = 64\pi^2. \tag{14}$$

For $k \ge 1$, we say that $\mathcal{H}_k$ holds if there exist $(x_{i,\varepsilon})_{i=1,\dots,k}$ k converging sequences of points in M and $(\mu_{i,\varepsilon})_{i=1,\dots,k}$ k sequences of positive real numbers going to 0 as $\varepsilon \to 0$ such that $f_\varepsilon(x_{i,\varepsilon})\,\mu_{i,\varepsilon}^4 e^{u_\varepsilon(x_{i,\varepsilon})} = 1$ and such that, after passing to a subsequence, the following assertions hold:

(A_k^1) $\frac{d_g(x_{i,\varepsilon},x_{j,\varepsilon})}{\mu_{i,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$ for all $i, j = 1, \dots, N$, $i \ne j$.

(A_k^2) We have that

$$v_{i,\varepsilon}(x) = u_\varepsilon\big(\exp_{x_{i,\varepsilon}}(\mu_{i,\varepsilon} x)\big) - u_\varepsilon(x_{i,\varepsilon}) \to V_0(x) = -4\ln\left(1 + \frac{|x|^2}{8\sqrt{6}}\right)$$

in $C^4_{loc}(\mathbb{R}^4)$ as $\varepsilon \to 0$ for all $i = 1, \dots, N$.

(A_k^3) For all $i = 1, \dots, N$, we have that

$$\lim_{R\to+\infty} \lim_{\varepsilon\to 0} \int_{B_{x_{i,\varepsilon}}(R\mu_{i,\varepsilon})} f_\varepsilon e^{u_\varepsilon} \, dv_g = 64\pi^2.$$

Clearly, with what we said above, $\mathcal{H}_1$ holds. Now we let $k \ge 1$ and assume that $\mathcal{H}_k$ holds. We also assume that

$$\sup_M R_{k,\varepsilon}(x)^4 e^{u_\varepsilon(x)} \to +\infty \quad \text{as } \varepsilon \to 0, \tag{15}$$

where

$$R_{k,\varepsilon}(x) = \min_{i=1,\dots,k} d_g(x_{i,\varepsilon}, x).$$

We prove in the following that, in this situation, $\mathcal{H}_{k+1}$ holds. For that purpose, we let $x_{k+1,\varepsilon} \in M$ be such that

$$R_{k,\varepsilon}(x_{k+1,\varepsilon})^4 e^{u_\varepsilon(x_{k+1,\varepsilon})} = \sup_M R_{k,\varepsilon}(x)^4 e^{u_\varepsilon(x)} \tag{16}$$

and we set

$$\mu_{k+1,\varepsilon} = \left(\frac{1}{f_\varepsilon(x_{k+1,\varepsilon})\, e^{u_\varepsilon(x_{k+1,\varepsilon})}}\right)^{\frac{1}{4}}.$$

Since M is compact, (15) implies that $\mu_{k+1,\varepsilon} \to 0$ as $\varepsilon \to 0$ and that

$$\frac{d_g(x_{i,\varepsilon}, x_{k+1,\varepsilon})}{\mu_{k+1,\varepsilon}} \to +\infty \quad \text{as } \varepsilon \to 0 \tag{17}$$

for all $i = 1, \dots, k$. Thanks to (A_k^2), it is also easily checked that $\frac{d_g(x_{i,\varepsilon},x_{k+1,\varepsilon})}{\mu_{i,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$ for all $i = 1, \dots, k$ so that (A_{k+1}^1) holds. It follows from (16) and (17) that

$$\lim_{\varepsilon\to 0} \sup_{z\in B_{x_{k+1,\varepsilon}}(R\mu_{k+1,\varepsilon})} \big(u_\varepsilon(z) - u_\varepsilon(x_{k+1,\varepsilon})\big) = 0.$$

Mimicking what we did above thanks to the Green representation formula, one then proves that, after passing to a subsequence,

$$u_\varepsilon\left(\exp_{x_{k+1,\varepsilon}}\left(\mu_{k+1,\varepsilon}x\right)\right)-u_\varepsilon\left(x_{k+1,\varepsilon}\right)\to V_0(x)$$

in $C^4_{loc}\left(\mathbb{R}^4\right)$ as $\varepsilon\to 0$. And, as a consequence,

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{B_{x_{k+1,\varepsilon}}\left(R\mu_{k+1,\varepsilon}\right)} f_\varepsilon e^{u_\varepsilon}\,dv_g=64\pi^2.$$

Recollecting the information above, one gets that $\mathcal{H}_{k+1}$ holds. Since (A^1_k) and (A^3_k) of $\mathcal{H}_k$ imply that

$$\int_M f_\varepsilon e^{u_\varepsilon}\,dv_g\geq 64\pi^2 k+o(1),$$

we easily get, thanks to (6) , that there exists a maximal k, $1\leq k\leq\frac{1}{64\pi^2}\int_M b_0\,dv_g$, such that $\mathcal{H}_k$ holds. Arriving to this maximal k, we get that (15) cannot hold. Writing $k=N$, we have finished the proof of Step 1. □

STEP 2 - *For $k=1,2,3$, there exists $C_k>0$ such that*

$$R_\varepsilon(x)^k\left|\nabla^k u_\varepsilon\right|_g(x)\leq C_k$$

for all $x\in M$ and all $\varepsilon>0$. Here,

$$R_\varepsilon(x)=\inf_{i=1,\dots,N} d_g\left(x_{i,\varepsilon},x\right)$$

where the $x_{i,\varepsilon}$'s are as in Step 1.

Proof of Step 2. We use again the Green representation for u_ε that we differentiate. We let $x_\varepsilon\in M$ be such that $x_\varepsilon\neq x_{i,\varepsilon}$ for all $i=1,\dots,N$. Note that, for $x_\varepsilon=x_{i,\varepsilon}$, the estimates of the proposition are obvious. We write, thanks to standard estimates on the Green function, that

$$\left|\nabla^k u_\varepsilon\right|_g(x_\varepsilon)=O\left(\int_M\frac{1}{d_g\left(x_\varepsilon,y\right)^k}e^{u_\varepsilon(y)}\,dv_g(y)\right)+O(1).$$

For $i=1,\dots,N$, we let

$$\Omega_{i,\varepsilon}=\left\{y\in M,\ R_\varepsilon(y)=d_g\left(x_{i,\varepsilon},y\right)\right\}$$

and we write that

$$\begin{aligned}&\int_{\Omega_{i,\varepsilon}}\frac{1}{d_g\left(x_\varepsilon,y\right)^k}e^{u_\varepsilon(y)}\,dv_g(y)\\&=O\left(\frac{1}{d_g\left(x_\varepsilon,x_{i,\varepsilon}\right)^k}\int_{\Omega_{i,\varepsilon}\cap B_{x_{i,\varepsilon}}\left(\frac{d_g(x_\varepsilon,x_{i,\varepsilon})}{2}\right)}e^{u_\varepsilon}\,dv_g\right)\\&\quad+O\left(\int_{\Omega_{i,\varepsilon}\setminus B_{x_{i,\varepsilon}}\left(\frac{d_g(x_{i,\varepsilon},x_\varepsilon)}{2}\right)}\frac{1}{d_g\left(x_\varepsilon,y\right)^k}\frac{1}{d_g\left(y,x_{i,\varepsilon}\right)^4}\,dv_g(y)\right)\\&=O\left(\frac{1}{d_g\left(x_\varepsilon,x_{i,\varepsilon}\right)^k}\right)\end{aligned}$$

thanks to assertion d) of Step 1, to (6) and to some straightforward computations. Step 2 clearly follows. □

STEP 3 - *For any $1 \le \nu < 2$, there exists $\delta_\nu > 0$ and $C_\nu > 0$ such that*

$$\mu_{i,\varepsilon}^{4(1-\nu)} d_g\left(x_{i,\varepsilon}, x\right)^{4\nu} e^{u_\varepsilon(x)} \le C_\nu$$

for all $i = 1, \dots, N$, all $\varepsilon > 0$ and all $x \in B_{x_{i,\varepsilon}}(\delta_\nu)$ where $x_{i,\varepsilon}$ and $\mu_{i,\varepsilon}$ are as in Step 1. In particular, we have that

$$d_g\left(x_{i,\varepsilon}, x_{j,\varepsilon}\right) \ge \delta_0$$

for all $i, j \in \{1, \dots, N\}$, $i \ne j$, where $\delta_0 > 0$ is independent of ε and i, j. At last, this implies that $\bar{u}_\varepsilon \to -\infty$ as $\varepsilon \to 0$.

Proof of Step 3. Fix $1 \le \nu < 2$. We set for $i = 1, \dots, N$

$$R_{i,\varepsilon} = \min_{j \ne i} d_g\left(x_{i,\varepsilon}, x_{j,\varepsilon}\right) \tag{18}$$

and we take some $i \in \{1, \dots, N\}$ such that there exists $\theta > 0$ such that

$$R_{i,\varepsilon} \le \theta R_{j,\varepsilon} \tag{19}$$

for all $j \in \{1, \dots, N\}$. We set

$$\varphi_{i,\varepsilon}(r) = r^{4\nu} \exp\left(\left(Vol_g\left(\partial B_{x_{i,\varepsilon}}(r)\right)\right)^{-1} \int_{\partial B_{x_{i,\varepsilon}}(r)} u_\varepsilon \, d\sigma_g\right) \tag{20}$$

for $0 \le r < inj_g(M)$. A simple consequence of assertion b) of Step 1 is that

$$\varphi'_{i,\varepsilon}\left(R\mu_{i,\varepsilon}\right) < 0 \tag{21}$$

for $\varepsilon > 0$ small and all $R \ge R_\nu$ where $R_\nu^2 = \frac{16\sqrt{6}\nu}{2-\nu}$. We define $r_{i,\varepsilon}$ by

$$r_{i,\varepsilon} = \inf\left\{R_\nu \mu_{i,\varepsilon} \le r \le \frac{R_{i,\varepsilon}}{2} \text{ s.t. } \varphi'_{i,\varepsilon}(r) < 0 \text{ in } \left[R_\nu \mu_{i,\varepsilon}, r\right)\right\}. \tag{22}$$

Note that, by (21), we have that

$$\frac{r_{i,\varepsilon}}{\mu_{i,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0. \tag{23}$$

Let us assume that

$$r_{i,\varepsilon} \to 0 \text{ as } \varepsilon \to 0. \tag{24}$$

For $x \in B_0\left(\delta r_{i,\varepsilon}^{-1}\right)$, $\delta > 0$ small fixed, we set

$$v_{i,\varepsilon}(x) = u_\varepsilon\left(\exp_{x_{i,\varepsilon}}\left(r_{i,\varepsilon} x\right)\right) - C_{i,\varepsilon}, \tag{25}$$

where

$$C_{i,\varepsilon} = \left(Vol_g\left(\partial B_{x_{i,\varepsilon}}(r_{i,\varepsilon})\right)\right)^{-1} \int_{\partial B_{x_{i,\varepsilon}}(r_{i,\varepsilon})} u_\varepsilon \, d\sigma_g. \tag{26}$$

We also set, for $j \in \mathcal{S}_i = \{j \ne i \text{ s.t. } d_g\left(x_{i,\varepsilon}, x_{j,\varepsilon}\right) = O\left(r_{i,\varepsilon}\right)\}$,

$$\tilde{x}_{j,\varepsilon} = r_{i,\varepsilon}^{-1} \exp_{x_{i,\varepsilon}}^{-1}\left(x_{j,\varepsilon}\right) \text{ and } \tilde{x}_j = \lim_{\varepsilon \to 0} \tilde{x}_{j,\varepsilon}, \tag{27}$$

after passing to a subsequence, if necessary. Note that, thanks to (18), to (22) and to the choice of i we made (see (19)), we have that $|\tilde{x}_j| \ge 2$ for all $j \in \mathcal{S}_i$ and that $|\tilde{x}_j - \tilde{x}_k| \ge \frac{2}{\theta}$ for all $j, k \in \mathcal{S}_i$, $j \ne k$. By equation (4), we have that

$$\Delta_{g_{i,\varepsilon}}^2 v_{i,\varepsilon} - r_{i,\varepsilon}^2 div_{g_{i,\varepsilon}}\left(A_{i,\varepsilon} \nabla v_{i,\varepsilon}\right) + r_{i,\varepsilon}^4 b_{i,\varepsilon} = f_{i,\varepsilon} \varphi_{i,\varepsilon}\left(r_{i,\varepsilon}\right) r_{i,\varepsilon}^{4(1-\nu)} e^{v_{i,\varepsilon}} \tag{28}$$

in $B_0\left(\delta r_{i,\varepsilon}^{-1}\right)$, where

$$
\begin{aligned}
&g_{i,\varepsilon}(x)=\left(\exp_{x_{i,\varepsilon}}^{\star} g\right)(r_{i,\varepsilon}x)\,, \quad A_{i,\varepsilon}(x)=\left(\exp_{x_{i,\varepsilon}}^{\star} A_\varepsilon\right)(r_{i,\varepsilon}x)\,,\\
&b_{i,\varepsilon}(x)=b_\varepsilon\left(\exp_{x_{i,\varepsilon}}(r_{i,\varepsilon}x)\right) \text{ and } f_{i,\varepsilon}(x)=f_\varepsilon\left(\exp_{x_{i,\varepsilon}}(r_{i,\varepsilon}x)\right).
\end{aligned}
\tag{29}
$$

Thanks to Step 2, we know that $(v_{i,\varepsilon})$ is uniformly bounded in $C^3(K)$ for all compact subsets K of $\mathbb{R}^4\setminus\{0,\tilde{x}_j\}_{j\in\mathcal{S}_i}$. Thanks to the definition (22) of $r_{i,\varepsilon}$ and to (23), we have that

$$\varphi_{i,\varepsilon}(r_{i,\varepsilon})\le\varphi_{i,\varepsilon}(R\mu_{i,\varepsilon})$$

for all $R>R_\nu$. Thanks to assertion b) of Step 1 and to (23), it is now rather easily checked that

$$\lim_{R\to+\infty}\lim_{\varepsilon\to0}\varphi_{i,\varepsilon}(R\mu_{i,\varepsilon})\,r_{i,\varepsilon}^{4(1-\nu)}=0$$

since $1\le\nu<2$. Thus standard elliptic theory leads, thanks to (28) and (29), that, after passing to a subsequence,

$$v_{i,\varepsilon}\to H_i \text{ in } C^4_{loc}\left(\mathbb{R}^4\setminus\{0,\tilde{x}_j\}_{j\in\mathcal{S}_i}\right) \text{ as } \varepsilon\to0, \tag{30}$$

where H_i satisfies

$$\Delta_\xi^2 H_i=0 \text{ in } \mathbb{R}^4\setminus\{0,\tilde{x}_j\}_{j\in\mathcal{S}_i}\,. \tag{31}$$

Moreover, thanks to Step 2, we have that, for $l=1,\,2,\,3$,

$$R(x)^l|\nabla^l H_i(x)|_\xi\le C_l \text{ in } \mathbb{R}^4\setminus\{0,\tilde{x}_j\}_{j\in\mathcal{S}_i}\,, \tag{32}$$

where

$$R(x)=\min\{|x|;|x-\tilde{x}_j|\}_{j\in\mathcal{S}_i}\,.$$

Equation (32) easily allows us to prove that

$$H_i(x)=\alpha\ln\frac{1}{|x|}+\sum_{j\in\mathcal{S}_i}\alpha_j\ln\frac{1}{|x-\tilde{x}_j|}+\beta, \tag{33}$$

where α, β and the α_j's are real numbers. Integrating equation (28) over $B_0(1)$ and passing to the limit as $\varepsilon\to0$ thanks to (29), (30) and (33), we obtain that

$$\lim_{\varepsilon\to0}\varphi_{i,\varepsilon}(r_{i,\varepsilon})\,r_{i,\varepsilon}^{4(1-\nu)}\int_{B_0(1)}f_{i,\varepsilon}e^{v_{i,\varepsilon}}\,dv_{g_{i,\varepsilon}}=-\int_{\partial B_0(1)}\partial_\nu\Delta_\xi H_i\,d\sigma_\xi=8\alpha\pi^2.$$

With a change of variable, we get that

$$\varphi_{i,\varepsilon}(r_{i,\varepsilon})\,r_{i,\varepsilon}^{4(1-\nu)}\int_{B_0(1)}f_{i,\varepsilon}e^{v_{i,\varepsilon}}\,dv_{g_{i,\varepsilon}}=\int_{B_{x_{i,\varepsilon}}(r_{i,\varepsilon})}f_\varepsilon e^{u_\varepsilon}\,dv_g$$

so that

$$\lim_{\varepsilon\to0}\int_{B_{x_{i,\varepsilon}}(r_{i,\varepsilon})}f_\varepsilon e^{u_\varepsilon}\,dv_g=8\alpha\pi^2. \tag{34}$$

Step 2 with $k=1$ together with the definitions of $R_{i,\varepsilon}$ and $r_{i,\varepsilon}$ gives the existence of some $C>0$ such that for any $0\le r\le 3/2$,

$$\left|u_\varepsilon\left(\exp_{x_{i,\varepsilon}}(r_{i,\varepsilon}x)\right)-u_\varepsilon\left(\exp_{x_{i,\varepsilon}}(r_{i,\varepsilon}y)\right)\right|\le C$$

for all $x, y \in \mathbb{R}^4$ such that $|x| = |y| = r$. With point b) of Step 1, (22) and (23), we then get that for any $\eta > 0$, there exists $R_\eta > 0$ such that for any $R > R_\eta$, we have that

$$d_g(x, x_{i,\varepsilon})^{4\nu} e^{u_\varepsilon(x)} \leq \eta \mu_{i,\varepsilon}^{4(\nu-1)} \tag{35}$$

for all $x \in B_{x_{i,\varepsilon}}(r_{i,\varepsilon}) \setminus B_{x_{i,\varepsilon}}(R\mu_{i,\varepsilon})$. With point b) of Step 1 and (35), we get that

$$\lim_{\varepsilon \to 0} \int_{B_{x_{i,\varepsilon}}(r_{i,\varepsilon})} f_\varepsilon e^{u_\varepsilon} \, dv_g = 64\pi^2.$$

With (34), we obtain that $\alpha = 8$. Integrating on $B_{\tilde{x}_j}(\delta)$ for $\delta > 0$ small instead of $B_0(1)$, one proves in the same way that $\alpha_j \geq 8$ for all $j \in \mathcal{S}_i$. We let

$$\bar{H}_i(r) = \frac{1}{2\pi^2 r^3} \int_{\partial B_0(r)} H_i(x) \, d\sigma.$$

A simple computation gives

$$\frac{d}{dr}\left(r^{4\nu} e^{\bar{H}_i(r)}\right) = 4\left(\nu - 2 - \left(\sum_{j \in \mathcal{S}_i} \frac{\alpha_j}{8|\tilde{x}_j|^2}\right) r^2\right) r^{4\nu-1} e^{\bar{H}_i(r)}$$

for $r \in \left(0, \frac{3}{2}\right)$. Since $\nu < 2$, we get in particular that

$$\frac{d}{dr}\left(r^{4\nu} e^{\bar{H}_i(r)}\right)(1) < 0.$$

This clearly proves that

$$r_{i,\varepsilon} = \frac{R_{i,\varepsilon}}{2} \tag{36}$$

for all i such that (19) holds. Thanks to (24), this in turn implies that $R_{i,\varepsilon} \to 0$ and that $\mathcal{S}_j \neq \emptyset$. Note that, for the moment, we have proved, with the help of Step 2 (see (35)), that the estimate of Step 3 holds if for any $i \in \{1, ..., N\}$, we have that $R_{i,\varepsilon} \not\to 0$ as $\varepsilon \to 0$. Indeed, if this is the case, there exists some $\delta > 0$ such that $R_{j,\varepsilon} \geq \delta$ for all $j \in \{1, \ldots, N\}$, and one can easily repeat the above arguments with any of the j's in $\{1, \ldots, N\}$. Thus, in order to end the proof of the step, it remains to prove that $R_{i,\varepsilon} \not\to 0$ as $\varepsilon \to 0$ for all $i \in \{1, ..., N\}$. We let $i_0 \in \{1, \ldots, N\}$ be such that, up to a subsequence,

$$R_{i_0,\varepsilon} = \min_{i=1,\ldots,N} R_{i,\varepsilon}.$$

We assume by contradiction that

$$\lim_{\varepsilon \to 0} R_{i_0,\varepsilon} = 0.$$

Clearly (19) holds for $i = i_0$, and (36) holds. It then follows from the definition of $\mathcal{S}_{i_0}$ that for any $i \in \mathcal{S}_{i_0}$, there exists $C(i) > 0$ such that

$$R_{i,\varepsilon} \leq C(i) R_{j,\varepsilon}$$

for all $j \in \{1, ..., N\}$. It follows that (19) holds for all $i \in \mathcal{S}_{i_0}$, and that the preceding analysis can be carried out. We pick up $i \in \mathcal{S}_{i_0}$ such that

$$d_g(x_{i,\varepsilon}, x_{i_0,\varepsilon}) \geq d_g(x_{j,\varepsilon}, x_{i_0,\varepsilon})$$

for all $j \in \mathcal{S}_{i_0}$ and all $\varepsilon > 0$. With (27), we get that $|\tilde{x}_{i_0}| \geq |\tilde{x}_j - \tilde{x}_{i_0}|$ for all $j \in \mathcal{S}_{i_0}$. Since $\mathcal{S}_i = (\mathcal{S}_{i_0} \setminus \{i\}) \cup \{i_0\}$, we have that

$$|\tilde{x}_{i_0}| \geq |\tilde{x}_j - \tilde{x}_{i_0}|$$

for all $j \in \mathcal{S}_i$. A consequence of this inequality is that

$$(\tilde{x}_{i_0}, \tilde{x}_j) > 0 \tag{37}$$

for all $j \in \mathcal{S}_i$, where $(\cdot, \cdot)$ denotes the Euclidean scalar product. This amounts to assuming that all the $\tilde{x}_j$'s, $j \in \mathcal{S}_i$, lie in the same half-space where boundary contains 0. Let $0 < \delta < 1$. We write thanks to equation (28) that

$$\begin{aligned} &\int_{B_0(\delta)} \nabla v_{i,\varepsilon} \Delta^2_{g_{i,\varepsilon}} v_{i,\varepsilon} dv_{g_{i,\varepsilon}} - r^2_{i,\varepsilon} \int_{B_0(\delta)} \nabla v_{i,\varepsilon} div_{g_{i,\varepsilon}} \left(A_{i,\varepsilon} \nabla v_{i,\varepsilon} \right) dv_{g_{i,\varepsilon}} \\ &\quad = \varphi_{i,\varepsilon} \left(r_{i,\varepsilon} \right) r_{i,\varepsilon}^{4(1-\nu)} \int_{B_0(\delta)} f_{i,\varepsilon} \nabla e^{v_{i,\varepsilon}} \, dv_{g_{i,\varepsilon}} - r^4_{i,\varepsilon} \int_{B_0(\delta)} b_{i,\varepsilon} \nabla v_{i,\varepsilon} \, dv_{g_{i,\varepsilon}}. \end{aligned}$$

Integrating by parts, using the estimates of Step 2, (6) and (30), one can easily estimate the different terms involved in this equation to arrive to

$$\int_{B_0(\delta)} \nabla v_{i,\varepsilon} \Delta^2_{g_{i,\varepsilon}} v_{i,\varepsilon} \, dv_{g_{i,\varepsilon}} \to 0 \text{ as } \varepsilon \to 0. \tag{38}$$

Using the Cartan expansion of the metric in the exponential chart and the estimates on the derivatives of $v_{i,\varepsilon}$, some integrations by parts then lead with (30) to

$$\begin{aligned} \left(\int_{B_0(\delta)} \nabla v_{i,\varepsilon} \Delta^2_{g_{i,\varepsilon}} v_{i,\varepsilon} \, dv_{g_{i,\varepsilon}} \right)_k \quad \to \quad & - \int_{\partial B_0(\delta)} \partial_k H_i \left(\nabla \Delta_\xi H_i, \nu \right)_\xi \, d\sigma_\xi \\ & + \int_{\partial B_0(\delta)} \partial_{lk} H_i \nu^l \Delta_\xi H_i \, d\sigma_\xi \\ & + \frac{1}{2} \int_{\partial B_0(\delta)} \left(\Delta_\xi H_i \right)^2 \nu_k \, d\sigma_\xi \end{aligned}$$

as $\varepsilon \to 0$. We let

$$H_i(x) = 8 \ln \frac{1}{|x|} + G_i(x).$$

Simple computations then give that

$$\int_{B_0(\delta)} \nabla v_{i,\varepsilon} \Delta^2_{g_{i,\varepsilon}} v_{i,\varepsilon} \, dv_\xi \to 64\pi^2 \nabla G_i(0)$$

as $\varepsilon \to 0$. Coming back to (38), we obtain that $\nabla G_i(0) = 0$, a contradiction with the choice of i we made in (37). This ends the proof of Step 3. Note that the fact that $\bar{u}_\varepsilon \to -\infty$ is a direct consequence of the estimate we just proved and of Step 2. □

We are now in a position to conclude the proof of Theorem 1. Using the estimates of Step 3, it is easily checked that

$$\int_M f_\varepsilon e^{u_\varepsilon} \, dv_g \to 64\pi^2 N \ \text{ as } \varepsilon \to 0,$$

which gives the first assertion of the theorem thanks to (6). Since we already proved that $\bar{u}_\varepsilon \to -\infty$ as $\varepsilon \to 0$, it remains to prove the convergence of $u_\varepsilon - \bar{u}_\varepsilon$ outside the concentration points and to prove the last property of the theorem concerning the location of concentration points. We let $\mathcal{S} = \{x_i\}_{i=1,\dots,N}$, where $x_i = \lim_{\varepsilon \to 0} x_{i,\varepsilon}$. We let $x_0 \in M \setminus \mathcal{S}$ and we write with the Green representation formula that

$$u_\varepsilon(x_0) - \bar{u}_\varepsilon = \int_M G_\varepsilon(x_0, y)\left(f_\varepsilon(y)e^{u_\varepsilon(y)} - b_\varepsilon(y)\right) dv_g(y),$$

where G_ε is the Green function of L_ε. It is then easy to compute an asymptotic expansion of the different terms involved to get that

$$u_\varepsilon(x_0) - \bar{u}_\varepsilon \to 64\pi^2 \sum_{i=1}^{N} G(x_0, x_i) - \int_M G(x_0, y)\, b_0(y)\, dv_g(y) \tag{39}$$

as $\varepsilon \to 0$, where G is the Green function of the limit operator L_0. The convergence result in the theorem easily follows. The last part of the theorem is a consequence of a Pohozaev-type identity. More precisely, we write in the exponential chart around $x_i \in \mathcal{S}$ and for $\delta > 0$ small enough that

$$\int_{B_{x_i}(\delta)} (L_\varepsilon u_\varepsilon + b_\varepsilon)\nabla u_\varepsilon\, dv_g = \int_{B_{x_i}(\delta)} f_\varepsilon e^{u_\varepsilon} \nabla u_\varepsilon\, dv_g$$

thanks to equation (4). Integration by parts together with the dominated convergence theorem then lead to

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{B_{x_i}(\delta)} f_\varepsilon e^{u_\varepsilon} \nabla u_\varepsilon\, dv_g = -64\pi^2 \frac{\nabla f_0(x_i)}{f_0(x_i)}$$

thanks to Steps 1 to 3 and to (39). On the other hand, after integration by parts, using (39), rather long but easy computations lead to

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{B_{x_i}(\delta)} (L_\varepsilon u_\varepsilon + b_\varepsilon)\nabla u_\varepsilon\, dv_g = 64\pi^2 \nabla G_i(x_i),$$

where

$$G_i(x) = 64\pi^2 \beta(x_i, x) + 64\pi^2 \sum_{j \neq i}^{N} G(x, x_j) - \int_M G(x, y)\, b_0(y)\, dv_g(y)$$

with β the regular part of G. The last assertion of the theorem follows.

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