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THE SET OF COMMON FIXED POINTS OF A ONE-PARAMETER CONTINUOUS SEMIGROUP OF MAPPINGS IS $F\big(T(1)\big) \cap F\big(T(\sqrt{2})\big)$

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Abstract. In this paper we prove the following theorem: Let $\{T(t) : t \geq 0\}$ be a one-parameter continuous semigroup of mappings on a subset C of a Banach space E. The set of all fixed points of $T(t)$ is denoted by $F\big(T(t)\big)$ for each $t \geq 0$. Then

$$\bigcap_{t \geq 0} F\big(T(t)\big) = F\big(T(1)\big) \cap F\big(T(\sqrt{2})\big)$$

holds. Using this theorem, we discuss convergence theorems to a common fixed point of $\{T(t) : t \geq 0\}$.

1. Introduction

Let C be a subset of a Banach space E, and let T be a *nonexpansive mapping* on C, i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We know that T has a fixed point in the case that E is uniformly convex and C is bounded, closed and convex; see Browder [5], Göhde [9], and Kirk [13]. We denote by $F(T)$ the set of all fixed points of T.

Let τ be a Hausdorff topology on E. A family of mappings $\{T(t) : t \geq 0\}$ is called a *one-parameter τ-continuous semigroup of mappings* on C if the following are satisfied:

- (sg 1) $T(s + t) = T(s) \circ T(t)$ for all $s, t \geq 0$;
- (sg 2) for each $x \in C$, the mapping $t \mapsto T(t)x$ from $[0, \infty)$ into C is continuous with respect to τ.

As the topology τ, we usually consider the strong topology of E. A family of mappings $\{T(t) : t \geq 0\}$ is called a *one-parameter τ-continuous semigroup of nonexpansive mappings* on C (*nonexpansive semigroup*, for short) if (sg 1), (sg 2) and the following (sg 3) are satisfied:

- (sg 3) for each $t \geq 0$, $T(t)$ is a nonexpansive mapping on C.

We know that $\{T(t) : t \geq 0\}$ has a common fixed point in the case that E is uniformly convex and C is bounded, closed and convex; see Browder [5]. Moreover,

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in 1974, Bruck [8] proved that $\{T(t) : t \geq 0\}$ has a common fixed point in the case that C is weakly compact, convex, and has the fixed point property for nonexpansive mappings.

In this paper we prove the following theorem: Let E be a Banach space and let τ be a Hausdorff topology on E. Let $\{T(t) : t \geq 0\}$ be a one-parameter τ-continuous semigroup of mappings on a subset C of E. Then

$$\bigcap_{t \geq 0} F\big(T(t)\big) = F\big(T(1)\big) \cap F\big(T(\sqrt{2})\big)$$

holds. Using this theorem, we discuss convergence theorems to a common fixed point of $\{T(t) : t \geq 0\}$.

2. PRELIMINARIES

Throughout this paper we denote by $\mathbb{Q}$ the set of all rational numbers, and by $\mathbb{N}$ the set of all positive integers. For a real number t, we denote by $[t]$ the maximum integer not exceeding t. It is obvious that for each real number t, there exists $\varepsilon \in [0, 1)$ such that $t = [t] + \varepsilon$.

We recall that a Banach space E is called *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. E is called *uniformly convex* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x + y\|/2 < 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. It is clear that a uniformly convex Banach space is strictly convex. The norm of E is called *Fréchet differentiable* if for each $x \in E$ with $\|x\| = 1$, $\lim_{t \to 0}(\|x + ty\| - \|x\|)/t$ exists and is attained uniformly in $y \in E$ with $\|y\| = 1$.

The following lemma is a corollary of Bruck's result in [7].

Lemma 1 (Bruck [7]). *Let C be a subset of a strictly convex Banach space E. Let S and T be nonexpansive mappings from C into E with a common fixed point. Then for each $\lambda \in (0, 1)$, a mapping U from C into E defined by $Ux = \lambda Sx + (1 - \lambda)Tx$ for $x \in C$ is nonexpansive and $F(U) = F(S) \cap F(T)$ holds.*

Proof. It is obvious that $F(U) \supset F(S) \cap F(T)$. Fix $x \in F(U)$ and $w \in F(S) \cap F(T)$. Then we have

$$\begin{aligned} \|x - w\| &= \|\lambda Sx + (1 - \lambda)Tx - w\| \\ &\leq \lambda\|Sx - w\| + (1 - \lambda)\|Tx - w\| \\ &\leq \lambda\|x - w\| + (1 - \lambda)\|x - w\| \\ &= \|x - w\| \end{aligned}$$

and hence

$$\|x - w\| = \|\lambda Sx + (1 - \lambda)Tx - w\| = \|Sx - w\| = \|Tx - w\|.$$

So, from the strict convexity of E, we obtain

$$x = \lambda Sx + (1 - \lambda)Tx = Sx = Tx.$$

That is, $x \in F(S) \cap F(T)$. This completes the proof. □

The following four convergence theorems for nonexpansive mappings are well known.

Theorem 1 (Baillon [2])**.** *Let T be a nonexpansive mapping on a bounded closed convex subset C of a Hilbert space E. Let $x \in C$ and define a sequence $\{x_n\}$ in C by $x_n = \sum_{k=1}^{n} T^k x/n$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Theorem 2 (Reich [17])**.** *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable. Let T be a nonexpansive mapping on a bounded closed convex subset C of E. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and $x_{n+1} = \alpha_n T x_n + (1-\alpha_n) x_n$ for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Theorem 3 (Browder [6])**.** *Let T be a nonexpansive mapping on a bounded closed convex subset C of a Hilbert space E. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ converging to 0. Fix $u \in C$ and define a sequence $\{x_n\}$ in C by $x_n = (1-\lambda_n) T x_n + \lambda_n u$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Theorem 4 (Wittmann [24])**.** *Let T be a nonexpansive mapping on a bounded closed convex subset C of a Hilbert space E. Let $u \in C$ and define a sequence $\{x_n\}$ in C by $x_1 \in C$ and $x_{n+1} = (1-\lambda_n) T x_n + \lambda_n u$ for $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence in $[0,1]$ satisfying $\lim_n \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

3. LEMMAS

In this section we prove two lemmas, which are used in Section 4.

Lemma 2. *Let t be a nonnegative real number and let $\{\beta_n\}$ be a sequence in $(0,\infty)$ converging to 0. Define sequences $\{\delta_n\}$ in $[0,\infty)$ and $\{k_n\}$ in $\mathbb{N} \cup \{0\}$ as follows:*

- $\delta_1 = t$;
- $k_n = [\delta_n/\beta_n]$ *for* $n \in \mathbb{N}$;
- $\delta_{n+1} = \delta_n - k_n\beta_n$ *for* $n \in \mathbb{N}$.

Then the following hold:

(i) $0 \leq \delta_{n+1} < \beta_n$ *for all* $n \in \mathbb{N}$;
(ii) $k_n \in \mathbb{N} \cup \{0\}$ *for all* $n \in \mathbb{N}$;
(iii) $\{\delta_n\}$ *converges to* 0;
(iv) $\sum_{j=1}^{n} k_j \beta_j + \delta_{n+1} = t$ *for all* $n \in \mathbb{N}$;
(v) $\sum_{j=1}^{\infty} k_j \beta_j = t$.

Proof. We put $\varepsilon_n \in [0,1)$ with $\delta_n / \beta_n = k_n + \varepsilon_n$ for $n \in \mathbb{N}$. We have

$$\delta_{n+1} = \delta_n - k_n\beta_n = \varepsilon_n \beta_n < \beta_n$$

for all $n \in \mathbb{N}$. From this, we also have $\delta_{n+1} = \varepsilon_n \beta_n \geq 0$. This implies (i). It is obvious that (ii) and (iii) follow from (i). Let us prove (iv) by induction. We have

$$k_1 \beta_1 + \delta_2 = k_1 \beta_1 + (\delta_1 - k_1 \beta_1) = \delta_1 = t.$$

We assume (iv) holds for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \sum_{j=1}^{n+1} k_j \beta_j + \delta_{n+2} &= \sum_{j=1}^{n+1} k_j \beta_j + (\delta_{n+1} - k_{n+1} \beta_{n+1}) \\ &= \sum_{j=1}^{n} k_j \beta_j + \delta_{n+1} = t. \end{aligned}$$

So, by induction, we obtain (iv). It is obvious that (v) follows from (iii) and (iv). This completes the proof. ☐

Lemma 3. *Let α and β be positive real numbers satisfying $\alpha/\beta \notin \mathbb{Q}$. Define sequences $\{\alpha_n\}$ in $(0,\infty)$ and $\{k_n\}$ in $\mathbb{N}$ as follows:*

- $\alpha_1 = \max\{\alpha, \beta\}$;
- $\alpha_2 = \min\{\alpha, \beta\}$;
- $k_n = [\, \alpha_n \,/\, \alpha_{n+1} \,]$ *for all* $n \in \mathbb{N}$;
- $\alpha_{n+2} = \alpha_n - k_n\, \alpha_{n+1}$ *for all* $n \in \mathbb{N}$.

Then the following hold:

(i) $0 < \alpha_{n+1} < \alpha_n$ *for all* $n \in \mathbb{N}$;
(ii) $k_n \in \mathbb{N}$ *for all* $n \in \mathbb{N}$;
(iii) $\alpha_n \,/\, \alpha_{n+1} \notin \mathbb{Q}$ *for all* $n \in \mathbb{N}$;
(iv) $\{\alpha_n\}$ *converges to* 0.

Proof. We note that (i) implies (ii). We shall prove (i), (ii) and (iii) by induction. By the assumption of $\alpha/\beta \notin \mathbb{Q}$, we have $\alpha \neq \beta$ and hence

$$\alpha_1 = \max\{\alpha, \beta\} > \min\{\alpha, \beta\} = \alpha_2 > 0.$$

It is obvious that $\alpha_1 \,/\, \alpha_2 \notin \mathbb{Q}$. We assume that $0 < \alpha_{j+1} < \alpha_j$ and $\alpha_j \,/\, \alpha_{j+1} \notin \mathbb{Q}$ for some $j \in \mathbb{N}$. Since $\alpha_{j+2} = \alpha_j - k_j\, \alpha_{j+1}$, we have

$$\frac{\alpha_{j+2}}{\alpha_{j+1}} = \frac{\alpha_j}{\alpha_{j+1}} - k_j \notin \mathbb{Q}$$

and hence $\alpha_{j+1} \,/\, \alpha_{j+2} \notin \mathbb{Q}$. Put $\varepsilon_j \in [0,1)$ satisfying $\alpha_j \,/\, \alpha_{j+1} = k_j + \varepsilon_j$. Since $\alpha_j/\alpha_{j+1} \notin \mathbb{Q}$, we note that $\varepsilon_j > 0$. We have

$$\alpha_{j+2} = \alpha_j - k_j\, \alpha_{j+1} = \varepsilon_j\, \alpha_{j+1} < \alpha_{j+1}.$$

From this, we also have $\alpha_{j+2} = \varepsilon_j\, \alpha_{j+1} > 0$. Therefore we have shown that $0 < \alpha_{j+2} < \alpha_{j+1}$ and $\alpha_{j+1} \,/\, \alpha_{j+2} \notin \mathbb{Q}$. By induction, we obtain (i), (ii) and (iii). Let us prove (iv). Since $\{\alpha_n\}$ is a sequence of positive real numbers and strictly decreasing, $\{\alpha_n\}$ converges to some $\alpha_\infty \in [0,\infty)$. Arguing by contradiction, we assume $\alpha_\infty > 0$. Then we can choose $j \in \mathbb{N}$ such that

$$\alpha_\infty < \alpha_{j+1} < \alpha_j < 2\, \alpha_\infty.$$

We have

$$k_j = \left[\frac{\alpha_j}{\alpha_{j+1}} \right] = 1 \quad \text{and} \quad \alpha_{j+2} = \alpha_j - k_j\, \alpha_{j+1} = \alpha_j - \alpha_{j+1} < \alpha_\infty.$$

This is a contradiction. Therefore $\alpha_\infty = 0$, that is, we obtain (iv). This completes the proof. ☐

4. Main results

In this section we give our main results. We first prove the following.

Proposition 1. *Let E be a Banach space and let τ be a Hausdorff topology on E. Let $\{T(t) : t \geq 0\}$ be a one-parameter τ-continuous semigroup of mappings on a subset C of E. Let $\{\alpha_n\}$ be a sequence in $[0,\infty)$ converging to $\alpha_\infty \in [0,\infty)$, and satisfying $\alpha_n \neq \alpha_\infty$ for all $n \in \mathbb{N}$. Suppose that $z \in C$ satisfies $T(\alpha_n)z = z$ for all $n \in \mathbb{N}$. Then z is a common fixed point of $\{T(t) : t \geq 0\}$.*

Proof. We note that

$$T(\alpha_\infty)z = \tau\text{-}\lim_{n\to\infty} T(\alpha_n)z = z.$$

We put $\beta_n = |\alpha_n - \alpha_\infty| > 0$ for $n \in \mathbb{N}$. By the assumption, $\{\beta_n\}$ is a sequence in $(0,\infty)$ converging to 0. Since

$$\max\{\alpha_n, \alpha_\infty\} = \min\{\alpha_n, \alpha_\infty\} + \beta_n,$$

we have

$$\begin{aligned} T(\beta_n)z &= T(\beta_n) \circ T\big(\min\{\alpha_n, \alpha_\infty\}\big)z \\ &= T\big(\beta_n + \min\{\alpha_n, \alpha_\infty\}\big)z = T\big(\max\{\alpha_n, \alpha_\infty\}\big)z \\ &= z \end{aligned}$$

for all $n \in \mathbb{N}$. We also have

$$T(0)z = T(0) \circ T(\alpha_1)z = T(0 + \alpha_1)z = T(\alpha_1)z = z.$$

Fix $t > 0$. Then by Lemma 2, there exists a sequence $\{k_n\}$ in $\mathbb{N} \cup \{0\}$ such that

$$\sum_{n=1}^{\infty} k_n \beta_n = t.$$

For each $n \in \mathbb{N}$ with $\sum_{j=1}^{n} k_j \beta_j > 0$, we obtain

$$\begin{aligned} T\left(\sum_{j=1}^{n} k_j \beta_j\right) z &= T(\beta_n)^{k_n} \circ T(\beta_{n-1})^{k_{n-1}} \circ \cdots \circ T(\beta_2)^{k_2} \circ T(\beta_1)^{k_1} z \\ &= T(\beta_n)^{k_n} \circ T(\beta_{n-1})^{k_{n-1}} \circ \cdots \circ T(\beta_2)^{k_2} z \\ &= \cdots = T(\beta_n)^{k_n} z \\ &= z, \end{aligned}$$

where $T(\beta_j)^0$ is the identity mapping on C. This implies

$$T(t)z = \tau\text{-}\lim_{n\to\infty} T\left(\sum_{j=1}^{n} k_j \beta_j\right) z = z.$$

This completes the proof. □

Now, we prove one of our main results.

Proposition 2. *Let E be a Banach space and let τ be a Hausdorff topology on E. Let $\{T(t) : t \geq 0\}$ be a one-parameter τ-continuous semigroup of mappings on a subset C of E. Let α and β be positive real numbers satisfying $\alpha/\beta \notin \mathbb{Q}$. Then*

$$\bigcap_{t\geq 0} F\big(T(t)\big) = F\big(T(\alpha)\big) \cap F\big(T(\beta)\big)$$

holds.

Proof. It is obvious that

$$\bigcap_{t\geq 0} F\big(T(t)\big) \subset F\big(T(\alpha)\big) \cap F\big(T(\beta)\big).$$

So we shall prove the converse inclusion. We fix $z \in F\big(T(\alpha)\big) \cap F\big(T(\beta)\big)$. Define sequences $\{\alpha_n\}$ in $(0,\infty)$ and $\{k_n\}$ in $\mathbb{N}$ as in Lemma 3. By the assumption, we have

$$T(\alpha_1)z = T(\max\{\alpha,\beta\})z = z \quad \text{and} \quad T(\alpha_2)z = T(\min\{\alpha,\beta\})z = z.$$

If $T(\alpha_j)z = T(\alpha_{j+1})z = z$, then we have

$$T(\alpha_{j+2})z = T(\alpha_{j+2}) \circ T(\alpha_{j+1})^{k_j} z = T(\alpha_{j+2} + k_j\,\alpha_{j+1})z = T(\alpha_j)z = z.$$

So, by induction, we obtain $T(\alpha_n)z = z$ for all $n \in \mathbb{N}$. Since $\{\alpha_n\}$ is a sequence of positive real numbers converging to 0, Proposition 1 yields that z is a common fixed point of $\{T(t) : t \geq 0\}$. This completes the proof. □

As a direct consequence of Proposition 2, we obtain the following.

Corollary 1. *Let E, τ, C and $\{T(t) : t \geq 0\}$ be the same as in Proposition* 2. *Then*

$$\bigcap_{t\geq 0} F\big(T(t)\big) = F\big(T(1)\big) \cap F\big(T(\sqrt{2})\big)$$

holds.

Using Lemma 1, we obtain the following.

Corollary 2. *Let E, τ, C, α and β be the same as in Proposition* 2. *Let $\{T(t) : t \geq 0\}$ be a τ-continuous semigroup of nonexpansive mappings on C. Assume that E is strictly convex, and $F\big(T(\alpha)\big) \cap F\big(T(\beta)\big) \neq \varnothing$. Then*

$$\bigcap_{t\geq 0} F\big(T(t)\big) = \{z \in C : \lambda\, T(\alpha)z + (1-\lambda)\, T(\beta)z = z\}$$

holds for every $\lambda \in (0,1)$.

Corollary 3. *Let E, τ, C, α and β be the same as in Proposition* 2. *Let $\{T(t) : t \geq 0\}$ be a τ-continuous semigroup of nonexpansive mappings on C. Assume that E is uniformly convex, and C is bounded, closed and convex. Then*

$$\bigcap_{t\geq 0} F\big(T(t)\big) = \{z \in C : \lambda T(\alpha)z + (1-\lambda)T(\beta)z = z\}$$

holds for every $\lambda \in (0,1)$.

5. Convergence theorems

Several authors have studied convergence theorems for nonexpansive semigroups; see [1, 3, 11, 16, 18, 20, 22] and others. For example, Suzuki and Takahashi in [22] proved the following: Let C be a compact convex subset of a Banach space E and let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on C. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = \frac{\lambda}{t_n} \int_0^{t_n} T(s)x_n\, ds + (1-\lambda)\, x_n$$

for $n \in \mathbb{N}$, where λ is a constant in $(0,1)$, and $\{t_n\}$ is a sequence in $(0,\infty)$ satisfying $\lim_n t_n = \infty$ and $\lim_n t_{n+1} / t_n = 1$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$.

Using Proposition 2, we can prove many convergence theorems for nonexpansive semigroups. In this section, we state some of them. In the following theorems, we make the assumption:

- Let E be a Banach space and let τ be a Hausdorff topology on E. Let $\{T(t) : t \geq 0\}$ be a one-parameter τ-continuous semigroup of nonexpansive mappings on a bounded closed convex subset C of E. Let α and β be positive real numbers satisfying $\alpha/\beta \notin \mathbb{Q}$.

We first state the following, which are connected with Baillon's type iteration [2]; see pages 63 and 83 in [23].

Theorem 5. *Assume that E is a Hilbert space. Let $x \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_n = \frac{\displaystyle\sum_{k=1}^{n}\sum_{\ell=1}^{n} T(k\,\alpha + \ell\,\beta)\,x}{n^2}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T(t) : t \geq 0\}$.

Proof. We note that

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n} T(k\,\alpha + \ell\,\beta)\,x = \sum_{k=1}^{n}\sum_{\ell=1}^{n} T(\alpha)^k \circ T(\beta)^\ell\,x.$$

Thus, $\{x_n\}$ converges weakly to a common fixed point z of $T(\alpha)$ and $T(\beta)$. Such z is a common fixed point of $\{T(t) : t \geq 0\}$ by Proposition 2. This completes the proof. □

Theorem 6. *Assume that E is a Hilbert space. Let $x \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_n = \frac{\displaystyle\sum_{k=1}^{n}\left(\frac{T(\alpha)+T(\beta)}{2}\right)^k x}{n}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T(t) : t \geq 0\}$.

Proof. By Theorem 1, $\{x_n\}$ converges weakly to some $z \in C$ which is a fixed point of $\big(T(\alpha)+T(\beta)\big)/2$. By Corollary 3, we obtain that z is a common fixed point of $\{T(t) : t \geq 0\}$. This completes the proof. □

We next state the following, which are connected with Krasnosel'skiĭ-Mann's type iteration [14, 15]; see Reich [17] and Suzuki [19, 21].

Theorem 7. *Assume that E is a uniformly convex Banach space whose norm is Fréchet differentiable. Fix $\kappa, \lambda > 0$ with $\kappa + \lambda < 1$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \kappa\,T(\alpha)x_n + \lambda\,T(\beta)x_n + (1-\kappa-\lambda)\,x_n,$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T(t) : t \geq 0\}$.

Proof. By Theorem 2, $\{x_n\}$ converges weakly to some $z \in C$ which is a fixed point of

$$\frac{\kappa}{\kappa+\lambda}\,T(\alpha) + \frac{\lambda}{\kappa+\lambda}\,T(\beta).$$

So, by Corollary 3, we obtain that z is a common fixed point of $\{T(t) : t \geq 0\}$. This completes the proof. □

Theorem 8. *Assume that C is compact. Fix $\lambda \in (0,1)$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \lambda \frac{\displaystyle\sum_{k=1}^{n}\sum_{\ell=1}^{n} T(k\,\alpha + \ell\,\beta)x_n}{n^2} + (1-\lambda)\,x_n,$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$.

Using Ishikawa's result in [12], we obtain the following.

Theorem 9. *Assume that C is compact. Fix $\kappa, \lambda \in (0,1)$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \big(\lambda\,T(\alpha) + (1-\lambda)\,I\big) \circ \big(\kappa\,T(\beta) + (1-\kappa)\,I\big)^n x_n$$

for $n \in \mathbb{N}$, where I is the identity mapping on C. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$.

We next state the following, which is connected with Browder's type implicit iteration [6]. We note that $x \mapsto (1-\lambda)\,Tx + \lambda\,u$ is a contractive mapping if T is a nonexpansive mapping and $\lambda \in (0,1)$. Therefore, the Banach contraction principle [4] yields that such a mapping has a unique fixed point.

Theorem 10. *Assume that E is a Hilbert space. Fix $u \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_n = \frac{1-\lambda_n}{2}\,T(\alpha)x_n + \frac{1-\lambda_n}{2}\,T(\beta)x_n + \lambda_n\,u$$

for $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence in $(0,1)$ converging to 0. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$.

We finally state the following, which is connected with Halpern's type explicit iteration [10]; see Wittmann [24].

Theorem 11. *Assume that E is a Hilbert space. Fix $u \in C$ and define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \frac{1-\lambda_n}{2}\,T(\alpha)x_n + \frac{1-\lambda_n}{2}\,T(\beta)x_n + \lambda_n\,u$$

for $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence in $[0,1]$ satisfying the following:

$$\lim_{n\to\infty} \lambda_n = 0; \quad \sum_{n=1}^{\infty} \lambda_n = \infty; \quad \textit{and} \quad \sum_{n=1}^{\infty} |\,\lambda_{n+1} - \lambda_n\,| < \infty.$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$.

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